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Mathematical Analysis of a Model for Syphilis Endemicity

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Abstract

In this paper, we formulate a compartmental model to investigate the dynamics of the spread of syphilis in a sexually active population with some measure of disease control. We quantitatively and qualitatively analyze the model amongst the other things found that the model undergoes the phenomenon of backward bifurcation and has a disease free equilibrium (DFE) which is globally asymptotically stable when the cause of backward bifurcation is removed, it also has an endemic equilibrium points (EEP) which is globally asymptotically stable (for a special case). Numerical simulation where carried out on the model and we proposed the effective treatment strategies of syphilis in its primary and secondary infected individuals which will help reduce the cases.

Keywords: Mathematical model, Syphilis, Endemicity and Mathematical Analysis

1. INTRODUCTION

Syphilis is a chronic bacterial disease that is contracted chiefly by infection during sexual intercourse which may be vaginal, oral or anal. It can also be transmitted congenitally by infection of a developing fetus. Syphilis is caused by a microorganism *Trepenomapallidum*.

Syphilis has been called “the great imitator” because of its many possible symptoms some of which are common to other diseases (CDC, 2014). A multi stage disease that progresses when left untreated from primary to secondary, early latent, late latent, and tertiary stages; its progression rate is very slow but it ends up killing or leading to various cardiovascular or nuerological diseases of one out of every four of those infected (Doherty L., *et al*, 2002)

The point of syphilis infection becomes characterized by an ulceratic chancre (painless sore) signaling the beginning of the primary stage of the disease (Aadland D., *et al*, 2013). Thereafter, the disease progresses to the secondary stage evident by skin rashes and (or) sores in the mouth, vagina or anus called mucous membrane lesions (CDC, 2014). The rashes may be red or reddish brown appearing on any part of the body especially on the palms, underneath the feet or at the back of an infected individual. This rash might even be unnoticed (NIAID, 2010).

The latent stage of the disease begins when all the symptoms described above have disappeared. This stage is divided into two:

- The early latent stage (infection less than 12months) during which patients can revert to either the secondary or primary stage(Ibio and Okuonghae, 2015), (Aadland D., *et al*, 2013).
- The late latent stage during which an individual is not infectious has three different possible outcomes. (i) the infection is biologically eradicated within the body over a number of years (ii) the infection remains in the individual over a long period of time (maybe a lifetime) (iii) the infection progresses gradually to cause organ damage which can be fatal (Cecil, 1948), (Aadland D., *et al*, 2013).

This progression however is called the tertiary stage. Without treatment, a third of infected people get to this stage (Bhatti M T, 2007). At this stage, the patient may no longer be infectious, but the bacteria reactivate, multiply and spread throughout the body, damaging the heart, eyes(ocular syphilis), brain nervous system, bones and joints. It may also result in degenerative central nervous system, dementia and paralysis; and damage is irreversible (NIAID, 2010).

Tertiary Syphilis may take the form of

(i) Gummatous syphilis characterized by the presence ofsoft tumor like gummas.
(ii) Neurosyphilis which is a condition of severe infection of the central nervous system.
(iii) Cardiovascular syphilis which causes aortic aneurysms and regurgitation
(Chen J.F.,*et al*, 2008).

A single intramuscular injection of penicillin is the standard treatment for primary, secondary and early latent syphilis(CDC, 2002). Follow-up is necessary for about one year until no bacteria is found in the blood test. This can also be used for the late latent and tertiary stage, but the damage done cannot be reversed. Other antibiotics such as macrolides and cephalosporins withcertain caveats can be used for those allergic to penicillin (Kevin F. A., 2008).

Though syphilis can be cured completely, one can be re-infected as immunity to syphilis is not permanent (Aadland D., *et al*, 2013). There are currently 36million people living with syphilis worldwide with 12million new reported cases every year (WHO 2012).

2 Model Formulation

The total sexually active population at time t is denoted by $N(t)$and it is divided into 8 mutually exclusive compartments. Let

$$S(t), E_s(t), I_p(t), I_s(t), L_{s1}(t), L_{s2}(t), I_T(t) and\ R_s(t)$$

be the eight(8) compartments which represent the populations of susceptible individuals, individuals with syphilis at incubation stage, individuals with primary syphilis, individuals with secondary syphilis, individuals with early latent syphilis, individuals with late latent syphilis, individuals with tertiary syphilis and individuals treated of syphilis

so that

$$N(t) = S(t) + E_s(t) + I_p(t) + I_s(t) + L_{s1}(t) + L_{s2}(t) + I_T(t) + R_s(t)$$

The susceptible population S(t) is generated by the recruitment of individuals (assumed susceptible) into the population at a rate Λ.

