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$\boldsymbol{\eta}$-Ricci solitons on 3-dimensional $\boldsymbol{\alpha}$-para Kenmotsu manifolds with semi-symmetric metric connection

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Absract

The object of the present paper is to study 3-dimensional α-para Kenmotsu manifolds with semi-symmetric metric connection. First, we study 3-dimensional α-para Kenmotsu manifolds with semi-symmetric metric connection satisfying the curvature conditions $\tilde{R}.\tilde{S}=0$ and $\tilde{S}.\tilde{R}=0$. Next, we study η-Ricci solitons on 3-dimensional α-para Kenmotsu manifolds with semi-symmetric metric connection and obtain some results.

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Key Words: 3-dimensional α-para Kenmotsu manifold, η-Ricci solitons, Semi-symmetric metric connection.

1 Introduction

On a Riemannian manifold, Ricci solitons represent a natural generalization of Einstein metric, which being generalized fixed points of hamiltons Ricci flow $\frac{\partial g}{\partial t}=-S$ [16]. This equation defines the Ricci flow, which is a kind of nonlinear diffusion equation, an analogue of the heat equation for metrics. A metric under the Ricci flow can be improved to evolve into a more canonical one by smoothing out its irregularities depending on the Ricci curvature of the manifold. The Ricci soliton will expand in the direction of negative Ricci curvature and shrink in the positive case. Ricci solitons have been studied in many contexts: On contact and Lorentzian manifolds [3], [6], [13], on Sasakian manifolds [4], and K-contact manifolds [17], on f-Kenmotsu manifolds [3] etc. Ricci soliton in para contact geometry firstly appeared in the paper of G. Calvaruso and D. Perrone [7]. C.L. Bejan and M. Crasmareanu [5] recently studied Ricci solitons on 3-dimensional normal para contact manifolds [5].

J.T. Cho and M. Kimura [9] introduced a more general notion of η-Ricci soliton, which was further treated by C. Călin on M. Crasmareanu [3] on hypersurfaces in complex spaces form.

In 1985, Kaneyuki and Williams [20] introduced almost para contact geometry and the further study of almost para contact metric manifolds was done by Zamkovoy [21]. The curvature identities for different classes of paracontact metric manifolds were obtained. A class of α-para Kenmotsu manifolds was studied by Srivastava and Srivastava [11] and Zhen et al [8] studied ξ-conformally flat contact metric manifolds.

Starting with the introduction in section 1, we have a brief introduction of 3-dimensional α-para Kenmotsu manifold in section 2. In section 3, we studied the relation between Riemannian connection and semi-symmetric metric connection in 3-dimensional α-para Kenmotsu manifold. Section 4 deals with 3-dimensional α-para Kenmotsu manifold with semi-symmetric metric connection satisfying the curvature conditions $\tilde{R}.\tilde{S}=0$ and $\tilde{S}.\tilde{R}=0$ respectively. In the last section, we have discussed η-Ricci soliton on 3-dimensional α-para Kenmotsu manifold with semi-symmetric metric connection. We obtained some results and discussed an example.

2 Preliminaries

An odd dimensional smooth manifold M together with an almost paracontact structure (ϕ, ξ, η), where ϕ is an (1,1) tensor field, ξ is a unique vector field, η is a 1-form satisfying

$$\phi^2 X = X - \eta(X)\xi,$$
$$\eta(\xi) = 1,$$
$$\eta \circ \phi = 0,$$
$$\phi\xi = 0, \tag{1}$$

for any vector field $X, Y \in \chi(M)$, is called an almost paracontact manifold. In addition, if g is a pseudo-Riemannian metric then M together with the structure (ϕ, ξ, η, g), satisfying

$$g(X, \xi) = \eta(X), \tag{2}$$

$$g(\phi X, Y) = -g(X, \phi Y) \tag{3}$$

and

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{4}$$

for any vector field $X, Y \in \chi(M)$, is called an almost paracontact metric manifold.

