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Lie Symmetry Analysis and the Optimal System of Nonlinear Fourth Order Evolution Equation.

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Abstract
In this paper, a nonlinear fourth order evolution equation is investigated by the Lie symmetry analysis approach. All the geometric vector fields of the evolution equation are obtained. Moreover the vector fields are shown to be closed under the Lie brackets. Finally the adjoint representation and the optimal systems are constructed.

Keywords: *Lie symmetry analysis, Vector fields, Lie brackets, Adjoint representation, Optimal systems.*
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1. Introduction

Problems involving nonlinear differential equations arise in various fields of science, mathematics and other related areas. Therefore the task of obtaining the exact solutions of such types of differential equations is of great importance. The theory of Lie symmetry group of differential equations, developed by Sophus Lie, has played a significant role in understanding and constructing solutions of differential equations.

For any given subgroup, an original differential equation can be reduced to a system with fewer independent variables which corresponds to group invariant solutions. The concept of optimal systems is thus very useful in minimizing the search for the group-invariant solutions in the event that a group leaves a PDE invariant. An optimal system provides precise insights into all possible invariant solutions, hence it is of great importance from mathematical point of view as well as constraining the system for the physical and engineering applications. In application, one usually construct the optimal systems of subalgebras from which the optimal systems of the subgroups and group invariant solutions are constructed.

The adjoint representation of a Lie group on its Lie algebra was known to Lie [13]. In [6], Ovsiannikov demonstrated the construction of the one-dimensional optimal system for the Lie algebra, using a global matrix for the adjoint transformation and sketched the construction of higher-dimensional optimal system with a simple example. For the higher-dimensional optimal systems of Lie algebra, Galas [3] also developed Ovsiannikov's idea of removing equivalent subalgebras and the problem of a nonsolvable algebra discussed. Some examples of optimal systems can also be found in [1], [2], [4], [5], [7], [8], [9], [10], [11] and [12].

In this paper, we have investigated the construction of the optimal system for the nonlinear fourth order evolution equation:

$$u_t - 2u_x u_{xx} - u^2 u_{xx} + u_{xxxx} = 0 \tag{1}$$

II. Lie symmetry and the geometric vector fields.

We let $\Delta = u_t - 2u_x u_{xx} - u^2 u_{xx} + u_{xxxx}$ and the infinitesimal generator X of (1) to be of the form

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \tag{2}$$

where the coefficient functions $\tau(x,t,u)$, $\xi(x,t,u)$, and $\eta(x,t,u)$ are to be determined.

For the symmetry condition to be satisfied by (1) , then: $X^{(4)}\Delta\big|_{\Delta=0} = 0$.

Here , $X^{(4)}$ is the fourth prolongation of (2) .

The following reduced determining equations for the infinitesimal transformation of (1) are obtained.

$$\xi_u = \xi_t = \tau_u = \tau_x = 0 \tag{3}$$

$$\xi_{xx} = 0 \tag{4}$$

$$\eta_{uu} = 0 \tag{5}$$

$$\tau_t - 4\xi_x = 0 \quad (6)$$

$$\left(12\eta_{xuu} - 18\xi_{xuu}\right) + \left(\eta_u + \tau_t - 3\xi_x\right) + 3\xi_u u^2 = 0 \quad (7)$$

From (3)-(7), we obtain the coefficient functions as:
$\xi = c_1 x + c_2$, $\tau = 4c_1 t + c_3$ and

$$\eta = -c_1 u + \alpha(x,t) \quad (8)$$

Substituting (8) into (2), the corresponding geometric vector fields are given by:

$$X_1 = x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial t}$$

III. The Lie brackets

The vector fields X_1, X_2 and X_3 are closed under the Lie brackets as shown:

