International Journal of Scientific Engineering and Applied Science (IJSEAS) – Volume-4, Issue-5, May 2018
ISSN: 2395-3470
www.ijseas.com

Tensor Formulation of Klein-Gordon Equation

Dr.Bulikunzira Sylvestre

Department of Physics, University of Rwanda

University Avenue, P.O.Box 117 Butare, Rwanda

Telephone number: +250788423056

Email: S.BULIKUNZIRA@ur.ac.rw

Abstract

In previous works, Dirac equation for half-spin particle has been written in tensor form, in the form of non-linear Maxwell's like equations for two "electromagnetic fields" $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$. It has been proved that the fields $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$ have the same properties as those of the vectors $(\vec{E},\vec{H})$, components of electromagnetic field.

In development of the above ideas, in the present work, we also wrote Klein-Gordon equation in tensor form, through strengths$(\vec{E},\vec{H})$.

***Keywords:* Klein-Gordon equation, Cartan map, tensor form.**

Introduction

In previous works, using Cartan map, relativistic Dirac equation for half-spin particle has been written in tensor form, in the form of non-linear Maxwell's like equations, through two complex isotropic vectors $\vec{F}=\vec{E}+i\vec{H}$ and $\vec{F}'=\vec{E}'-i\vec{H}'$. Complex vectors $\vec{F}=\vec{E}+i\vec{H}$ and $\vec{F}'=\vec{E}'-i\vec{H}'$ satisfy non-linear condition $\vec{F}^2=0$, equivalent to two conditions for real quantities $\vec{E}^2-\vec{H}^2=0$ and $\vec{E}.\vec{H}=0$, obtained by separating real and imaginary parts in equality $\vec{F}^2=0$. It has been proved that vectors $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$ have the same properties as those of $(\vec{E},\vec{H})$, components of electromagnetic field. For example, under relativistic Lorentz transformations, they are transformed as components of a second rank tensor $F_{\mu\nu}$. In addition, it has been proved that the solution of Dirac equation for free particle as well fulfils Maxwell's equations for vacuum (with zero at the right side).

In development of the above ideas, in this work, we shall prove that, Klein-Gordon equation for zero spin particle can also be written in tensor form, through isotropic vector $\vec{F}=\vec{E}+i\vec{H}$.

Research Method

In this work, we shall investigate Klein-Gordon equation for zero spin particles. In previous works, using Cartan map, Dirac equation for half –spin particle has been written in tensor form, in the form of non-linear Maxwell's like equations for two "electromagnetic fields" $(\vec{E}, \vec{H})$ and $(\vec{E}', \vec{H}')$. In this work, using the same method, we shall prove that, Klein-Gordon equation for zero spin particles can also be written in tensor form, through one complex isotropic vector $\vec{F} = \vec{E} + i\vec{H}$ or equivalently through strengths $\vec{E}$ and $\vec{H}$.

1. Cartan Map

1.1. Definition and algebraic properties

We shall denote by C^n, the complex vector space of dimension "n".

We shall consider only C^2, C^3 and C^4.

Elements of C^2 will be denoted by Geek syllables

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \tag{1}$$

And will be called spinors.

Elements of C^3 will be denoted by Latin syllables

$$\vec{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}, \tag{2}$$

And will be called vectors.

Finally, elements of C^4 will be denoted by Latin syllables

$$j_\mu = \begin{bmatrix} j_0 \\ j_x \\ j_y \\ j_z \end{bmatrix}, \tag{3}$$

And will be called four vectors.

Definition1: Cartan map is a bilinear transformation b from space $C^2 \times C^2$ in space C^4, defined as follows:

$$b^0(\xi,\tau) = -(\xi_1\tau_2 - \xi_2\tau_1) \tag{4}$$

$$\vec{b}(\xi,\tau) = \begin{bmatrix} \xi_1\tau_1 - \xi_2\tau_2 \\ i(\xi_1\tau_1 + \xi_2\tau_2) \\ -(\xi_1\tau_2 + \xi_2\tau_1) \end{bmatrix}. \tag{5}$$

From the definitions Eq(4) and Eq(5) follows that b^0 is antisymmetric and $\vec{b}$ is symmetric relative to the permutation ξ by τ, i.e.,

$$b^0(\xi,\tau) = -b^0(\tau,\xi) \tag{6}$$

$$\vec{b}(\xi,\tau) = \vec{b}(\tau,\xi). \tag{7}$$

In particular, for any spinor ξ

$$b^0(\xi,\xi) = 0 \tag{8}$$

Using the definitions Eqs(4)-(5), one can prove the following properties of Cartan map:

