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Every Simple Planar Graph Is 4-Colorable

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Abstract:
A simple mathematical proof of 4-color conjecture is proposed.

1. Introduction

The 4 – color conjecture has been referred to as the most famous unsolved problem in graph theory. The first proof of the conjecture was given by Kempe[4]. The proof was found to be erroneous by Heawood [3]. However, he proved the five color theorem. Since then many mathematicians have been trying to settle the conjecture [5]. Apple and Haken [1],[2] have given a proof of the conjecture by using a large scale computer. A simple proof still using a computer has been given by Robertson et. al.[6]. A simple proof of five color theorem for planar graphs is given by Thomessen [7] by showing them to be five choosable. In this paper we present a mathematical proof of the 4-color conjecture for the simple planar graphs.

2. Definitions and Notation:

Let $G=(V,E)$ be a near triangulation that is G is a simple planar graph which consists of a cycle C and vertices and edges inside C, such that each bounded face is bounded by a triangle. Let x be a vertex of C and $\psi=\{P_1,P_2,...,P_k\}$ $k\geq 2$ be any arbitrary partition of vertices of C into paths P_i where $|V(P_i)|\geq 2$ taken in clockwise order such that x is the initial vertex of P_1 and $\overset{k}{\underset{i=1}{U}} P_i = C$. The intersection $P_i \cap P_{i+1}$ for $1\leq i\leq k-1$ and k ≥ 2 contains exactly one vertex of P_i and P_{i+1} which is end vertex of P_i and the initial vertex of P_{i+1}. We say that ψ is the sequence of paths bounding the graph G.

Suppose a near triangulation G is colored with 4 colors α,β,γ and δ. We write $v\to\alpha$ if $v\in V(G)$ is assigned the color α and a path $P_i\to\overline{\alpha}$ if the vertices of $P_i\in\psi$ are colored with at most three colors β,γ and δ. The path $P_i\to\overline{\beta}$ or $\overline{\gamma}$ or $\overline{\delta}$ is similarly defined. If P_i is colored with only two colors say α and β, then both the statements $P_i\to\overline{\gamma}$ and $P_i\to\overline{\delta}$ are true. We shall use $P_i\to\overline{\gamma}$ or $P_i\to\overline{\delta}$ according to our convenience.

3. Results:

Theorem: Let G be a near triangulation with C, x, ψ as defined in the Section 2 above with $|V(G)|\geq 3$. Then there exists a 4 coloring of G such that –

(i) $x\to\alpha$,

(ii) $P_i\to\overline{\beta}$ or $\overline{\gamma}$ or $\overline{\delta}$ for $1\leq i\leq k$,

(iii) If $P_i\to\overline{\beta}$ then $P_{i+1}\to\overline{\gamma}$ or $\overline{\delta}$ and similarly if $P_i\to\overline{\gamma}$ (or $\overline{\delta}$) then $P_{i+1}\to\overline{\delta}$ or $\overline{\beta}$ (or $\overline{\beta}$ or $\overline{\gamma}$) for $1\leq i\leq k-1$.

Proof : Let $y_1\in V(P_1)$ be the vertex adjacent to the vertex x. We have following two cases:

Case A: There is no chord through y_1 in G,

Case B: G has a chord or chords through y_1.

Proof of Case A: We prove the case by induction on $|V(G)|$. Suppose $|V(G)|$ = 3. Then C is a triangle xy_1y_2x. Let $x \to \alpha$, $y_1 \to \beta$ and $y_2 \to \gamma$. We have following three possibilities for ψ :

$\psi_1 = \{P_1, P_2\}$ where $P_1 = xy_1$ and $P_2 = y_1y_2x$,
$\psi_2 = \{P_1, P_2, P_3\}$ where $P_1 = xy_1$, $P_2 = y_1y_2$ and $P_3 = y_2x$,
$\psi_3 = \{P_1, P_2\}$ where $P_1 = xy_1y_2$ and $P_2 = y_2x$.