Infectious interactions in the population are modeled using a standard incidence function typically written in the form $\frac{\beta I}{N}$ where the effective contact rate β increases linearly with the population size N.

2.1 Transmission by singly infected individuals

Susceptible individuals acquire syphilis infection from individuals with primary and secondary syphilis at a rate given by

$$\lambda_S = \beta_s \frac{(I_p + \theta_1 I_s)}{N}$$

In this case, β_s is the effective contact rate for the transmission of syphilis and the modification parameter; $\theta_1 > 1$; accounting for the relative infectiousness of those in the I_s class in comparison to those in the I_p class.

2.2 Derivation of Model Equations

The population of individuals in the $E_s, I_p, I_s, L_{s1}, and\ L_{s2}$ classes progress to the $I_p, I_s, L_{s1}, L_{s2}\ and\ I_T$ classes respectively at a rate $\gamma_1, \gamma_2, \gamma_3 \gamma_4\ and\ \gamma_5$ respectively. The treatment rates for the classes $I_p, I_s, L_{s1}, L_{s2}\ and\ I_T$ are given as $r_1, r_2, r_3, r_4\ and\ r_5$ respectively. The likelihood of being re-infected with syphilis after a previous infection is given by α_1. Combining all the afore mentioned assumptions and definitions, the model for the syphilis in a sexually active population is given by the following system of differential equations (1)

$$\dot{S} = \Lambda - \lambda_S S - \mu S \qquad (1a)$$

$$\dot{E}_S = \lambda_S S - (\gamma_1 + \mu) E_S + \alpha_1 \lambda_S R_s \qquad (1b)$$

$$\dot{I}_P = \gamma_1 E_S - (\gamma_2 + r_1 + \mu) I_P \qquad (1c)$$

$$\dot{I}_S = \gamma_2 I_P - (\gamma_3 + r_2 + \mu) I_S \qquad (1d)$$

$$\dot{L}_{S1} = \gamma_3 I_S - (\gamma_4 + r_3 + \mu) L_{S1} \qquad (1e)$$

$$\dot{L}_{S2} = \gamma_4 L_{S1} - (\gamma_5 + r_4 + \mu) L_{S2} \qquad (1f)$$

$$\dot{I}_T = \gamma_5 L_{S2} - (r_5 + \mu) I_T \qquad (1g)$$

$$\dot{R}_S = r_1 I_P + r_2 I_S + r_3 L_{S1} + r_4 L_{S2} + r_5 I_T - \mu R_S - \alpha_1 \lambda_S R_S \qquad (1g)$$

$$with\ \lambda_S = \beta_S \frac{(I_P + \theta_1 I_S)}{N}\ and\ N = S + E_S + I_P + I_S + L_{S1} + L_{S2} + I_T + R_S$$

Since the model (**1**) monitors human population, all variables and parameters of the model are non-negative. Natural death mortality occurs in all of the classes at a rate μ.

Table 1: Description of the State variables of Model (1)

State Variables	Description
S	Susceptible individuals
E_s	Individuals with syphilis at incubation stage.
I_p	Individuals with syphilis at primary stage.
I_s	Individuals with syphilis at secondary stage.
L_{s1}	Individuals with syphilis at early latent stage.
L_{s2}	Individuals with syphilis at late latent stage.
I_T	Individuals with syphilis at tertiary stage.
R_s	Individuals treated of syphilis

Table 2: Description of Parameters of Model (3.1)

Parameters	Description
Λ	Recruitment rate into the sexually active population
μ	Natural death rate
β_s	Effective contact rate for the transmission of syphilis
r_1, r_2, r_3, r_4, r_5	Treatment rate for infected individuals
$\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$	Progression rates for infected individuals
α_1	Modification parameter

