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Duality: "An integrative approach."

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Abstract

The concept of duality is one of the oldest concepts in mathematics and is needed to explain various mathematical concepts in many branches. Moreover, this concept is widely used in mathematics while solving different real-world problems. This article aims to analyze how the different approaches or concepts are interrelated and how we can handle naively using an integrative approach.

Keywords: *Dual, L^p space, inequality, linear programming problem.*

1. Introduction

We first begin with a famous statement of Sir Michael F. Atiyah [1] on duality. According to him, duality in mathematics is not a theorem, but a principle. Though the duality in mathematics has a very simple origin, it has been strongly used in various context and has helped greatly in understanding the number of mathematical concepts. Thus the concept of duality is widely needed in mathematics and will be used in various situations in the days to come. We recall the dictionary meaning of duality. It defines duality as a division of the domain of reality in terms of two opposed and contrasted aspects or the state of being so divided one. The two aspects are then called the duals of each other. Hence when the structure of the domain is evident, analysis and the interpretation of the problem or concept can be made employing its counterpart. These suggest that in principle, duality gives two entirely different points of view of visualizing the same object. Consequently, we need the concept of duality whenever we need to deal the things that have two different points of view. In mathematics, duality translates concepts, namely theorems or structures, into the other concepts (theorems or structures) in a one-to-one fashion with the help of operation called involution where the involution operation means that if the dual of structure A is B, then the dual of the structure B is A. The concept of duality is used in many branches of mathematics such as geometry, algebra, analysis to name just a few. Moreover it is also widely needed in physics. For more about the concept of duality in various contexts, we suggest the readers to refer [2], [3], [4], [5] and [6]. Here in this article, we exemplify using different domains of reality and use them to serve as tools to generalize the concept of duality.

If we represent the totality of the domain at hand by the numbers 1, the parts dividing it are each positive and less than 1. So if we divide the domain into two parts and represent one of them by t, the others must be 1-t, i.e., t+ (1-t) =1. When t increases in magnitude, $1-t$ decreases, and vice versa as shown in Figure 1. Moreover, the division into equal halves holds when t=0.5.

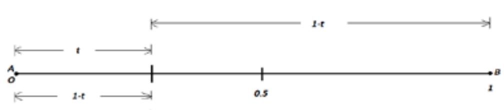

Figure 1

In case, the domain (of reality) is a line segment joining two points A and B, this represents the section formula $\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)$ in a plane or $\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n},\frac{mz_2+nz_1}{m+n}\right)$ in three dimensional space where (x_1,y_1) and (x_2,y_2) or (x_1,y_1,z_1) and (x_2,y_2,z_2) are the coordinates of the points A and B in plane or in space respectively. When the point under consideration divides the line joining AB in the ratio m: n.

Assuming $\frac{m}{m+n}=$ t and $\frac{n}{m+n}=1-$ t, then

$$\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)=\left(\frac{mx_2}{m+n}+\frac{nx_1}{m+n},\frac{my_2}{m+n}+\frac{ny_1}{m+n}\right)$$
$$=(tx_2+(1-t)x_1,ty_2+\ (1-t)y_1)$$

is a point lying between (x_1,y_1) and (x_2,y_2). The midpoint of the line is

$$(0.5x_2+0.5x_1,0.5y_2+0.5y_1)=\left(\frac{x_2+x_1}{2},\frac{y_2+y_1}{2}\right).$$

Here $t=0$ and $t=1$ refers to the end points A and B.

Indeed, the section formula says that a point lies between the line joining two points if each x_i coordinate of the point lies between the endpoints corresponding x_i^{th} coordinate.

In case when $t<0$ or $t>1$, t and $1-t$ have opposite signs. Considering the case $t=0$ as the point A and $t=1$ as the point B, $t<0$ refers to the point lying leftwards to A and $t>1$ refers to AB or refers to the external division [i.e., as points move leftward and as t increases point move rightwards]

Expressed in the other way, if $A(x_1,y_1)$ and $B(x_2,y_2)$ are two points, then the coordinates of ant point dividing line joining AB is the ratio $m{:}\,n$ is

$$\frac{m(x_2,y_2\,)+n(x_1,y_1)}{m+n}.$$

Expressed vectorically $\vec{a}$ and $\vec{b}$ refer to the position vectors of two points A and B, then position vector of any point P on the line joining AB is given by

$$\overrightarrow{OP}=\frac{\mathrm{t}\vec{b}+(1-\mathrm{t})\,\vec{a}}{t+(1-t)}=t\vec{b}+\ (1-t)\vec{a}.$$

In other words, we have $\vec{r}=t\vec{b}+(1-\mathrm{t})\,\vec{a}$ which is the vectorial equation of a straight line. In our discussion we have used t and $1-t$ as a pair of contrasted concepts that serve as complement of each other. Expressed otherwise, t and 1-t are duals of each other diving the totality 1 into two parts. As aforementioned our goal is to explore how the interpretation of the universe (domain) at hand and the symbol t therein give rise to the various concept of duality in different situations.