In ψ_1, $P_1 \to \overline{\gamma}$ or $\overline{\delta}$, $P_2 \to \overline{\delta}$. Let $P_1 \to \overline{\gamma}$ and $P_2 \to \overline{\delta}$. In ψ_2, $P_1 \to \overline{\gamma}$ or $\overline{\delta}$, $P_2 \to \overline{\delta}$ and $P_3 \to \overline{\beta}$ or $\overline{\delta}$ Let $P_1 \to \overline{\gamma}$, $P_2 \to \overline{\delta}$ and $P_3 \to \overline{\beta}$. In ψ_3, $P_1 \to \overline{\delta}$ and $P_2 \to \overline{\beta}$ or $\overline{\delta}$. Let $P_1 \to \overline{\delta}$ and $P_2 \to \overline{\beta}$. This shows that the theorem holds for $|V(G)|$ = 3. in case A.

Suppose the theorem holds for $|V(G)|$ = $n-1$ in case A. We show that the theorem holds for $|V(G)|$ = n in case A.

Let $\psi = \{P_1, P_2, \ldots\ldots, P_k\}$ where $k \geq 2$ and $P_1 = xy_1y_2\ldots.y_{m_1}$ and $P_i = y_{m_{i-1}}y_{m_{i-1}+1}\ldots..y_{m_i}, i = 2,3\ldots.k, y_{m_k} = x$. If the edge $xy_2 \notin E(G)$ then let $x, w_1, w_2, \ldots.w_r, y_2$ where $r \geq 1$ be the vertices adjacent to the vertex y_1 taken in anticlockwise direction around y_1. As the interior of C is triangulated, the graph G contains the path $Q = xy_2$ or $xw_1w_2\ldots w_ry_2$. Thus the graph G-y_1 is also a near triangulation. Corresponding to C and ψ for G, we have C* and ψ* for G-y_1.

Depending on $|V(P_1)|$ we have following three cases:

Case I : $|V(P_1)| \geq 4$,

Case II : $|V(P_1)| = 3$,

Case III : $|V(P_1)| = 2$,

Case I : For G-y_1, we take

$\psi^* = \{Q, P_1^*, P_2, \ldots.., P_k\}$ where $P_1^* = y_2y_3\ldots..y_{m_1}$, $m_1 \geq 3$ *and* $k \geq 2$.

By induction hypothesis, we have a 4-coloring of G-y_1 satisfying the conditions of the theorem. Suppose $x \to \alpha, Q \to \overline{\beta}, P_1^* \to \overline{\gamma}, P_2 \to \overline{\beta}$ if k =2 and so on if k > 2 say. Then $y_{m_1} \to \delta$ (*or* α) . If $y_1 \to \beta$ then we get a 4 – coloring of G with $x \to \alpha, P_1 \to \overline{\gamma}, P_2 \to \overline{\beta}$ if k = 2 and so on for k > 2. The theorem holds in this case, when $|V(P_1)| = n$.

Case II : $|V(P_1)| = 3$

We have following three cases :

(i) $|V(P_1)| = 3$, $|V(P_2)| = 2$ and k = 2,

(ii) $|V(P_1)| = 3$, $|V(P_2)| > 2$ and k = 2,

(iii) $|V(P_1)| = 3$, $|V(P_2)| \geq 2$ and k ≥ 3.

In case (i), $P_1 = xy_1y_2, P_2 = y_2x$. Let $\psi^* = \{Q, P_2\}$ in G – y_1. There is a 4-coloring of G-y_1, by induction hypothesis, such that $Q \to \overline{\beta}, P_2 \to \overline{\gamma}$ say, satisfying the conditions of the theorem. Then $y_2 \to \delta$. We extend the 4-coloring of G-y_1 to that of G by $y_1 \to \beta$ so that $P_1 \to \overline{\gamma}$ and $P_2 \to \overline{\beta}$. Note that since $|V(P_2)|$ = 2, the statements $P_2 \to \overline{\gamma}$, $P_2 \to \overline{\beta}$ are true.