Lemma1: For any spinors ρ, ξ, τ of space C^2, the following identities are verified

$$\vec{b}(\rho,\xi)\,\vec{b}(\tau,\tau) = -2\, b^0(\rho,\tau)\, b^0(\xi,\xi) \tag{9}$$

$$\vec{b}(\rho,\xi)\,\vec{b}(\xi,\tau) = -2\, b^0(\rho,\xi)\, b^0(\xi,\tau) \tag{10}$$

$$\vec{b}(\rho,\tau)\,\vec{b}(\xi,\tau) = \quad b^0(\rho,\tau)\, b^0(\xi,\tau) \tag{11}$$

$$\vec{b}(\xi,\xi)\,\vec{b}(\tau,\tau) = -2 b^0(\xi,\tau)^{\,2} \tag{12}$$

$$\vec{b}(\xi,\tau)\,\vec{b}(\tau,\xi) = \quad b^0(\xi,\tau)^{\,2} \tag{13}$$

$$\vec{b}(\xi,\xi)\,\vec{b}(\tau,\xi) = 0 \tag{14}$$

Lemma2: For any two spinors ξ and τ of space C^2, the following identity is verified

$$\vec{b}(\xi,\xi) \times \vec{b}(\tau,\tau) = 2i\, b^0(\xi,\tau)\, \vec{b}(\xi,\tau) \tag{15}$$

Definition2: If

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \in C^2 \tag{16}$$

Is a spinor, then the conjugate spinor ξ^* of the spinor ξ is defined as follows

$$\xi^* = \begin{bmatrix} -\overline{\xi_2} \\ \overline{\xi_1} \end{bmatrix} \in C^2 , \qquad (17)$$

Where $\overline{\xi_1}, \overline{\xi_2}$ are complex conjugates of spinor components $\xi_1$and ξ_2.

Lemma3: For any two spinors ξ and τ of space C^2, the following identities are verified

$$b^0(\xi, \tau^*) = \overline{b^0(\tau, \xi^*)} , \qquad (18)$$

$$\vec{b}(\xi, \tau^*) = \overline{\vec{b}(\tau, \xi^*)} , \qquad (19)$$

$$b^0(\xi^*, \tau^*) = \overline{b^0(\xi, \tau)}, \qquad (20)$$

$$\vec{b}(\xi^*, \tau^*) = -\overline{\vec{b}(\xi, \tau)}. \qquad (21)$$

Let introduce vectors $\vec{F} \in C^3$ and $j_\mu \in C^4$ as follows:

$$\vec{F} = \vec{E} + i\vec{H} = i\vec{b}(\xi, \xi), \qquad (22)$$

$$j_\mu = b_\mu(\xi, \xi^*) . \qquad (23)$$

Here $\vec{E}$ and $\vec{H}$ are real vectors.

From Eq(21) follows, that

$$\vec{F^*} = \vec{E} - i\vec{H} = \overline{i\vec{b}(\xi, \xi)} = i\, \vec{b}(\xi^*, \xi^*). \qquad (24)$$

Lemma4: From Eq(5) and Eq(22) follows identity

$$\vec{F}^2 = \vec{F}.\vec{F} = 0. \qquad (25)$$

i.e., $\vec{F}$ is isotropic vector.

Eq(25) is equivalent to two conditions, obtained by equating to zero real and imaginary parts of equality $\vec{F}^2 = 0$

$$\vec{E}^2 = \vec{H}^2 , \qquad (26)$$

$$\vec{E}.\vec{H} = 0. \qquad (27)$$

One can also prove, that

$$j_0=\left[\frac{\vec{F}.\vec{F}^*}{2}\right]^{1/2}=|\vec{E}|,$$

$$\vec{j}=i\,\frac{\vec{F}\times\vec{F}^*}{2j_0}=\frac{\vec{E}\times\vec{H}}{|\vec{E}|}\,. \qquad (28)$$

Lemma5: For any spinor $\xi\in C^2$, the following identities are verified

$$j_0=|\vec{E}| = |\xi|^2, \qquad (29)$$

$$\vec{j}=\frac{\vec{E}\times\vec{H}}{|\vec{E}|}=\bar{\xi}^T\vec{\sigma}\xi. \qquad (30)$$

Where $\bar{\xi}^T$ is the transposed conjugate of the spinor ξ and $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ are Pauli spin matrices.