$$[X_1, X_1] = [X_2, X_2] = [X_3, X_3] = 0$$

$$\begin{aligned}[X_1, X_2] &= X_1X_2 - X_2X_1 \\ &= \left(x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\right)\left(\frac{\partial}{\partial x}\right) \\ &\quad - \left(\frac{\partial}{\partial t}\right)\left(x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\right) \\ &= -\frac{\partial}{\partial x} \\ &= -X_2 \\ &= -[X_2, X_1]\end{aligned}$$

$$\begin{aligned}[X_1, X_3] &= X_1X_3 - X_3X_1 \\ &= \left(x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\right)\left(\frac{\partial}{\partial t}\right) \\ &\quad - \left(\frac{\partial}{\partial t}\right)\left(x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\right) \\ &= -4\frac{\partial}{\partial t} \\ &= -4X_3 \\ &= -[X_3, X_1]\end{aligned}$$

$$\begin{aligned}[X_2, X_3] &= X_2X_3 - X_3X_2 \\ &= \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}\right) - \left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x}\right) \\ &= 0 \\ &= [X_3, X_2]\end{aligned}$$

This can be summarized as shown in the commutator table below :

Table 1: Commutator table

$\left[X_i, X_j\right]$	X_1	X_2	X_2
X_1	0	$-X_2$	$-4X_3$
X_2	X_2	0	0
X_3	$4X_3$	0	0

IV. Adjoint representation and the optimal system of (1)

4.1 Adjoint representation and transformation matrix.

To compute the adjoint representation, we use the Lie series :

$$\begin{aligned}Ad\left(\exp(\varepsilon v)\right)w_0 &= \sum_{n=0}^{\infty}\left(ad\, v\right)^n (w_0) \\ &= w_0 - \varepsilon\left[v, w_0\right] + \frac{\varepsilon^2}{2}[v,[v,w_0]] - \ldots\ldots\end{aligned}$$

in conjunction with the commutator table above.
The computation is done as follows:

$$\begin{aligned}Ad\left(\exp(\varepsilon_1 X_1)\right)X_1 &= X_1 - \varepsilon_1\left[X_1, X_1\right] + \frac{1}{2}\varepsilon^2\left[X_1,[X_1, X_1]\right] - \\ &= X_1\end{aligned}$$

$$Ad\left(\exp(\varepsilon_1 X_1)\right) X_2 = X_2 - \varepsilon_1 [X_1, X_2] + \frac{1}{2}\varepsilon_1^{\,2}[X_1,[X_1,X_2]] -$$
$$= X_2 - \varepsilon_1(-X_2) + \frac{1}{2}\varepsilon_1^{\,2}[X_1,-X_2] -$$
$$= X_2 + \varepsilon_1 X_2 + \frac{1}{2}\varepsilon^2{}_1 X_2 -$$
$$= X_2\left(1+\varepsilon_1 + \frac{1}{2}\varepsilon_1^{\,2} + \ldots\right)$$
$$= e^{\varepsilon_1} X_2$$

$$Ad\left(\exp(\varepsilon_1 X_1)\right) X_3 = X_3 - \varepsilon_1 [X_1, X_3] + \frac{1}{2}\varepsilon_1^{\,2}[X_1,[X_1,X_3]] -$$
$$= X_3 - \varepsilon_1(-4X_3) + \frac{1}{2}\varepsilon_1^{\,2}[X_1,-4X_3] -$$
$$= X_3 + 4\varepsilon_1 X_3 + \frac{16}{2}\varepsilon^2{}_1 X_3 -$$
$$= X_3\left(1+4\varepsilon_1 + \frac{16}{2}\varepsilon_1^{\,2} + \ldots\right)$$
$$= e^{4\varepsilon_1} X_3$$

$$Ad\left(\exp(\varepsilon_2 X_2)\right) X_1 = X_1 - \varepsilon_2 [X_2, X_1] + \frac{1}{2}\varepsilon_2^{\,2}[X_2,[X_2,X_1]] -$$
$$= X_1 - \varepsilon_2(X_2) + \frac{1}{2}\varepsilon_2^{\,2}[X_2,X_2]$$
$$= X_1 - \varepsilon_2 X_2$$