In case (ii), we have $P_1 = xy_1y_2$ and $P_2 = y_2y_3....y_{m_2}$ where $m_2 > 3$ and $y_{m_2} = x$. Let $\psi^* = \{Q^*, P_2 - P_2^/\}$ where Q*= $QUP_2^/$ and $P_2^/ = P_2 - y_{m_2-1}x$ in G – y_1. The graph G-y_1 has a 4-coloring such that Q*$\rightarrow \overline{\beta}$ and $P_2 - P_2^/ \rightarrow \overline{\gamma}$ so that $y_{m_2-1} \rightarrow \delta$ say, by induction hypothesis. We extend the 4-coloring of G-y_1 to that of G by $y_1 \rightarrow \beta$, $y_2 \rightarrow \gamma$ (or δ or α) so that $P_1 \rightarrow \overline{\delta}$ or $\overline{\gamma}$ and $P_2 \rightarrow \overline{\beta}$.

In Case (iii), take $\psi^* = \{Q', P_3, ..., P_k\}$ where, $Q' = QUP_2$ in G-y_1. By induction hypothesis we have a 4-coloring of G-y_1 such that $x \rightarrow \alpha$, $Q' \rightarrow \overline{\beta}$, $P_3 \rightarrow \overline{\gamma}$ if *k=3* and so on for *k>3*, say , so that $y_{m_2} \rightarrow \delta(or\ \alpha)$. Extend this coloring of G-y_1 to that of G by $y_1 \rightarrow \beta$, $y_2 \rightarrow \gamma$ (or α) so that $P_1 \rightarrow \overline{\delta}$, $P_2 \rightarrow \overline{\beta}$ and $P_3 \rightarrow \overline{\gamma}$ if k = 3 and so on for k > 3. Thus the theorem holds in this case also if $|V(G)|$ = n.

Case III : $|V(P_1)| = 2$

In this case we have following two cases for ψ in G:

$\psi_1 = \{P_1, P_2, ..., P_k\}$, where $|V(P_1)| = |V(P_2)| = 2, |V(P_3)| \geq 2$ *and* $k \geq 3$,

$\psi_2 = \{P_1, P_2, ..., P_k\}$, where $|V(P_1)| = 2, |V(P_2)| \geq 3$ *and* $k \geq 2$.

If $P' = P_1 \cup P_2$ then $|V(P')| = 3$ and the first case reduces to case II with $\psi_1^{'} = \{P', P_3, .., P_k\}$, $k \geq 3$. By case II, we have a 4-coloring of G such that $x \rightarrow \alpha, y_1 \rightarrow \beta, P' \rightarrow \overline{\gamma}, P_3 \rightarrow \overline{\beta}$ if k=3 and so on for k > 3 then $y_2 \rightarrow \delta(or\ \alpha)$. Thus we have a desired 4-coloring of G with $x \rightarrow \alpha, P_1 \rightarrow \overline{\delta}, P_2 \rightarrow \overline{\gamma}$ *and* $P_3 \rightarrow \overline{\beta}$ if k = 3 and so on for k>3.

If $P'' = P_1 \cup P_2$ then $|V(P'')| \geq 4$ in the second case so that this case reduces to case I for k≥ 3. For G, let $\psi_2^{'} = \{P'', P_3, ..., P_k\}$, By case I for k$\geq 3$, G has a 4-coloring such that $x \rightarrow \alpha, y_1 \rightarrow \beta, P'' \rightarrow \overline{\gamma}$, $P_3 \rightarrow \overline{\beta}$ if k=3 and so on for k>3. Thus the desired 4-coloring of G is $x \rightarrow \alpha, y_1 \rightarrow \beta, P_1 \rightarrow \overline{\delta}, P_2 \rightarrow \overline{\gamma}$ and $P_3 \rightarrow \overline{\beta}$ for $k = 3$ and so on for k>3.