From Eqs(29)-(30) follows, that under Lorentz relativistic transformations, $j_\mu$transforms as four vector. Vectors $\vec{E}$ and $\vec{H}$ transform as components of electromagnetic field, i.e., form a second rank tensor $F_{\mu\nu}$.

Lemma6: For any pair of spinors ξ and τ of space C^2 and any vector $\vec{v}$ the following identities are verified

$$b^0(\vec{v}.\vec{\sigma}\xi,\tau) = \vec{v}.\vec{b}(\xi,\tau), \qquad (31)$$

$$\vec{b}(\vec{v}.\vec{\sigma}\xi,\tau) = \vec{v}b^0(\xi,\tau) + i\,\vec{v}\times\vec{b}(\xi,\tau), \qquad (32)$$

$$\vec{b}(\vec{v}.\vec{\sigma}\xi,\xi)= (\vec{v}.\vec{s})\,\vec{b}(\xi,\xi). \qquad (33)$$

Here $\vec{s}=(s_x,s_y,s_z)$ are Proka spin matrices, with $s_i=i(\varepsilon_i)_{jk}$, where ε_{ijk} is the tridimensional antisymmetric tensor Levi-Cevita.

From Eq(33) follows, that if ξ is eigenvector of operator $(\vec{v}.\vec{\sigma})$ with eigenvalue λ, then $\vec{b}(\xi,\xi)$ is eigenvector of operator $(\vec{v}.\vec{s})$ with the same eigenvalue λ.

Definition3: Let ξ be a spinor field and $\widetilde{A}$, an operator acting on ξ. Let $\vec{b}$ maps spinor ξ on isotropic vector $\vec{F}=i\vec{b}(\xi,\xi)$. We shall say, that operator $\widetilde{A}$ commutes with Cartan map and becomes $\widehat{A}$, acting on $\vec{F}$, if:

$$\widehat{A}\,\vec{F} = i\widehat{A}\,\vec{b}(\xi,\xi) = i\vec{b}(\widetilde{A}\xi,\xi). \qquad (34)$$

From Eq(34) follows, that if ξ is eigenvector of operator $\widetilde{A}$ with eigenvalue λ, then $\vec{F}$ is eigenvector of operator $\widehat{A}$ with the same eigenvalue λ; i.e., Cartan map conserves eigenvectors and eigenvalues.

Lemma7: For any spinor ξ of space C^2, the following identities are verified

$$b^0(\vec{p}\xi,\xi) = -i\{\vec{D}\vec{b}(\xi,\xi)\}.\vec{v}, \tag{35}$$

$$\vec{b}(\vec{p}\xi,\xi) = \vec{D}\vec{b}(\xi,\xi), \tag{36}$$

Where $\vec{v} = \frac{\vec{j}}{j_0} = \frac{\vec{E}\times\vec{H}}{\vec{E}^2}$

$$\vec{D} = -i\frac{\hbar}{2}\vec{\nabla}.$$

2. Klein-Gordon Equation in Tensor Form

Relativistic particle with zero spin and rest mass m is described by Klein-Gordon equation

$$(\square + m^2)\Psi = 0 \tag{37}$$

Here $\square = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$ is D'Alembert operator, Ψ is a scalar function.

Let us prove, that this equation can be written in tensor form. For this, let us introduce a two component wave function

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{38}$$

and second rank matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{39}$$

Where

$$\phi_1 = \frac{1}{\sqrt{2}}\left[\Psi + \frac{i\hbar}{mc}\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{ie}{\hbar c}\varphi\right)\Psi\right] \tag{40}$$

$$\phi_2 = \frac{1}{\sqrt{2}}\left[\Psi - \frac{i\hbar}{mc}\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{ie}{\hbar c}\varphi\right)\Psi\right]. \tag{41}$$

Then, the following pair of first order equations,

$$i\hbar\frac{\partial\phi_1}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2\{\phi_1 + \phi_2\} + (e\varphi - mc^2)\phi_1, \tag{42}$$

$$i\hbar \frac{\partial \phi_2}{\partial t} = -\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 \{\phi_1 + \phi_2\} + (e\varphi - mc^2)\phi_2, \tag{43}$$

is exactly equivalent to Klein-Gordon equation, Eq(37). Here $A_\mu = (\varphi, \vec{A})$is external electromagnetic potential.