Further, the structure (ϕ, ξ, η, g), satisfying

$$d\eta(X, Y) = g(X, \phi Y) \tag{5}$$

is called paracontact structure and the corresponding manifold is called paracontact metric manifold [21].

On an almost paracontact metric manifold, we define the $(1,2)$ tensor field N_ϕ, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ, by

$$N_\phi = [\phi, \phi] - 2d\eta \otimes \xi. \tag{6}$$

If N_ϕ vanishes identically, then we say that the manifold M is normal almost paracontact metric manifold. The normality condition implies that the almost paracomplex structure J defined on $M \times R$ by

$$J(X, \lambda \tfrac{d}{dt}) = (\phi X + \lambda\xi, \eta(X) \tfrac{d}{dt}),$$

is integrable. Here t is coordinate on R, X is tangent vector to M, and λ be the differentiable function on $M \times R$.

Let M be 3-dimensional almost paracontact metric manifold, then these following conditions are mutually equivalent [10]

1. M is normal.

2.For any vector fields $X,\ Y \in \chi(M)$, the covarient derivative of ϕ is

$$(\nabla_X \phi)Y = \alpha[g(\phi X, Y)\xi - \eta(Y)\phi X] + \beta[g(X, Y)\xi - \eta(Y)X], \tag{7}$$

3. ∃ smooth functions α, β on M such that

$$\nabla_X \xi = \alpha(X - \eta(X)\xi) + \beta\phi X, \tag{8}$$

Consider a 3-dimensional normal almost contact metric manifold
(A) If $\alpha = \beta = 0$, the manifold is called Para-Cosymplectic manifold [15].
(B) The manifold is quasi-para Sasakian manifold iff $\alpha = 0$ and $\beta \neq 0$ [19].
(C) β-para Sasakian manifod iff α=0 and $\beta \neq 0$. If $\beta = 1$, it is called para Sasakian manifold [21].
(D) If $\alpha \neq 0$ and $\beta = 0$, the manifold M is α-para Kenmotsu manifold [8]. Particularly, for $\alpha = 1$, the manifold is para Kenmotsu manifold [1].

A linear connection $\tilde{\nabla}$ on M is said to be semi-symmetric metric connection if its tortion tensor $\tilde{T}$ is of the form

$$\tilde{T}(X, Y) = \eta(Y)X - \eta(X)Y, \tag{9}$$

where η is 1-form defined by $\eta(X) = g(X, \xi)$, ξ being the vector field and all vector fields $X \in \chi(M)$ where $\chi(M)$ is the set of all differentiable vector fields on M.

3 3-dimensional α-para Kenmotsu manifold with semi-symmetric metric connection

An almost paracontact metric 3-dimensional manifold M with the structure (ϕ, ξ, η, g) is an α-para Kenmotsu manifold if the covarient derivative of ϕ satisfies

$$(\nabla_X \phi)Y = \alpha[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{10}$$

for any vector fields $X, Y \in \chi(M)$.

In an 3-dimensional α-para Kenmotsu manifold [11], the following results hold

$$\nabla_X \xi = \alpha(X - \eta(X)\xi), \tag{11}$$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{12}$$

$$\begin{aligned} R(X, Y)Z = {} & (\frac{r}{2} + 2\alpha^2)[g(Y, Z)X - g(X, Z)Y] \\ & -(\frac{r}{2} + 3\alpha^2)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]\xi \\ & +(\frac{r}{2} + 3\alpha^2)[\eta(X)Y - \eta(Y)X]\eta(Z). \end{aligned} \tag{13}$$

Taking $Z = \xi$ in (13) we have

$$R(X, Y)\xi = \alpha^2[\eta(X)Y - \eta(Y)X], \tag{14}$$

and replacing $X = \xi, Y = X \quad and \quad Z = Y$ in equation (13) and using (1), and (2) we have

$$R(\xi, X)Y = -\alpha^2[g(X, Y)\xi - \eta(Y)X]. \tag{15}$$

Also

$$S(X, Y) = (\frac{r}{2} + \alpha^2)g(X, Y) - (\frac{r}{2} + 3\alpha^2)\eta(X)\eta(Y). \tag{16}$$

