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Solutions of Polynomial Equation of the 3rd degree using Vedic Method

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Introduction

This paper deals with an exemplary evaluation of all solutions in case of a Cubic Polynomial (one variable) equation using Vedic Sutrams "Straight Division Method" and also "Purana Apuranabhyam" as envisaged by Jagadguru Shankaracharya of Puri Mutt, Shri Bharati Krishna Thirtha Swamiji.

Jagadguru Shankaracharya of Puri Mutt had applied (A) a Straight Division method using Vedic Sutrams for solving the polynomial equations of any degree. In principle, basically, it is similar to that of the method as applied to determine the roots of numbers but with a few other Vedic Sutrams such as Vilokanam, Adyamadyena Antyamantyena, Gunita Samuchayah Samuchaya Gunakaha, Vinculam being incorporated in, to arrive at all the solutions of a 3rd degree Polynomial Equation - in general, Polynomial Equations of any degree of one variable.

In addition to the above, Shankaracharya had introduced also a novel method called (B) "Purana Apuranabhyam", a Vedic Sutram for the evaluation of all the solutions of a polynomial equation of any degree. This is also exemplified in this work for the Cubic Polynomial Equation.

Abstract

The working is briefly - as a first step, a number of attempts are carried out to pick out one most probable solution using which the rest of the solutions are aimed out.

1) From the given polynomial equation, the difference between the RHS and LHS is considered against a series of values for the variable "x". From this set, one can locate one most probable value for "x=a" together with its corresponding RHS-LHS value which is the starting dividend to work with. The solution is considered as a.bcde… a, as the integer & b,c,d,e, are the decimals.

2) Using Straight Division method, an expansion table of the type $(a+b+c+d+\ldots)^n$ where "n" is the highest order of the given polynomial, is to be prepared.

3) The first solution is to be checked and if the error is considerable, then one has to change the variable "x" in terms of $x=\frac{z}{2}, \frac{z}{10}, \frac{z}{100}$, etc. and carry out the work with this new variable, z.

4) One has to identify which terms of the expansion are to be deducted from the respective dividends i.e., those terms that give rise to the specific decimal values and corresponding remainders to form the next dividends (new dividends)

5) One has to follow the procedure of subtraction as equal to the terms as groups of the derivatives of the polynomial and other decimals. For example, as the method is Straight Division, Swamiji had given the following details to be adopted.

a) The Common Divisor is from the 1st derivative of the polynomial, the value of which is reckoned at the integer value "x=a" and as determined from RHS and LHS.

b) As the expansion of $(a.bcde..)^n$ contains many expressions depending on the decimals considered, one has to read the expansion terms as functions of certain representation (derivatives) cubic and lower of the given polynomial, for example the expansion terms contributing to the various decimals are to be sorted out in different derivatives which are shown in the working details of the specific problem.

c) After deducting those terms corresponding to a particular decimal, from the dividends the results are to be divided by the Common Divisor (CD), to get the respective values of the decimal and remainders, which have to be added to the next successive dividends, to form new dividends. The procedure is continued digit by digit until all the decimals are evaluated. When the solution can be read from the result, one can write down the remaining form of the polynomial to be the quadratic in case of cubic polynomial. One has to solve the quadratic form separately using again the corresponding vedic sutrams.

d) The details of the method Purana Apuranabhyam have been clearly described step wise. As per the solving of the remaining equation, it is just the same as described earlier.

e) The details of the expansion terms belonging to the different decimals; which need to be subtracted are shown in one of the tables.

f) The identification of groups of terms in expansion formed in the derivatives are given in tables.

g) The individual decimals outside the derivative groups are adopted as per the expansion terms while forming the deductions.

h) In order to get the first decimal "b", the first Dividend is to be directly divided by the CD, the consequent quotient of which is "b" and the remainder to be added to the next to form the new dividend. This procedure is continued digit by digit to arrive at all the decimals, after the proper deductions, followed by division by CD.

i) For all other decimals, one has to follow in succession that after the deductions, the result has to be divided by CD to get the other decimals.

j) The full details of working are clearly explained at the end of which, all the solutions are consolidated in the table "Conclusion".

The results of both the methods A and B are well comparable.

Solve the Cubic Equation E= x^3-$2x^2$-51x-110=0. The given polynomial is of the 3rd order equation.

Method – A (A Straight Division)

In order to find out the solutions, the method of obtaining one approximate value of "x" from a series of values, which when substituted for in the given polynomial equation, gives differences between RHS and LHS from which, one can locate "a" value for "x" as the most probable integer, as suggested by Swamiji, using this approximate solution, one can aim at the remaining solutions. This procedure is called Vilokanam (mere inspection) Table – (1). The form of the solution is x=a.bcde, where x is the integer value followed by a number of decimals and the corresponding difference (RHS – LHS) is the starting value for calculation.

