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Empirical Evaluation of Weibull Distribution in Black-Scholes Call Option Pricing Model

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Abstract

This paper evaluates empirically the effect of Weibull Distribution in Black-Scholes Option Pricing Model. The data for this study were gotten from Australian Clearing House of Australian Securities Exchange (ASX) which consists of 50 enlisted stocks in the clearing house as products of monthly market summary for long term options from 3rd January, 2017 to 31st December, 2019. The data were arranged according to 25, 27, 28, 29 and 30 maturity days. The maximum Likelihood Estimate (MLE) was used to obtain the shape and scale parameters of Weibull Distribution and the goodness of fit test was conducted to ascertain the how fit is Weibull Distribution in Black-Scholes Option Pricing Model and the result revealed that Weibull Distribution is not a good fit in Black-Scholes Model with ($P<0.05$), hence, the null hypothesis is accepted since the ($P>0.05$).

Key words: Black-Scholes, Option Pricing, Weibull Distribution, Goodness – of – Fit Test

1.0 Introduction

The Black-Scholes Option pricing model is one of the most important concepts in modern financial theory. It is a method of pricing and valuation of options. The major breakthrough in option pricing was provided by Black and Scholes in 1973 where they introduced the model that significantly changed the pricing and improved options trading in finance. The original Black-Scholes (1973) model has undergone several theoretical developments such as Black (1976); Black-Scholes-Merton model (1979); German-Kolhagen Black-Scholes model Extension (1983); Practitioners Black-Scholes Model (1998) and other alternative option pricing models, see, for example, Shannon (1987); Fang (2000); Heston and Nandi (2000); Savickas (2002); Madan and Unal (2004); Baek (2006); Jang, et al (2014); Singh (2015).

Hence, in this paper, we empirically evaluate the effect of Weibull distribution in Black-Scholes Option Pricing Model.

2.0 Weibull Distribution and its Properties

The Weibull distribution is a continuous probability distribution originally proposed in 1951 by a Swedish Mathematician named Wallodi Webull as a model for breaking strength of materials. The wide applicability of this model in today's research has find its common use in the assessment of product reliability, analysis of life data and model failures (see for example Rinne, (2007). The Weibull Distribution can also be used in deriving the well-known traditional Black and Scholes Option Pricing Model of 1973.

If random variable X_t is Weibull distributed, its probability density function is given as

$$f(x)=\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{X_t}{\alpha}\right)^{\beta-1}\exp\left\{-\left(\frac{X_t}{\alpha}\right)^{\beta} \quad x_t>0,\beta>0,\alpha>0,\nu=0 \qquad (1)$$

Here, α and β are positive constants, and are the parameters of the Weibull distribution.

The rth moment of the Weibull Distribution is given as

$$\mu_r = \mathrm{E}\left(\mathrm{X}_t^r\right) = \int_0^\infty \beta x_t^{\beta+r-1} \exp\left(-x_t^{\beta}\right) dx_t \qquad (2)$$

and the gamma function is defined as

$$\Gamma(\beta) = \int_0^\infty x_t^{\beta-1} \exp\left\{-x_t^{\beta}\right\} dx_t \qquad (3)$$

Let $U = x_t^{\beta} \Rightarrow x_t = U^{\frac{1}{\beta}}$ which results in $du = \beta x_t^{\beta-1} dx_t \Rightarrow dx_t = \dfrac{du}{\beta x_t^{\beta-1}}$

$$= \frac{du}{\beta U^{\frac{\beta-1}{\beta}}} = U^{\left(\frac{1}{\beta}-1\right)} \frac{du}{\beta} \qquad (4)$$

Hence, the rth moments of the Weibull distribution are obtained as in the properties of Weibull Distribution.