Replacing $Y = \xi$ in equation (16), we have

$$S(X, \xi) = -2\alpha^2\eta(X). \tag{17}$$

Let $\widetilde{\nabla}$ be linear connection and ∇ be Riemannian connection of an α-para Kenmotsu manifold M. This linear connection $\widetilde{\nabla}$ defined by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi, \tag{18}$$

where η is 1-form and $X, Y \in \chi(M)$, denotes the semi-symmetric metric connection [12]. Now using (18), we have

$$\widetilde{\nabla}_X \xi = (1 + \alpha)(X - \eta(X)\xi). \tag{19}$$

In a 3-dimensional α-para Kenmotsu manifold M, let $\tilde{R}$ be curvature tensor with respect to semi-symmetric metric connection $\widetilde{\nabla}$, then

$$\tilde{R}(X, Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z. \tag{20}$$

Now using (18), (1), (2), (11) and (19) we have

$$\begin{aligned}\tilde{R}(X, Y)Z = R(X, Y)Z - (1 + 2\alpha)[g(Y, Z)X - g(X, Z)Y] \\ +(1 + \alpha)[\eta(Y)X - \eta(X)Y]\eta(Z) \\ +(1 + \alpha)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]\xi.\end{aligned} \tag{21}$$

Taking inner product of equation (21) with W and using equation (2), we have

$$\begin{aligned}g(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) \\ -(1 + 2\alpha)[g(Y, Z)g(X, W) - g(X, Z)g(Y, W] \\ +(1 + \alpha)[\eta(Y)g(X, W) - \eta(X)g(Y, W)]\eta(Z) \\ +(1 + \alpha)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]\eta(W)\end{aligned} \tag{22}$$

where $g(\tilde{R}(X, Y)Z, W) = \tilde{R}(X, Y, Z, W)$ and $g(R(X, Y)Z, W) = R(X, Y, Z, W)$.

Now, contracting equation (22) over X and W, we have

$$\tilde{S}(Y, Z) = S(Y, Z) - (1 + 3\alpha)g(Y, Z) + (1 + \alpha)\eta(Y)\eta(Z) \tag{23}$$

Let $\{e_1, \phi e_1, \xi\}$ be a local orthonormal basis of vector fields on 3-dimensional vector field M. Then we have

$$\tilde{S}(Y,Z) = (-1+\frac{r}{2}-3\alpha+\alpha^2)g(Y,Z) + (1-\frac{r}{2}+\alpha-3\alpha^2)\eta(Y)\eta(Z). \quad (24)$$

Replacing $Z = \xi$ in equation (24) and using (1) and (2)

$$\tilde{S}(Y,\xi) = -2\alpha(1+\alpha)\eta(Y). (25)$$

Also by equation (24) we have

$$\tilde{r} = r - 2 - 8\alpha. \quad (26)$$

Again replacing $Z = \xi$ in equation (21) and using (2) and (14), we get

$$\tilde{R}(X,Y)\xi = \alpha(1+\alpha)[\eta(X)Y - \eta(Y)X], \quad (27)$$

and replacing $X = \xi$ in equation (27) and using (2)

$$\tilde{R}(\xi,Y)\xi = \alpha(1+\alpha)(Y - \eta(Y)\xi). \quad (28)$$

Again replacing $X = \xi$ in equation (21) and using (2) we have

$$\tilde{R}(\xi,Y)Z = -\alpha(1+\alpha)g(Y,Z)\xi + (\alpha^2+3\alpha+2)\eta(Z)Y \quad (29)$$

4 3-dimensional α-para Kenmotsu manifolds with semi-symmetric metric connection satisfying $\tilde{R}.\tilde{S} = 0$ and $\tilde{S}.\tilde{R} = 0$