$$Ad\left(\exp(\varepsilon_2 X_2)\right) X_2 = X_2 - \varepsilon_2 [X_2, X_2] + \frac{1}{2}\varepsilon_2^{\,2}[X_2,[X_2,X_2]] -$$
$$= X_2$$

$$Ad\left(\exp(\varepsilon_2 X_2)\right) X_3 = X_3 - \varepsilon_2 [X_2, X_3] + \frac{1}{2}\varepsilon_2^{\,2}[X_2,[X_2,X_3]] -$$
$$= X_3$$

$$Ad\left(\exp(\varepsilon_3 X_3)\right) X_1 = X_1 - \varepsilon_3 [X_3, X_1] + \frac{1}{2}\varepsilon_3^{\,2}[X_3,[X_3,X_1]] -$$
$$= X_1 - \varepsilon_3(4X_3) + \frac{1}{2}\varepsilon_3^{\,2}[X_3,4X_3]$$
$$= X_1 - 4\varepsilon_3 X_3$$

$$Ad\left(\exp(\varepsilon_3 X_3)\right) X_2 = X_2 - \varepsilon_3 [X_3, X_2] + \frac{1}{2}\varepsilon_3^{\,2}[X_3,[X_3,X_2]] -$$
$$= X_2$$

$$Ad\left(\exp(\varepsilon_3 X_3)\right) X_3 = X_3 - \varepsilon_3 [X_3, X_3] + \frac{1}{2}\varepsilon_3^{\,2}[X_3,[X_3,X_3]] -$$
$$= X_3$$

We therefore construct the adjoint table as shown

Table 2: Adjoint table

$Ad\left(\exp(\varepsilon_i X_i) X_j\right)$	X_1	X_2	X_2
X_1	X_1	$e^{\varepsilon_1} X_2$	$e^{4\varepsilon_1} X_3$
X_2	$X_1 - \varepsilon_2 X_2$	X_2	X_3
X_3	$X_1 - 4\varepsilon_3 X_3$	X_2	X_3

Using the adjoint table 2 above, we obtain the adjoint matrix A by applying the adjoint actions of X_1, X_2 and X_3 to

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 \qquad (9)$$

as illustrated below.

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$$\begin{aligned}
&Ad\exp(\varepsilon_1 X_1)(a_1X_1+a_2X_2+a_3X_3)\\
&=a_1Ad\exp(\varepsilon_1X_1)X_1+a_2Ad\exp(\varepsilon_1X_1)X_2+a_3Ad\exp(\varepsilon_1X_1)X_3\\
&=a_1X_1+a_2e^{\varepsilon_1}X_2+a_3e^{4\varepsilon_1}X_3\\
&=\begin{pmatrix}a_1 & a_2 & a_3\end{pmatrix}\begin{pmatrix}1&0&0\\0&e^{\varepsilon_1}&0\\0&0&e^{4\varepsilon_1}\end{pmatrix}\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}
\end{aligned} \tag{10}$$

$$\begin{aligned}
&Ad\exp(\varepsilon_2 X_2)(a_1X_1+a_2X_2+a_3X_3)\\
&=a_1Ad\exp(\varepsilon_2X_2)X_1+a_2Ad\exp(\varepsilon_2X_2)X_2+a_3Ad\exp(\varepsilon_2X_2)X_3\\
&=a_1X_1+(a_2-a_1\varepsilon_2)X_2+a_3X_3\\
&=\begin{pmatrix}a_1 & a_2 & a_3\end{pmatrix}\begin{pmatrix}1&-\varepsilon_2&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}
\end{aligned} \tag{11}$$

$$\begin{aligned}
&Ad\exp(\varepsilon_3 X_3)(a_1X_1+a_2X_2+a_3X_3)\\
&=a_1Ad\exp(\varepsilon_3X_3)X_1+a_2Ad\exp(\varepsilon_3X_3)X_2+a_3Ad\exp(\varepsilon_3X_3)X_3\\
&=a_1X_1+a_2X_2+(a_3-4a_1\varepsilon_3)X_3\\
&=\begin{pmatrix}a_1 & a_2 & a_3\end{pmatrix}\begin{pmatrix}1&0&-4\varepsilon_3\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}
\end{aligned} \tag{12}$$