2.1 The Properties of Weibull Distribution

(i) The Mean is obtained as follows:

$$E\left(X_t\right) = \mu$$

$$E\left(X_t^r\right) = \int_0^\infty \beta x_t^{\beta+r-1} \exp\left\{-x_t^{\beta}\right\} dx_t \qquad (5)$$

$$= \int_0^\infty \beta U \left(1 + \frac{r}{\beta}\right)^{-1} - \frac{1}{\beta} \exp\left\{-U\right\} \left(\mu^{\frac{1}{\beta}-1}\right) \frac{du}{\beta}$$

$$\therefore E\left(X_t\right) = \alpha \Gamma\left(1 + \frac{1}{\beta}\right) \qquad (6)$$

(ii) The variance is also obtained as

$$Var(X_t) = E\left(X_t^2\right) - \left[E\left(X_t\right)\right]^2$$

$$\therefore Var\left(X_t\right) = \alpha^2 \Gamma\left(\frac{2}{\beta} + 1\right) - \Gamma^2\left(\frac{1}{\beta} + 1\right) \qquad (7)$$

(iii) Skewness of Weibull Distribution

$$Skew(X_t) = \left[E\left(X_t\right) - E\left(X_t\right)\right]^3 = 2\mu_1^3 - 3\mu_2\mu_1 + \mu_3$$

$$= \frac{1}{\alpha^{\frac{1}{\beta}}}\left[\left[2\gamma_1\right]^3 - 3\gamma_1 \Gamma\left(1 + \frac{2}{\beta}\right) + \Gamma\left(1 + \frac{3}{\beta}\right)\right] \qquad (8)$$

(iv) Kurtosis of Weibull Distribution

$$kurt\left(X_t\right) = \frac{\Gamma\left(1 + \frac{4}{\beta}\right) - 4\gamma_1 \Gamma\left(1 + \frac{3}{\beta}\right) + 6\gamma_1^2 \Gamma\left(1 + \frac{2}{\beta}\right) - 3\gamma_1^4}{\gamma_2} \qquad (9)$$

Where $\gamma_1 = \Gamma\left(1+\frac{1}{\beta}\right)$ and $\gamma_2 = \left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)^2\right]$

(v) The survival and failure (hazard) rates of 2 – parameter Weibull Distribution are respectively given as

$$\bar{F}_{web}(X_t) = 1 - F(X_t) = \exp\left[-\left(\frac{X_t}{\alpha}\right)^{\beta-1}\right], X_t > 0 \quad (10)$$

$$h_{web}(X_t) = \frac{f(X_t)}{\bar{F}(X_t)} = \frac{\beta}{\alpha}\left(\frac{X_t}{\alpha}\right)^{\beta-1}, X_t > 0 \quad (11)$$

2.2 Estimation of the Weibull Parameters using Maximum Likelihood Estimator

The method of maximum likelihood estimation is a commonly used method for estimating parameters of Weibull distribution, see, for example, Cohen (1965), Harter and Moore (1965), Al-fawzan (2000), Cheng and Chen (1988), Johnson, et al (1994) and Nwobi and Ugomma (2014).

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from a population with probability density function $f(x,\theta)$ where $\theta = (\beta,\alpha)$ is an unknown parameters, so that the likelihood function is define by

$$L = f(x_i,\beta,\alpha) = \prod_{i=1}^{n} f(x_i,\beta,\alpha) \quad (12)$$

The maximum likelihood of $\theta = (\beta,\alpha)$, maximizes L or equivalently, the logarithm of L when

$$\frac{\partial \ln L}{\partial \theta} = 0 \quad (13)$$

Consider the Weibull distribution pdf given in equation (1), then, its likelihood function is given as:

$$L(x_1,x_2,\ldots,x_n;\beta,\alpha) = \prod_{i=1}^{n}\left(\frac{\beta}{\alpha}\right)\left(\frac{X_t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{X_t}{\alpha}\right)^{\beta}\right] \quad (14)$$

$$= \left(\frac{\beta}{\alpha}\right)\left(\frac{X_t}{\alpha}\right)^{n\beta-n}\sum_{i=1}^{n} x_i^{(\beta-1)} - \ln\left(\alpha^{\beta-1}\right)\exp\left[-\sum_{i=1}^{n}\left(\frac{X_t}{\alpha}\right)^{\beta}\right]$$

Taking the natural logarithm of both sides, we obtain

$$\ln L = n\ln\left(\frac{\beta}{\alpha}\right) + (\beta-1)\sum_{i=1}^{n} x_i - \ln\left(\alpha^{\beta-1}\right) - \sum_{i=1}^{n}\left(\frac{X_t}{\alpha}\right)^{\beta} \quad (15)$$