Consider a 3-dimensional α-para Kenmotsu manifold with semi-symmetric metric connection $\tilde{\nabla}$ satisfying the condition

$$\tilde{R}(X,Y).\tilde{S}(U,V) = 0, \quad (30)$$

then we have

$$\tilde{S}(\tilde{R}(X,Y)U,V) + \tilde{S}(U,\tilde{R}(X,Y)V) = 0, \quad (31)$$

for any vector fields $X,Y,U,V \in \chi(M)$. Now put $X = \xi$ in (31),we have

$$\tilde{S}(\tilde{R}(\xi,Y)U,V) + \tilde{S}(U,\tilde{R}(\xi,Y)V) = 0. \quad (32)$$

From (25) and (29) in (32), we have

$$-2\alpha(1+\alpha)[g(Y,U)\eta(V) + g(Y,V)\eta(U)] - [\eta(U)\tilde{S}(Y,V) + \eta(V)\tilde{S}(U,Y)] = 0. \quad (33)$$

Now puting $U = \xi$ in (33) and by using (1) and (2), we have

$$\tilde{S}(Y,V) = -2\alpha(1+\alpha)g(Y,V). \quad (34)$$

Now by equation (23), the above equation (34) takes the form

$$S(Y,V) = -(2\alpha^2-\alpha-1)g(Y,V) - (1+\alpha)\eta(Y)\eta(V) \quad (35)$$

Thus we have,

Theorem 1 For a 3-dimensional α-para Kenmotsu manifold with semi-symmetric metric connection satisfying the condition $\tilde{R}.\tilde{S} = 0$, the Ricci tensor S is given by

$$S(Y,V) = -(2\alpha^2 - \alpha - 1)g(Y,V) - (1+\alpha)\eta(Y)\eta(V).$$

Again consider a 3-dimensional α-para Kenmotsu manifold with semi-symmetric metric connection $\tilde{\nabla}$ satisfying the condition

$$(\tilde{S}(X,Y).\tilde{R}(U,V)Z) = 0, \tag{36}$$

for any vector fields $X,Y,Z,U,V \in \chi(M)$.
Then we have

$$(X \wedge_{\tilde{S}} Y)\tilde{R}(U,V)Z + \tilde{R}((X \wedge_{\tilde{S}} Y)U,V)Z + \tilde{R}(U,(X \wedge_{\tilde{S}} Y)V)Z + \tilde{R}(U,V)(X \wedge_{\tilde{S}} Y)Z = 0, \tag{37}$$

where the endomorphism $X \wedge_{\tilde{S}} Y$ is defined as

$$(X \wedge_{\tilde{S}} Y)Z = \tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y. \tag{38}$$

By taking $Y = \xi$ in (37), we get

$$(X \wedge_{\tilde{S}} \xi)\tilde{R}(U,V)Z + \tilde{R}((X \wedge_{\tilde{S}} \xi)U,V)Z + \tilde{R}(U,(X \wedge_{\tilde{S}} \xi)V)Z + \tilde{R}(U,V)(X \wedge_{\tilde{S}} \xi)Z = 0. \tag{39}$$

Now using (25), (26) and (38) in (39), we have

$$\begin{aligned}&-2\alpha(1+\alpha)[\eta(\tilde{R}(U,V)Z)X + \eta(U)\tilde{R}(X,V)Z + \eta(V)\tilde{R}(U,X)Z\\ &+\eta(Z)\tilde{R}(U,V)X] - \tilde{S}(X,\tilde{R}(U,V)Z)\xi - \tilde{S}(X,U)\tilde{R}(\xi,V)Z\\ &-\tilde{S}(X,V)\tilde{R}(U,\xi)Z - \tilde{S}(X,Z)\tilde{R}(U,V)\xi = 0.\end{aligned} \tag{40}$$

Taking inner product with ξ in equation (40), we have

$$\begin{aligned}&-2\alpha(1+\alpha)[\eta(\tilde{R}(U,V)Z)\eta(X) + \eta(U)\eta(\tilde{R}(X,V)Z) + \eta(V)\eta(\tilde{R}(U,X)Z)\\ &+\eta(Z)\eta(\tilde{R}(U,V)X)] - \tilde{S}(X,\tilde{R}(U,V)Z) - \tilde{S}(X,U)\eta(\tilde{R}(\xi,V)Z)\\ &-\tilde{S}(X,V)\eta(\tilde{R}(U,\xi)Z) - \tilde{S}(X,Z)\eta(\tilde{R}(U,V)\xi) = 0.\end{aligned} \tag{41}$$