Assuming that $t=\frac{1}{p}$ and $(1-\mathrm{t})\ =\frac{1}{q}$ then $\mathrm{t}\in(0,1)\Rightarrow(1-\mathrm{t})\in(0,1)$. Then we have $p>1,q>1$. Also, $\frac{1}{p}+\frac{1}{q}=t+(1-t)=1$. Here $t=1$ implies $1-t=0$. Moreover, $p=0$ implies $q=\infty$. This exactly is the basic idea what holder's conjugate mean. Hence the association of the concept in related areas appear a natural consequence. The use of $\frac{1}{p}$ and $\frac{1}{q}$ instead of t and 1-t, however reveals more interesting facts regarding the structural behavior and about real number themselves.

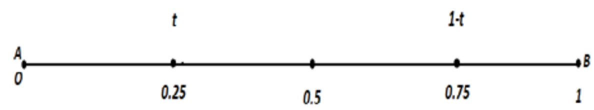

Figure 2

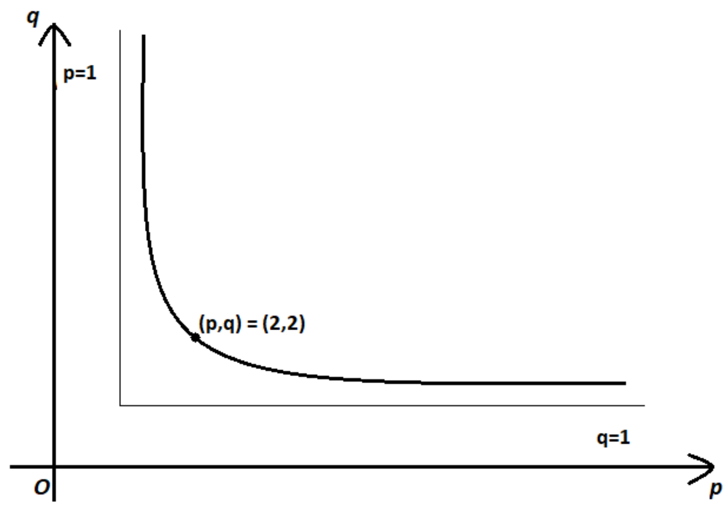

Figure 3

When t =0.5 then 1- t =1-0.5 =0.5 i.e. t =1-t when t =0.5. This gives, $\frac{1}{t} = \frac{1}{1-t}$ when t = 0.5 i.e., $p = q$ when $p = 2 \; or \; q = 2$. Here, 0.5 is the mid-point of the line joining 0 and 1 on the sum of any two points equidistant from 0.5 on the line segment as shown in the figure must sum to 1. We note that 0.25 and 0.75, and 0.1 and 0.9, etc. are some exemplifying pairs.

The reciprocal of 1 is 1, and 0.5 is 2 and the reciprocal of others numbers between 1 and 0.5 lies between 1 and 2. But this contributes to the half part of the line segment is when $t \to 0^+$, $\frac{1}{t} \to \infty$. It means that when the value of t ranges in the rest half $\frac{1}{t}$ (the reciprocal) assumes all possible values of real numbers greater than 2 i.e., ranges from 2 to ∞ i.e., a single jump from 1 to 2 i.e. one step balance the infinite steps. A little more generalization of this helps to develop and intuition how the internal (0, 1) and the whole $\mathbb{R}$ are equinumerous. This can be proved by the use of the function $f: (0,1) \to \mathbb{R}$ defined by f(t) = $\frac{1}{ln2} \ln\left(\frac{1}{1-t} - 1\right) \leftrightarrow \frac{1}{ln2} \ln\left(\frac{t}{1-t}\right)$ i.e. by the use of components t and 1-t of the duality 1.

Dividing both sides by t (1-t) gives

$$\frac{1}{(1-t)} + \frac{1}{t} = \frac{1}{t(1-t)} = \frac{1}{t} - \frac{1}{(1-t)}.$$

i.e. $q + p = pq$ which means that p and q are two numbers such that their sum equals their product. Also, $p = (p-1)\, q$ and $q = (q-1)\, p \Rightarrow q = \frac{p}{p-1}$ and $q = \frac{q}{q-1}$. So that if one of p or q is x the other must be $\frac{x}{x-1}$. So if $p > 2, p-1 > 1$, in such a case q less than p and decreasing accordingly as p increases and vice versa. If $1 < p < 2, 0 < p-1 < 1$ so that q is greater than p in this case and increases as p decreases and vice versa as shown in Figure 3

When p moves from 1 to 2, q decreases from ∞ to 2, but remains greater than p and when p increases from 2 to ∞, q decreases from 2 to 1 and p at this stage is greater than q .The change is symmetrical and $t = 0.5$ or $p = q = 2$ being the point of symmetry. If a graph is plotted then this is of exponential nature. It is symmetric about the point $(2, 2)$, $p = 1$ and $q = 1$ are the asymptotes. Therefore in the planer domain pq, p and q are complementary. Therefore the pair (p, q) meets the requirements demanded to be the duals of each other. It will therefore natural to say that if L^p is a space, the corresponding space it has the pre knowledge to be its dual. For each fixed p there corresponds a fixed q. In case $p^2 = q = 2$, the space l^2 must therefore be its own dual. As $p \to 1, q \to \infty$ and as $p \to \infty, q \to 1$ which then suggest the line dual of l^∞ is l^1 and vice versa. Now we discuss the concept of duality in various context. We begin with discussion of association of duality in probability.