$E=x^3-2x^2-51x-110=0$

LHS $\Rightarrow x^3-2x^2-51x=110 \Leftarrow$ RHS

The solution is considered to be of the form a.bcde..

Table – 1

	LHS	RHS	RHS – LHS
x = 1	– 52	110	162
x = 2	– 102		212
x = 3	– 144		254
x = – 1	48		62
x = – 2	86		24
x = – 3	108		2
x = – 4	108		2
x = – 5	80		30
x = – 6	18		92

One can try for the value of "x" to be "a", the integer value of one solution under consideration to start with, as the first one of the solutions as either -3 or -4 with the corresponding RHS-LHS as equal to 2. This is considered as the first dividend 2-0 as shown in the Table – 2.

1) As per the application of Straight Division method, one should evaluate the Common Divisor (CD) which can be derived from the first derivative of the polynomial. Here it is 1st derivative of the polynomial with x=-3, of $x^3-2x^2-51x-110 \Rightarrow -12$.

One can evaluate the equation under $3a^2$ representation, as the first derivative of the entire equation with the value of "a" as -3, being incorporated. This will be the Common Divisor (CD)

2) As the solution is in the form of a.bcd and also of cubic nature, we have to consider the cubic expansion. Considering the value of the answer atleast upto 3 decimals i.e. b,c,d, the general cubic expansion is

   $(a+b+c+d+e+f)^3$ up to 5th decimal= $a^3+3a^2b+3ab^2+b^3+3a^2d+6abc+3ac^2+3b^2c+6abd+3b^2d+3bc^2+6ace+3ad^2+6bcd+c^3+3a^2c+3a^2e+3a^2f+6abe+6adc+6abf+3b^2e+3a^2g$
   and upto "f" with a different value for a=-32, when $x=\frac{z}{10}$
   The terms that are to be deducted successively to get the values of decimals as quotients after dividing the new dividends by the Common Divisor (CD) are given in the Table – 3.

3) The $3a^2$ representation is the first derivative of the polynomial $x^3-2x^2-51x-110$, when the value of x=a=-3, is -12. This is the value of Common Divisor (CD)

4) On further differentiation of the polynomial, we will obtain 3a representation of the equation when x =a=-3, being substituted for , which is equal to $\frac{1}{2}(6x-4)$ = -11

With the help of the 3a representation of the given equation, (3a = -11), b^3+6abc (i.e. 3a.2bc) = -155

These two $3ab^2$ and b^3+3a.2bc terms to be deducted from the 2nd and 3rd dividend positions of the given equation to form the new dividends and the decimals c and d.

The value of "b" is obtained by direct division of the 1st dividend 20 by the CD =$\overline{12}$, and its quotient is $\bar{1}$.

Let us consider the RHS – LHS value against the value of x=-3 as the starting point to work with. On the basis of determining the successive new dividends 20, 80 and 70 when subjected to Common Divisor give rise to the quotients as the decimals and further remainders.

Table – 2

	0	0	0	0
CD = $\overline{12}$	2	8	7	9
		11	1 + 154 = 155	
		$3ab^2$	b^3 + 3a.2bc	
– 3	$\bar{1}$	$\bar{7}$	$\overline{18}$	
a	b	c	d	

Upto d (3 decimals), the value of x = $\bar{3}.\bar{1}\ \bar{7}\ \overline{18} = -3.188$

Verification with the value -3.188 :

Let us verify if x=-3.188 will give a zero value when substituted in the equation

$E=x^3-2x^2-51x-110=0$

$E=(-3.188)^3-2(-3.188)^2-51(-3.188)-110=-0.13943$

Table – 3

for determination of b, $3a^2b$	-	from $3a^2$ representation
for determination of c, $3ab^2$	-	from 3a representation
for determination of d, $b^3+6abc = b^3+3a.2bc$	-	from 3a representation

It is noticed that the value needs more refinement which can be derived by adopting multiple of "x" as the variable for obtaining more accuracy in the decimal part.

So one can try that $x = \frac{z}{10}$ with another variable "z" as 10x, the solution is attempted to obtain more accuracy in the decimal value as the "x' value is lying between -3 and -4 with $x = \frac{z}{10}$, $z = 10x$

The procedure for the determination of x value is repeated. With the equation written in “z”, the new variable.

One obtains the equation in “z”, as

$\frac{z^3}{1000} - 2\frac{z^2}{100} - 51\frac{z}{10} - 110 = 0 \Rightarrow F(z) = z^3 - 20z^2 - 5100z = 110000$

Table RHS – LHS (Z)

	F(z)	RHS – LHS
z = – 32	109952	48
z = – 33	110583	– 583

It is noticed from the difference between RHS and LHS, at z=-32 as 48 is, more probable a solution.