By taking $U = Z = \xi$ in (41) and using (25), (27), (28), (29), we have

$$\tilde{S}(X,V) = 2\alpha(1+\alpha)g(V,X) - 2(\alpha^2 + \alpha + 1)\eta(V)\eta(X). \tag{42}$$

Using equation (23) in (42), we have

$$S(X,V) = (2\alpha^2 + 5\alpha + 1)g(X,V) - (2\alpha^2 + 3\alpha + 3)\eta(X)\eta(V) \tag{43}$$

Thus we have a following theorem:

Theorem 2 For a 3-dimensional α-para Kenmotsu manifold with semi-symmetric metric connection satisfying the condition $\tilde{S}.\tilde{R}=0$, the Ricci tensor S is given by

$$S(X,V)=(2\alpha^2+5\alpha+1)g(X,V)-(2\alpha^2+3\alpha+3)\eta(X)\eta(V)$$

5 η-Ricci soliton on 3-dimensional α-para Kenmotsu manifold with semi-symmetric metric connection

Let (M,ϕ,ξ,η,g) be an α-para Kenmotsu manifold. Consider the equation

$$L_\xi g+2\tilde{S}+2\lambda g+2\mu\eta\otimes\eta=0 \tag{44}$$

where L_ξ is the Lie derivative operator along the vector field ξ, $\tilde{S}$ is the Ricci curvature tensor field of the metric g, and λ and μ are real constants. We have $L_\xi g$ in terms of the semi-symmetric metric connection ?, so

$$2\tilde{S}(X,Y)=-g(\tilde{\nabla}_X\xi,Y)-g(X,\tilde{\nabla}_Y\xi)-2\lambda g(X,Y)-2\mu\eta(X)\eta(Y), \tag{45}$$

for any $X,Y\in\chi(M)$.

The (g,ξ,λ,μ), satisfying the equation (44) is called η-Ricci soliton on M [9]. If $\mu=0$, the (g,ξ,λ) is a Ricci soliton [16] which is called expanding, steady or shrinking as λ is positive, zero or negative respectively [2].

Consider α', a symmetric $(0,2)$-tensor field which is parallel with respect to the semi-symmetric metric connection $(\tilde{\nabla}\alpha'=0)$. So from the Ricci identity

$$\tilde{\nabla}^2\alpha'(X,Y;Z,W)-\tilde{\nabla}^2\alpha'(X,Y;W,Z)=0,$$

we obtain

$$\alpha'(\tilde{R}(X,Y)Z,W)+\alpha'(Z,\tilde{R}(X,Y)W)=0, \tag{46}$$

for any $X,Y,Z,W\in\chi(M)$ [18].

Now, for $Z=W=\xi$ and from the symmetry of α', we have that

$$\alpha'(\tilde{R}(X,Y)\xi,\xi)=0,$$

for any $X,Y\in\chi(M)$

If (ϕ,ξ,η,g) is a α-para Kenmotsu structure on M, from (25), (27), we have $\tilde{R}(X,Y)\xi=\alpha(1+\alpha)[\eta(X)Y-\eta(Y)X]$ and solving the expression $\alpha'\big(\tilde{R}(X,Y)\xi,\xi\big)=0$, we get

$$\alpha'(Y,\xi)-\alpha'(\xi,\xi)g(Y,\xi)=0, \tag{47}$$

for any $Y\in\chi(M)$. Now differentiating the above equation covariantly with respect to the vector field X, we have

$$\alpha'(\tilde{\nabla}_X Y,\xi)+(1+\alpha)\alpha'(Y,X)-(1+\alpha)\eta(X)\alpha'(Y,\xi)$$
$$=\alpha'(\xi,\xi)g(\tilde{\nabla}_X Y,\xi)+(1+\alpha)\alpha'(\xi,\xi)[g(Y,X)-\eta(X)\eta(Y)], \quad (48)$$

again using (19) and (48), we have

$$\alpha'(Y,X)=\alpha'(\xi,\xi)g(Y,X), \quad (49)$$

for any $X,Y\in\chi(M)$. As α' is parallel, gives that $\alpha'(\xi,\xi)$ is constant and hence we have

Proposition 5.1 Any parallel symmetric (0,2)-tensor field α' with semi-symmetric metric connection, is a constant multiple of metric g.