2. Association with probability

This concept of duality can also be interpreted in terms of the concept of probability. In probability the total universe (sample space) at hand is divided into two parts, which includes the opp or sample points divided into two parts, one in favor of the events and the other being complement comprises of the remaining cases or sample space . If E denotes an event, S the sample space, E the complementary event then

$$n(\bar{E}) = n(S) - n(E)$$

$$n(\bar{E}) = 1 - n(E)$$

This gives

$q = 1 - p$ or $p + q = 1.$

where p is the probability of success and q the probability failure of the event E. So if t denotes p then $q = 1 - t$, so that $p + q = t + t - 1 = 1$. In either case we consider the total universe as 1 and divide it into two parts (fractional) as two contrasted and complementary parts, so that the knowledge of one can be used to describe the other.

3. Association with Sequence Spaces.

Let us write $p' = \frac{1}{t} = \frac{1}{p}$ and $q' = \frac{1}{1-t} = \frac{1}{q}$.

Then $p' \geq 1$ and $q' \geq 1$, $p' + q' = p'q'$. As $p' = 1$, $q' \to \infty$ and vice versa. Here $p'$and q' are called the conjugate or complementary of each other. The pair of numbers p' and q' so generated from a basis in the proof of Holder's inequality and Minkowski's inequality (opp the inequalities).

Holder's Inequality: Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_1, a_2, a_3 \ldots a_n \geq 0$ and $b_1, b_2, b_3 \ldots b_n \geq 0$. Then

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} {a_k}^p\right)^{\frac{1}{p}} \left(\left(\sum_{k=1}^{n} {b_k}^q\right)^{\frac{1}{q}}\right)$$

Minkowski's Inequality: Let $p \geq 1$, $a_1, a_2, a_3. ..a_n \geq 0$ and $b_1, b_2, b_3 ... b_n \geq 0$. Then

$$\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} {a_k}^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} {b_k}^p\right)^{\frac{1}{p}}$$

The results can be generated to the sums that run from 1 to ∞. The conjugate exponents p and q are used in defining norms in l^pspace (sequence space).

Space l^p: Let $p \geq 1$ be a fixed real numbers. Then space l^p is the space of the sequence

$x = (x_k) = (x_1, x_2, ...)$ of the numbers such that $|x_1|^p + |x_2|^p + ...$ converges; thus

$\sum_{k=1}^{\infty} |x_k|^p < \infty$ and the metric is defined by

$$d(x, y) = \left(\sum (x_k - y_k)^p\right)^{\frac{1}{p}}$$

where $p \geq 1$ is fixed.

Corresponding to the values of $p = 1, p = 2$ and $p \to \infty$ the spaces l^1, l^2 and l^∞ and their matrices are accordingly defined. The cases $p = 2 = q$ refers to Hilbert space the corresponding norm being the Euclidean norm. We observe that for $0 < p < 1, 1 < t < \infty \Rightarrow 1 - t < 0$ be $t \notin (0, 1)$ which means that t and $1 - t$ do not lie within the parent space to the universe under consideration and hence the corresponding matrix or norm is not defined.

4. Association with Hyperplane

We use conjugates p and q or t and $1 - t$ to study the hyperplanes. For a fixed p, q has a fixed value. Expressed otherwise for a fixed $t, 1 - t$ has a fixed value. Inline, they refer to points; in the plane, they refer to lines, and in 3-dimensional space, a reference is a plane; they refer to a space of dimension one less than the parent or ambient space. It means that they refer to hyperplanes.

5. Association with linear programming problems (LPP)

In linear programming problem (LPP), we generally deal with optimizing the function under consideration. We find the linear combination of the variable in the function subjected to a particular set of constraints. In due course of associating it with t and $1 - t$, we observe as aforementioned that $t = 0.5$ implies that $1 - t = 0.5$ the mid-way; it must refer to breakeven point in profit maximization when t increases 1-t decreases and vice versa means that increment in profit must be due to the corresponding decrement of the corresponding associated cost. Since t refers to a space of one dimension less than the parent, we can use this concept to handle optimization problems related to more than two variables.

Conclusion:

The study of duals enables us to know the parent space's very structure, so dealing with the problem can be effectively dealt with its counterpart or conjugate. We can observe the effect of variation of one of them and interpret the other's corresponding effect. As the dissection of an object reveals its very structure, which is beneficial to know opp it is a whole, a problem a dual play similar role. The generalization of the concept motivates us towards structural analysis and treatment. The reduction of problems in a different field to one single approach simplifies the study and observes how things or problems that appear different at a glance are interrelated. In other words, it gives a bird's eye view to deal with real words situations.

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