Now in the revised condition with z=-32, which is the value for “a”,

The Common Divisor (CD) : $3a^2$ representation when considered with the equation in z ($3z^2$-40z-5100) is -748

The 3a representation of the equation is $\frac{1}{2}$[6z-40], with z=-32 is -116

With the values of $3a^2$ and 3a representations, one can work out 5 decimals for a better accuracy. In the expansion of $(a+b+c+d+e+f)^3$, the terms of the expansion which are to be deducted from the respective dividends to obtain corresponding decimals after the division by the CD are, for b, the dividend is to be directly divided by the $3a^2$= -748, for others, after deduction of $3ab^2$ (for c), b^3+3a.2bc (for d), $3b^2c$+3a.2bd+$3ac^2$ (for e) and $3b^2d$+$3bc^2$+3a.2bc+3a.2cd (for f), the details of terms shown in the Table, are to be deducted from the respective dividend and the results are divided by CD to obtain the decimals and remainders.

Table – 4

	0	0	0	0	0	0
CD = $\overline{748}$	48	480	312	128	220	288
		0	0	0 + 0 + 4176 = 4176	0 + 0 + 0 + 5568 = 5568	Subtraction Terms
		$3ab^2$	b^3 + 3a.2bc	$3b^2c$ + 3a.2bd + $3a.c^2$	$3b^2d$ + $3bc^2$ + 3a.2be + 3a.2cd	
a – 32	0	$\bar{6}$	$\bar{4}$	$\bar{7}$	$\overline{10}$	
	b	c	d	e	f	

Table – 4 gives the details when $x = \frac{z}{10}$ is used for evaluation of the first solution

$z = -32.0\bar{6}\bar{4}\bar{7}\overline{10}$ = -32.06480. $\therefore x = \frac{z}{10} = -3.20648$

With the 1st value as x=-3.188, E=0.1394287.

With the 2nd value when $x=\frac{z}{10}$ as x=-3.20648, E=0.000017

Using this 2nd value for x, we can write the given cubic equation, E=(x+3.20648)A, where A is a quadratic equation, and -3.20648 is one solution of the polynomial.

Swamiji at this stage, introduced an application of vedic sutram Adyamadyena Antyamantyena to arrive at the values of the quadratic equation.

$E= x^3-2x^2-51x-110=(x+3.20648)A$ … (A is quadratic equation $x^2+\alpha x+b$)

The sutram reads how to write down A and is as follows :

1) Adyamadyena comparision of this form suggests that
   $\frac{x^3(of\ E)}{x\ (of\ the\ probable\ solution\ considered)} = x^2$ is the first tem of A. i.e. the first by first.
   (Adyam – to be written as the 1st of the given equation as x^3, Adyena, to be written as the 1st of the obtained factor "x", the ratio of which gives the x^2 – as the first quadratic part of A)

2) Antyamantyena – the last by last. i.e. $\frac{-110}{3.20648} = -34.30553$.
   (Antyam the last of the given equation, i.e. -110 and Antyena, the last value of the obtained factor 3.20648, the ratio of which is -34.30553)

3) In between, we have in A, an x term, whose coefficient is to be determined. Let it be α.
   $\therefore E=(x+3.20648)(x^2+\alpha x-34.30553)=0$
   α is to be determined by way of equating the coefficients of x on both sides.
   $-51x=-\alpha x-34.30553x \quad \therefore \alpha=-28.13349911$

   $\therefore E=(x+3.20648)(x^2-28.13349911x-34.30553)=0$

   By solving the quadratic equation through swamiji's relation for $f(x)=(x^2+bx+c)=0$

$f'(x) = \pm \sqrt{Discri\min ant} = \pm\sqrt{b^2 - 4ac}$, f(x) is the quadratic polynomial (ax^2+bx+c)

x = + 9.012793411, – 3.80631506

$\therefore$ The solutions of E are : $E = x^3-2x^2-51x-110 = (x + 3.20648)(x – 9.012793411)(x + 3.80631506)$

Lastly, Swaamiji has introduced another sutram by means of which the correctness of the solutions determined are sanctified i.e.

Gunita samuchayah Samuchaya Gunakaha

Adding the coefficients and the number = adding the coefficients and numbers in each solution and multiplying accordingly

$\therefore E = X^3-2x^2-51x-110 = (x + 3.20648)(x – 9.012793411)(x + 3.80631506)$

$S_c=-162=(1 + 3.20648)(1 – 9.012793411)(1 + 3.80631506)=-161.999998\sim162$

The equality of the values, indicate the correctness, of course with the first solution being checked in the beginning itself so that, the variable can be modified for a better accuracy of necessary.