In the α-para Kenmotsu manifolds (M,ϕ,ξ,η,g) with semi-symmetric metric connection, $L_\xi g=2\alpha(g-\eta\otimes\eta)$,and using (19), the equation (45) becomes

$$\tilde{S}(X,Y)=-(\lambda+\alpha)g(X,Y)-(\mu-\alpha)\eta(X)\eta(Y). \quad (50)$$

Thus we have,

Proposition 5.2 *If (44) defines an η-Ricci soliton on the 3-dimensional α-para Kenmotsu manifold (M,ϕ,ξ,η,g), then (M,g) is quasi-Einstein.*

The manifold M is called $quasi-Einstein$ if the Ricci curvature tensor field $\tilde{S}$ is a linear combination (with the real scalars λ and μ respectively where $\mu\neq 0$) of g and the tensor product of a non zero 1-form η satisfying $\eta(X)=g(X,\xi)$, for ξ a unit vector field [13] and called $Einstein$ if $\tilde{S}$ is collinear with g.

In particular, $\tilde{S}(X,\xi)=\tilde{S}(\xi,X)=-(\lambda+\mu)\eta(X)$,
but it is known that on (2n+1)-dimensional para-contact manifold M, $\tilde{S}(X,\xi)=-(\dim M-1)\eta(X)=-2n\eta(X)$. So in 3-dimensional α-para Kenmotsu manifold,

$$\lambda+\mu=2 \quad (51)$$

Now Ricci operator $\tilde{Q}$, defined as $g(\tilde{Q}X,Y)=\tilde{S}(X,Y)$ and using (50) ,which gives

$$\tilde{Q}X=-(\lambda+\alpha)X-(\mu-\alpha)\eta(X)\xi. \quad (52)$$

Theorem 3 Let (M,ϕ,ξ,η,g) be a 3-dimensional α-para Kenmotsu manifold. Suppose that the symmetric (0,2)-tensor field $\alpha'(X,Y)=L_\xi g(X,Y)+2\tilde{S}(X,Y)+2\mu\eta(X)\eta(Y)$ is parallel with the semi-symmetric metric connection associated to g. Then (g,ξ,μ) gives an η-Ricci soliton.

Proof. We have

$$\alpha'(\xi,\xi)=(L_\xi g)(\xi,\xi)+2\tilde{S}(\xi,\xi)+2\mu\eta(\xi)\eta(\xi),$$

by using $L_\xi g(X,Y)=2\alpha(g(X,Y)-\eta(X)\eta(Y))$ and equation (50), we have

$$\alpha'(\xi,\xi)=-2\lambda.$$

From (49), we have

$$\alpha'(X,Y) = -2\lambda g(X,Y), \text{ for any } X,Y \in \chi(M).$$

This shows that a symmetric (0,2) type tensor field is parallel with semi-symmetric metric connection associated to g.

Thus,

$$L_\xi g + 2\tilde{S} + 2\mu\eta \otimes \eta = -2\lambda g.$$

For $\mu = 0$ follows $L_\xi g + 2\tilde{S} + 4g = 0$. Now we conclude

Corollary 5.1 *On a 3-dimensional α-para Kenmotsu manifold (M,ϕ,ξ,η,g) with symmetric $(0,2)$-tensor field $\alpha' = L_\xi g + 2\tilde{S}$, is parallel with the semi-symmetric metric connection associated to g, the relation (44), for $\mu = 0$ and $\lambda = 2$, defines a Ricci solitons on M.*

We have an example of η-Ricci soliton on α-para Kenmotsu manifold.