From (10), (11) and (12), the following matrices are obtained.

$$A_1=\begin{pmatrix}1&0&0\\0&e^{\varepsilon_1}&0\\0&0&e^{4\varepsilon_1}\end{pmatrix},\quad A_2=\begin{pmatrix}1&-\varepsilon_1&0\\0&1&0\\0&0&1\end{pmatrix}$$

and $A_3=\begin{pmatrix}1&0&-4\varepsilon_3\\0&1&0\\0&0&1\end{pmatrix}$

Thus the general adjoint transformation matrix A is given by :

$$A=A_1A_2A_3=\begin{pmatrix}1&-\varepsilon_2&-4\varepsilon_3\\0&e^{\varepsilon_1}&0\\0&0&e^{4\varepsilon_1}\end{pmatrix}$$

4.2 Optimal System

The general adjoint transformation equation of (1) is given by :

$$\frac{1}{a}A(\alpha_1,\alpha_2,\alpha_3)^T=(\beta_1,\beta_2,\beta_3)^T \tag{13}$$

where $a\neq 0$ and $\beta=(\beta_1,\beta_2,\beta_3)\rightarrow(a_1,a_2,a_3)$.

According to (13), we have

$$\frac{1}{a}\begin{pmatrix}1&-\varepsilon_2&-4\varepsilon_3\\0&e^{\varepsilon_1}&0\\0&0&e^{4\varepsilon_1}\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}=\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}$$

This reduces to

$$\frac{1}{a}\begin{pmatrix}\alpha_1-\alpha_2\varepsilon_2-4\alpha_3\varepsilon_3\\\alpha_2e^{\varepsilon_1}\\\alpha_3e^{4\varepsilon_1}\end{pmatrix}=\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix} \tag{14}$$

Case 1:

From (14), when $\alpha_1 \neq 0$, $\alpha_2 = 0$ and $\alpha_3 = 0$, then by letting $\alpha_1 = a$, we find that $\beta = (1,0,0)$.
Thus, we obtain the generator X_1.

Case 2:
When $\alpha_1 = 0$, $\alpha_2 \neq 0$, $\alpha_3 = 0$ and by choosing $\varepsilon_2 = 0$, $\varepsilon_1 = -\ln\frac{\alpha_2}{a}$, we find $\beta = (0,1,0)$
Therefore, we obtain the generator X_2.

Case 3:
When $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 \neq 0$ and by choosing $\varepsilon_3 = 0$, $\varepsilon_1 = -\frac{1}{4}\ln\left(\frac{\alpha_3}{a}\right)$, we get $\beta = (0,0,1)$
Thus we obtain the generator X_3.

Case 4:
Let $\beta_1 = 0$ by choosing $\varepsilon_2 = \frac{\alpha_1}{2\alpha_2}$ and $\varepsilon_3 = \frac{\alpha_1}{8\alpha_3}$.
Then by letting $\varepsilon_1 = -\frac{1}{4}\ln\left(\frac{\alpha_3}{a}\right)$, we find $\beta = (0,\lambda,1)$ with $\lambda = \frac{1}{a}\left(\frac{a}{\alpha_3}\right)^{\frac{1}{4}}$.
Thus we obtain the generator $X_3 + \lambda X_2$.

Hence the optimal system of (1) consists of X_1, X_2, X_3 and $X_3 + \lambda X_2$, where $\lambda \neq 0$ is a constant.

V. Conclusion

In this paper, we have investigate the geometric vector fields of the nonlinear fourth order evolution equation (1) using Lie symmetry analysis approach. Furthermore, we have shown that the vector fields are closed under the Lie brackets. Finally we have constructed the adjoint representation and the optimal system of the equation.

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