Differentiating (15) partially w.r.t β and α in turn equating to zero, we obtain the following estimating equations as:

$$\frac{\partial}{\partial\beta}\ln L = \tfrac{n}{\beta} + \sum_{i=1}^{n}\ln x_i - \frac{1}{\alpha}\sum_{i=1}^{n} x_i^{\beta}\ln x_i = 0 \quad (16)$$

$$\frac{\partial}{\partial\alpha}\ln L = -\tfrac{n}{\alpha} + \frac{1}{\alpha^2}\sum_{i=1}^{n} x_i^{\beta} = 0 \quad (17)$$

From (17), we obtain an estimator of α as

$$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} x_i^{\beta} \tag{18}$$

and on substitution of (17) in (16) we obtain

$$\frac{1}{\beta} + \frac{1}{\alpha}\sum_{i=1}^{n}\ln x_i - \frac{\sum_{i=1}^{n} x_i^{\beta}\ln x_i}{\sum_{i=1}^{n} x_i^{\beta}} = 0 \tag{19}$$

$$\hat{\beta} = \frac{1}{\ln x_i - \sum_{t=1}^{n}\ln x_i} \tag{20}$$

2.3 The Black-Scholes Option Pricing Model under Weibull Distribution

The price of the underlying asset can be expressed as the Weibull Call Option pricing in our case given by

$$C_{web} = e^{-rT}\max\int_{0}^{\infty}(X_t - K) f(X_t)\, dX_t \tag{21}$$

where $f(*)$ is a density function of lognormal distribution in Black – Scholes of 1976, X_t is the underlying price and K is the underlying price of the option.

Substituting equation (21) into equation (1) gives the Weibull call option as

$$f(x_t) = C_{web} = e^{-rT} Max\int_{K}^{\infty}(X_t - K)\left(\frac{\beta}{\alpha}\right)\left(\frac{X_t}{\alpha}\right)\exp\left(-\frac{X_t}{\alpha}\right)^{\beta} dX_t \tag{22}$$

Equation (22) can also be written as

$$f(X_t) = e^{-rT}\int_{K}^{\infty} X_t \exp\left\{-\left(\frac{X_t}{\alpha}\right)^{\beta}\frac{\beta X_t^{\beta-1}}{\alpha^{\beta}}\right\} dX_t - K\exp\left\{-\frac{X_t}{\alpha}\right\}^{\beta} \tag{23}$$

The first part of equation (23) is further simplified as

$$\int_{K}^{\infty} X_t \exp\left\{-\left(\frac{X_t}{\alpha}\right)^{\beta}\frac{\beta X_t^{\beta-1}}{\alpha^{\beta}}\right\} dX_t = \frac{\beta}{\alpha^{\beta}}\int_{K}^{\infty} X_t^{\beta}\exp\left\{-\left(\frac{X_t}{\alpha}\right)^{\beta}\right\} dX_t$$

$$= \frac{\beta}{\alpha^{\beta}}\int_{K}^{\infty}\left(\frac{X_t}{\alpha}\right)^{\beta}\exp\left\{-\left(\frac{X_t}{\alpha}\right)^{\beta}\right\}\frac{\alpha}{\beta}\left(\frac{X_t}{\alpha}\right)^{\beta\left(\frac{1}{\beta-1}\right)} dX_t \tag{24}$$

Let $y = \left(\frac{X_t}{\alpha}\right)^{\beta}$, $Z = \alpha y^{\frac{1}{\beta}}$, and $dz = \frac{\alpha}{\beta} y^{\frac{1}{\beta}-1}$

Then we have

$$\frac{\beta}{\alpha}\int_{\left(\frac{K}{\alpha}\right)^{\beta}}^{\infty}\alpha^{\beta} y\left(\exp(-y)\right)\frac{\alpha}{\beta} y^{\frac{1}{\beta}-1} dy \tag{25}$$

$$= \alpha\Gamma\left(1+\frac{1}{\beta}\right) - \alpha\Gamma\left(1+\frac{1}{\beta}\right)\frac{\int_0^{\left(\frac{K}{\alpha}\right)^{\beta}} y^{\frac{1}{\beta}}\left(\exp(-y)\right)dy}{\Gamma\left(1+\frac{1}{\beta}\right)} \tag{26}$$

where $\Gamma_t(*) = \int_0^t y^{\frac{1}{\beta}-1}\frac{\exp(-y)\,dy}{\Gamma\left(1+\frac{1}{\beta}\right)}$ (27)