Suppose k=2 in ψ_2. Let $P''' = y_{m_1}y_{m_1+1}.......y_{m_2-1}$. For G, let $\psi_2^{'} = \{P_1 \cup P''', P_2 - P'''\}$. As $|V(P_1 \cup P''')| \geq 3$, by cases I and II, G has a 4-coloring such that $x \rightarrow \alpha, y_1 \rightarrow \beta, P_1 \cup P''' \rightarrow \overline{\gamma}, P_2 - P''' \rightarrow \overline{\beta}$ so that $y_{m_2-1} \rightarrow \delta$. This gives the desired 4-coloring of G with $x \rightarrow \alpha, P_1 \rightarrow \overline{\delta}, P_2 \rightarrow \overline{\gamma}$. Thus the theorem holds in case III also if $|V(G)| = n$, Hence the theorem holds for all n in case A.

Case B: G has a chord or chords through y_1

Let z be a vertex in the unbounded region of G near y_1. Join z to the vertices of P_1 that is to $x, y_1, y_2, ...y_{m_1}$. Thus we get a new near triangulation G' with $C' = P_1^{'} \cup P_2 \cup .... \cup P_k$ where $P_1^{'} = xzy_{m_1}$. As G' has no chord through z the graph G' has a 4-coloring with $x \rightarrow \alpha$ satisfying the conditions of the theorem in cases I, II and III of case A by case A. Then G'-z gives the desired 4-coloring of G in each of the cases.

We now show explicitly that from 4-coloring of G' we can get the desired 4-coloring of G with $x \rightarrow \alpha$ satisfying the conditions of the theorem.

If $k \geq 3$ then $|V(P_1^{'}) \cup V(P_2)| \geq 4$. Let ψ' for G' be given by $\psi' = \{P_1^{''}, P_3, P_4, ..., P_k\}$ where $P_1^{''} = P_1^{'} \cup P_2$ and $k \geq 3$ in G. Then by case I of case A, G'has a 4-coloring such that $x \to \alpha, z \to \beta, P_1^{''} \to \overline{\gamma}, P_3 \to \overline{\delta}$ so that $y_{m_1} \to \delta (or\, \alpha)$ and $y_{m_2} \to \beta (or\, \alpha)$ if k=3 and so on if k > 3. Then remove z to get the desired 4-coloring of G with $x \to \alpha, P_1 \to \overline{\beta}, P_2 \to \overline{\gamma}$ and $P_3 \to \overline{\delta}$ if $k = 3$ and so on for $k > 3$ in all cases I, II and III of case A when $k \geq 3$.

If $k = 2$ take ψ' for G' as $\psi' = \{P_1^{'} \cup P_2^{''}, P_2 - P_2^{''}\}$ where $P_2^{''} = P_2 - y_{m_2 - 1} x$ if $|V(P_2)| \geq 3$ ($\psi' = \{P_1^{'}, P_2\}$ if $|V(P_2)| = 2$). The graph G' has a 4-coloring such that $x \to \alpha, z \to \beta, P_1^{'} \cup P'' \to \overline{\gamma} (P_1^{'} \to \overline{\gamma})$ and $P_2 - P_2^{''} \to \overline{\delta}\ (P_2 \to \overline{\beta})$ so that $y_{m_2 - 1} \to \beta(\delta)$ by case I (case II) of case A. Then remove z to get the desired 4-coloring of G with $x \to \alpha, P_1 \to \overline{\beta}$ and $P_2 \to \overline{\gamma}$.

Thus the theorem holds in Case B also. This complete the proof of the theorem.
As a consequence of theorem1, we get the following result.
Theorem 2 : Every simple planar graph is 4 colorable.

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