In compact form, Eqs(42)-(43) can be written as follows,

$$i\hbar \frac{\partial \Phi}{\partial t} = \frac{(\tau_3 + i\tau_2)}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 \Phi + mc^2\tau_3\Phi + e\varphi\Phi \tag{44}$$

Let us consider a free particle (φ=0 and $\vec{A}$=0) and use the natural system of units in which c=ħ=1. Then, we obtain

$$i\frac{\partial \Phi}{\partial t} = \frac{(\tau_3 + i\tau_2)}{2m}\vec{p}^2\Phi + m\tau_3\Phi. \tag{45}$$

Now, let us transform Eq(45) according to Cartan map. We have

$$\vec{b}(p_0\Phi, \Phi) = \frac{1}{2m}\vec{b}\left[(\tau_3 + i\tau_2)\vec{p}^2\Phi, \Phi\right] + m\vec{b}(\tau_3\Phi, \Phi). \tag{46}$$

Here $p_0 = i\frac{\partial}{\partial t}$, $\vec{p} = -i\vec{\nabla}$.

From Eq(46), we obtain

$$iD_0\vec{b}(\Phi, \Phi) = \frac{1}{2m}(S_3 + iS_2)2\vec{D}^2 i\,\vec{b}(\Phi, \Phi) + mS_3\, i\vec{b}(\Phi, \Phi)\,. \tag{47}$$

Here $D_0 = \frac{i}{2}\frac{\partial}{\partial t}$, $\vec{D} = -\frac{i}{2}\vec{\nabla}$, $\vec{S}=(S_1, S_2, S_3)$ are Proka spin matrices, where $S_i = i\varepsilon_{ijk}$, ε_{ijk} is tridimensional antisymmetric tensor Levi-Cevita.

Calculating matrices $\vec{S}$, we obtain

$$S_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix},\ S_2 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix},\ S_3 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{48}$$

Introducing in Eq(47) isotropic vector

$$\vec{F} = \vec{E} + i\vec{H} = i\,\vec{b}(\Phi, \Phi), \tag{49}$$

we obtain

$$D_0\vec{F} = \frac{1}{m}(S_3 + iS_2)\,\vec{D}^2\vec{F} + m\,S_3\,\vec{F}\,. \tag{50}$$

Or in components

$$\begin{cases} -\frac{\partial H_1}{\partial t} = -\frac{1}{2m}\vec{\nabla}^2(E_3 - H_2) - m\,H_2 \\ -\frac{\partial H_2}{\partial t} = -\frac{1}{2m}\vec{\nabla}^2 H_1 + m\,H_1 \\ -\frac{\partial H_3}{\partial t} = \frac{1}{2m}\vec{\nabla}^2 E_1 \\ \frac{\partial E_1}{\partial t} = -\frac{1}{2m}\vec{\nabla}^2(E_2 + H_3) + m\,E_2 \\ \frac{\partial E_2}{\partial t} = \frac{1}{2m}\vec{\nabla}^2 E_1 - m\,E_1 \\ \frac{\partial E_3}{\partial t} = \frac{1}{2m}\vec{\nabla}^2 H_1 \end{cases} \tag{51}$$

Discussion and Conclusion

In this work, we investigated the possibility of representing Klein-Gordon equation in tensor form, through strengths $\vec{E}$ and $\vec{H}$. In previous works, similar problem has been solved, when spinor Dirac equation for half-spin particle has been written in tensor form, in the form of non-linear Maxwell's like equations for two electromagnetic fields $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$. Applying Cartan map, we wrote Klein-Gordon equation in tensor form, in the form of one linear complex vector equation for one isotropic vector $\vec{F} = \vec{E} + i\vec{H}$. This equation, of course, is equivalent to six scalar equations for six components of two real vectors $\vec{E}$ and $\vec{H}$. Therefore, we can conclude that, scalar particles can also be described in terms of strengths $\vec{E}$ and $\vec{H}$, components of an antisymmetric tensor $F_{\mu\nu}$.

References

[1] Bulikunzira S, "Tensor formulation of Dirac equation through divisors",Asian Journal of Fuzzy and Applied Mathematics, Vol.2, No.6, 2014, pp.195-197.
[2] Bulikunzira S, "Tensor formulation of Dirac equation in standard representation", Asian Journal of Fuzzy and Applied Mathematics, Vol.2, No.6, 2014, pp. 203-208.
[3] Reifler F, "Vector wave equation for neutrinos", Journal of mathematical physics, Vol.25, No.4, 1984, pp.1088-1092.
[4] Sommers P, "Space spinors", Journal of mathematical physics, Vol.21, No.10,1980, pp.2567-2571.