Substituting equation (27) into equation (23) and discounting it as the risk free rate yields

$$C_{web} = e^{-rT}\left[\alpha\Gamma\left(1+\frac{1}{\beta}\right)\left(1-\frac{\Gamma\left(\frac{X_t}{\alpha}\right)^{\beta}+\left(1+\frac{1}{\beta}\right)}{\Gamma\left(1+\frac{1}{\beta}\right)}\right)-K\exp\left(-\left(\frac{X_t}{\alpha}\right)^{\beta}\right)\right]$$

$$= e^{-rT}\left[X_0\left(1-\frac{\Gamma y+\left(1+\frac{1}{\beta}\right)}{\Gamma\left(1+\frac{1}{\beta}\right)}\right)-K\exp(-y)\right] \tag{28}$$

Multiplying e^{-rT} to both sides of (28), we obtain the Weibull Call Option as

$$C_{web} = X_0\Phi(d_1) - Ke^{-rT}\Phi(d_2) \tag{29}$$

where $\Phi(*)$ is the standardized normal value obtained from normal distribution table.

$$= X_0 = E(X) = \left(1+\frac{1}{\beta}\right), \quad d_1 = \left[1-\frac{\Gamma y\left(1+\frac{1}{\beta}\right)}{\Gamma\left(1+\frac{1}{\beta}\right)}\right]$$

3.0 Data Description and Method of Analysis

3.1 Data Description

The data for this study were obtained from Australian Clearing House of Australian Securities Exchange (ASX). The sample consists of fifty (50) enlisted stocks in the clearing house as products of monthly market summaries for long term options which consists of the period of January, 3rd 2017 to December, 31st 2019 when there are no significant structural changes among the products. For each transaction, our sample contains the following information: the opening and closing dates of the options, option prices comprising opening and closing prices otherwise referred in our case as the underlying and strike prices respectively. The final sample consists of 50 stocks for the period of 36 months (720 trading days).
The maturity period of the options was gotten from the difference between the opening date and closing date of the options over the trading days. The data for the analysis obtained from http//www.asx.com/au/product/equity_options/options_statistics.htm were arranged in accordance to the maturity days of 25, 27, 28, 29 and 30 days.

3.2 Methodology

The data in each of the maturity days (expiration time) were tested in accordance 252 trading days.
The Computation of the Annualized Standard Deviation (Implied Volatility) is illustrated as follows:

Let $X_i = ABS\ln\left(\frac{X_t}{X_{t-1}}\right)$, X_t is the underlying option price at time t.

$\bar{x} = \frac{1}{n}\sum_{i-1}^{n} x_i$ and $\sigma_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

So that the implied volatility is obtained by

$$\hat{\sigma}_{im} = \sqrt{\frac{\sigma_x^2}{\Delta t}}$$

where, $\Delta t = t_i, t_{i-1}, ..$

$$= \sqrt{\frac{T}{n}\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} = \sqrt{\frac{T}{n}\operatorname{var}(x)} \quad (31)$$

where T is 252 trading days per annum and n is number of stocks.

and the rate of return is estimated by

$$r = \frac{1}{T}\ln\left(\frac{K}{X_0}\right) \quad (32)$$

3.3 Goodness – of – Fit – Test of Weibull Option Pricing

In this study, we wish to test the hypothesis whether Weibull option pricing model is a good fit for pricing options against that it does not.

The test statistic approximately follows the Chi – Square distribution with $k-1$ degrees of freedom at 5% level of significance given by

$$\chi_c^2 = \sum_{j=1}^{N}\frac{\left(P_j - P_j(\hat{\sigma})\right)^2}{P_j(\hat{\sigma})} \quad (33)$$

where

P_j is the observed option price of the jth category and $P_j(\hat{\sigma})$ is the expected (predicted) option pricing model of the jth category and n is the sample size.

Reject the null hypothesis that it is not a good fit if ($p<0.05$) otherwise accept the null hypothesis.

4.0 Application and Empirical Performance

4.1 Application

In this section, we illustrate with the absolute returns of the log difference of the underlying prices of ASX option where we obtained the descriptive statistics of the data, obtain the Maximum Likelihood Estimate of the Weibull parameters and the Black – Scholes Option Pricing Model using Weibull distribution. All the computations were performed using R – Statistical Software.