Example 5.1 Let $M = R^3$ be 3-dimensional manifold with the standard cartesian coordinates (x,y,z). Let (ϕ,ξ,η) be an almost paracontact structure with semi-symmetric metric connection. Then we define this structure on M as

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0, \quad \xi = e_3, \quad \eta = d(Z). \tag{53}$$

where e_1, e_2, e_3 are the linearly independent system of vector fields such that

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

Let g be pseudo-Riemannian metric defined by

$$\begin{aligned} &g(e_1,e_1) = e^{2z},\ g(e_2,e_2) = -e^{2z},\ g(e_3,e_3) = 1, \\ &g(e_1,e_2) = g(e_2,e_3) = g(e_1,e_3) = 0. \end{aligned} \tag{54}$$

Now, by direct computation, we have

$$[e_1,e_2] = 0, \quad [e_1,e_3] = 0, \quad [e_2,e_3] = 0.$$

Again let ∇ be Levi-Civita connection with metric g, then using Koszuls formula, we have

$$\begin{aligned} &\nabla_{e_1}e_1 = -e^{2z}e_3, \quad \nabla_{e_2}e_2 = e^{2z}e_3, \quad \nabla_{e_3}e_3 = 0, \\ &\nabla_{e_1}e_2 = 0, \quad \nabla_{e_2}e_1 = 0, \quad \nabla_{e_1}e_3 = e_1, \\ &\nabla_{e_3}e_1 = e_1, \quad \nabla_{e_3}e_2 = e_2, \quad \nabla_{e_2}e_3 = e_2, \end{aligned} \tag{55}$$

Now, semi-symmetric metric connection $\tilde{\nabla}$ on M is given as

$$\begin{aligned} &\tilde{\nabla}_{e_1}e_1 = -2e^{2z}e_3, \quad \tilde{\nabla}_{e_2}e_2 = 2e^{2z}e_3, \quad \tilde{\nabla}_{e_3}e_3 = 0, \\ &\tilde{\nabla}_{e_1}e_2 = 0, \quad \tilde{\nabla}_{e_2}e_1 = 0, \quad \tilde{\nabla}_{e_1}e_3 = 2e_1, \\ &\tilde{\nabla}_{e_3}e_1 = e_1, \quad \tilde{\nabla}_{e_3}e_2 = e_2, \quad \tilde{\nabla}_{e_2}e_3 = 2e_2, \end{aligned} \tag{56}$$

And using equation (56), we have the Riemannian curvature tensor as

$$\tilde{R}(e_1,e_2)e_2 = 4e^{2z}e_1, \quad \tilde{R}(e_1,e_3)e_3 = -2e_1, \quad \tilde{R}(e_2,e_1)e_1 = -4e^{2z}e_2,$$
$$\tilde{R}(e_2,e_3)e_3 = -2e_2, \quad \tilde{R}(e_3,e_1)e_1 = -2e^{2z}e_3, \quad \tilde{R}(e_3,e_2)e_2 = 2e^{2z}e_3.$$

Using above results in $\tilde{S}(X,Y) = \sum_1^3 g(\tilde{R}(e_i,X)Y,e_i)$, we verify that

$$\tilde{S}(e_1,e_1) = 2(2e^{2z}-1)e^{2z}, \quad \tilde{S}(e_2,e_2) = 2(2e^{2z}+1)e^{2z}, \quad \tilde{S}(e_3,e_3) = 0.$$

In this case, $\lambda = 0$ and $\mu = 0$. Therefore the structure (g,ξ,λ,μ) is an η-Ricci soliton on the α-para Kenmotsu manifold with semi-symmetric metric connection with non zero scalar curvature $r = 8e^{4z}$.

Conclusion

In this paper, we have given 3-dimensional α-para Kenmotsu manifold with semi-symmetric metric connection satisfying the curvature conditions $\tilde{R}.\tilde{S} = 0$ and $\tilde{S}.\tilde{R} = 0$. Further study of η-Ricci solitons on 3-dimensional α-para Kenmotsu manifold with semi-symmetric metric connection is being done and obtained some results.

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