Method B (Purana Apuranabhyam)

The method involves to identify the first two terms of the given polynomial equation (in the descending order) as the first two expressions of its perfect (standard form of the cubic nature) or to identify the first two terms of the given polynomial (cubic) equation taken in the descending order as the first two terms of the standard cubic equation in the same order.

For example, the given polynomial cubic equation is **x^3-2x^2-51x-110=0 ---(1)** whose solutions are to be determined.

1) To consider the first two terms x^3-2x^2 as forming the first two terms of its standard cubic form (i.e. $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$, where a=x, $b=-\frac{2}{3}=-0.667$)⇒(x-0.667)
2) To complete its standard form as $(x-0.667)^3=x^3-3x^2(0.667)+3x(0.667)^2-(0.667)^3$
3) $(x-0.667)^3==x^3-2x^2+1.330467x-0.29674$
4) To substitute for x^3-2x^2 From the given equation, $x^3-2x^2=51x+110$

   $\therefore (x-0.667)^3=51x+110++1.330467x-0.29674 = 52.3346x+109.7033$

   Let y=x-0.667; x=y+0.667. This amounts to solving i.e. y^3 equation

   $y^3=52.3346y+144.6104 \Rightarrow y^3-52.3346y-144.6104=0$ with y^2 as non existence.

   If the equation is considered upto 1st decimal value, $y^3-52.3y-144.6=0$

   To avoid the decimal, by multiplying the equation by 10, we get $10y^3-523y-1446=0$ Considering $10y^3-523y-1446=0$ for evaluating y. Table 5 shows one of the probable values of the equation with y^3 by vilokanam (Table-5)

Table – 5

RHS-LHS is shown below for various values of y.

Y value	RHS-LHS
Y=3	2745
Y=4	2898
Y=5	2811
Y=6	2824
Y=7	1677
Y=8	510
Y=9	-1137

From the above values, one can significantly assume for one value as lying between 8 and 9. 8 is considered as one value of y to start with.

RHS-LHS=510 is the starting dividend with y=8

The first derivative of the y^3 equation, $3a^2$ representation of the y^3 equation = $30y^2-523$. With y=8, $3a^2$ representation is 1397. This is the Common Divisor (CD).

3a representation $=\frac{1}{2}(60y)$=30y, with y=8, is 240.
Let y=a.bcde be the solution

Table - 6

CD=1397	0 510	0 909		0 1342	0 405
		$\frac{3ab^2}{2160}$	$\frac{10(b^3)+3a.2bc}{6030}$		
8 y= a	3 b	4 c	5 d		

$\therefore$ y=8.345

As y=(x-0.667)$\Rightarrow$x=y+0.667, x=8.345+0.667=9.012

After obtaining one value for y, the remaining two are to be evaluated or can be read as in "x" using the relation between "x" and "y".

The given equation can be written in the form (x-9.012)(A), where A is a quadratic part of the polynomial.

Swamiji had used two sutrams
(1) Adyamadyena Antyamantyena
(2) Differential value for finding out the two values of the quadratic part of the polynomial

$\therefore$ Using the first solution, Given polynomial is $x^3-2x^2-51x-110 = (x-9.012)(\frac{x^3}{x}+\alpha x+\frac{-110}{-9.012})$

$\Rightarrow(x-9.012)(x^2+\alpha x+12.2059)$. Comparing the coefficients of like terms, the value of α is determined.

$\Rightarrow -9.012\alpha+12.2059=-51 \Rightarrow \alpha=7.0135$

$\therefore$ $x^3-2x^2-51x-110 = \Rightarrow(x-9.012)(x^2+7.0135x+12.2059)$

To evaluate the two solutions of $(x^2+7.0135x+12.2059)$,

By applying 1st derivative = $\pm\sqrt{discriminant}$

$2x+7.0135=\pm\sqrt{(7.0135)^2-4(12.2059)} = \pm\sqrt{49.1891-48.8236}$

$2x+7.0135=\pm\sqrt{0.3655} \Rightarrow 2x+7.0135=\pm 0.6045$

X=-3.2045 & -3.809

$\therefore$ $x^3-2x^2-51x-110 = (x-9.012)(x+3.2045)(x+3.809)$

Conclusion : The three solutions of the given polynomial of 3rd order are given in Table – 7.

Table – 7 (Conclusion)

Solution	Straight Division Mehtod	Purana Apuranabhyam Method
1	9.012	9.012
2	-3.206	-3.205
3	-3.806	-3.809

Expansion Tables

Table - 8
$(a+b+c+d+e+f+g)^3$

Coefficients →	1	3		6
10^0	a^3			
10^1		a^2b		
10^2		a^2c b^2a		
10^3	b^3	a^2d	abc	
10^4		a^2e b^2c, c^2a	abd	
10^5		a^2f, c^2b b^2d	abe acd	
10^6	c^3	a^2g, d^2a b^2e	abf ace	bcd

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