4.2 Empirical Performance

Table 1: Summary Statistics of Absolute Returns of ASX Original Data

Maturity Days	Sample Size	Sample Mean	Sample Standard Dev	Skewness	Kurtosis	Implied Volatility	Rate of Return (r)
25	99	1.1983	1.0483	1.5652	3.2483	1.67	0.01
27	199	1.2150	1.0801	1.5371	2.9098	1.22	0.02

28	399	1.2408	1.0780	1.4211	2.2671	0.86	0.01
29	449	1.2593	1.1117	1.4916	2.6866	0.83	0.03
30	499	1.2466	1.0890	1.4374	2.3707	0.77	-0.01

The results from Table 1 showed an approximately equal sample means and standard deviations for all the maturity days. The result further indicates that all skewness are positive, thereby showing the right tail of the distribution is longer than left tail. Hence, this result proved that the absolute returns of ASX data for the period of study are normally distributed.

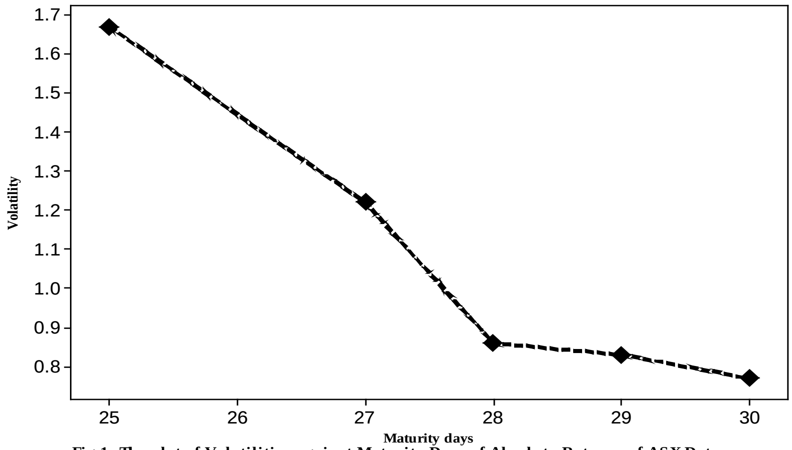

Fig 1: The plot of Volatilities against Maturity Days of Absolute Returns of ASX Data

The implied volatility in Table 1 shows the significant effect it has on the option prices. From Fig 1, we observed that the implied volatility for 25 and 27 maturity days were higher than 28, 29 and 30 maturity days, hence, the lesser the maturity days, the higher the volatility, and vice-versa.

Table 2: Summary Statistics of Weibull Parameters using the Absolute Returns of ASX Data

Maturity Days	Sample Size	$(\hat{\alpha})$	$(\hat{\beta})$	Mean	Std. Dev.	Skewness	Kurtosis
25	99	25.86	3.97	24.43	6.62	-1.43	60.32
27	199	17.07	3.60	15.38	4.75	-1.37	66.53
28	399	41.12	4.26	37.41	9.91	-1.47	56.52
29	449	35.23	4.04	31.95	8.88	-1.45	58.42
30	499	25.77	3.89	23.32	6.73	-1.42	60.32

From the results in Table 2, we observed that Weibull distribution has negative skewness and excess kurtosis. It indicates that the distribution is left tailed. This result also indicates that the distribution is leptokurtic in nature. Since the kurtosis is higher, it showed that the overall risk of financing in these options is determined by a few shocks (outliers).

Table 3: Goodness – of – Test for Weibull Parameters using the Absolute Returns of ASX Data

Maturity Days	Sample Size	χ^2	df	P_value	Decision
25	99	9603	9506	0.2401	Accept
27	199	38400	38208	0.2433	Accept
28	399	149600	149226	0.2466	Accept
29	449	187650	187233	0.2476	Accept
30	499	22055	221610	0.2518	Accept

5.0 Conclusion

In this study, we evaluate empirically the effect of Weibull Distribution in Black-Scholes Option Pricing Model using the goodness – of – fit test and we observed that the null hypothesis of not a good fit was accepted (P>0.05), hence, we conclude that Weibull Distribution is not good fit in Black- Scholes Option Pricing Model using ASX Data for the period under study.

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