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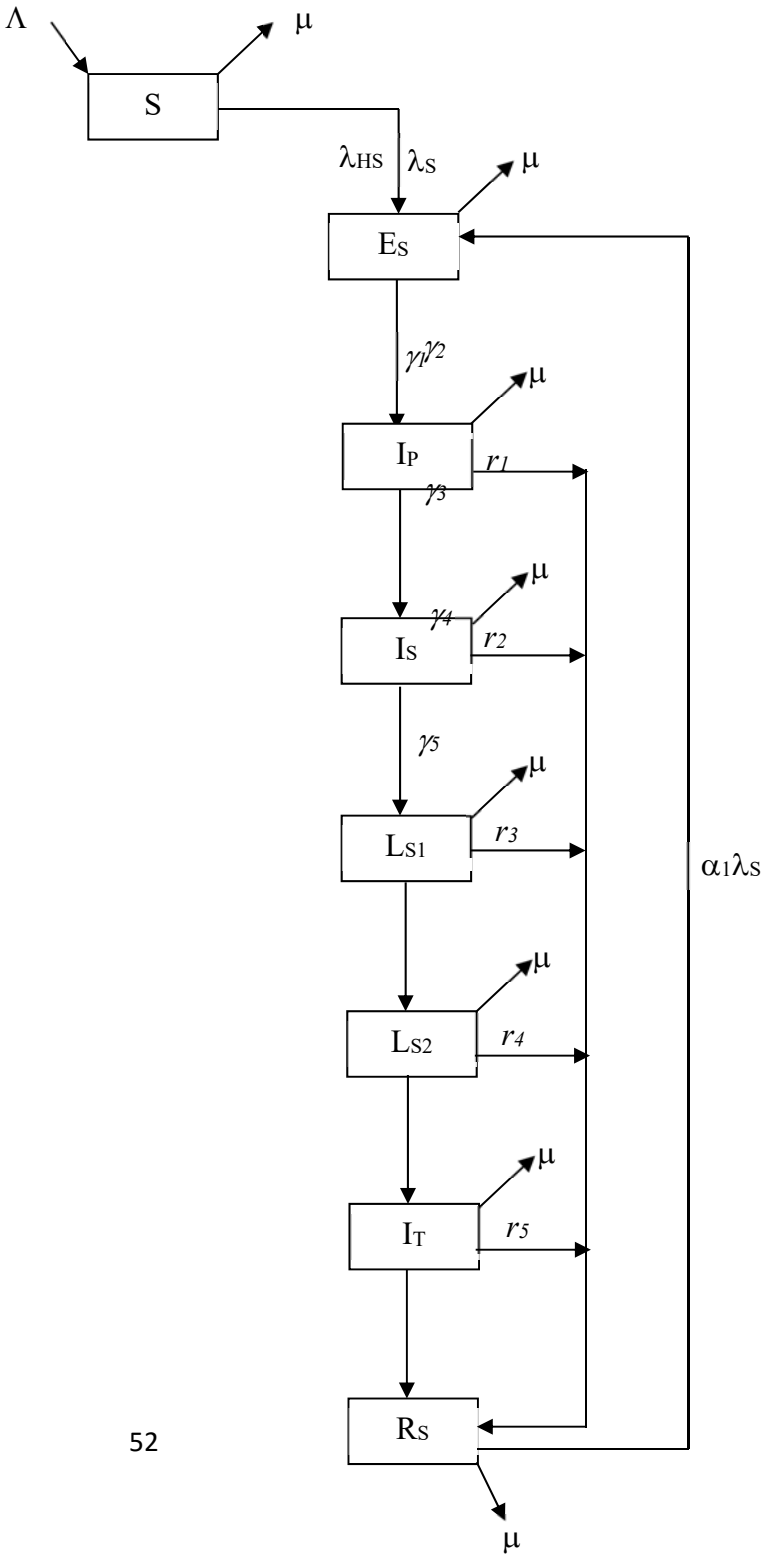
FIGURE 1: SCHEMATIC DIAGRAM OF THE MODEL (1)



2.3 Boundedness and Positivity of Solutions.

Consider the region $D_2 = \left\{(S, E_S, I_P, I_S, L_{S1}, L_{S2}, I_T, R_S) \in \mathbb{R}_+^8 : N \leq \frac{\Lambda}{\mu}\right\}$. It can be shown that the set D_2 is positively invariant and an attractor of all positive solution of the system **(1).**

Lemma 1*The region D_2 is positively invariant for the system (1)*

Proof: The rate of change of the total population is give as

$$\dot{N} = \dot{S} + \dot{E}_S + \dot{I}_P + \dot{I}_S + \dot{L}_{S1} + \dot{L}_{S2} + \dot{I}_T + \dot{R}_S = \Lambda - \mu N$$

Using the integrating factor method, we have

$$\dot{N} + \mu N = \Lambda$$

$$\dot{N}e^{\mu t} + \mu N e^{\mu t} = \Lambda e^{\mu t}$$

$$\frac{d}{dt}(Ne^{\mu t}) = \Lambda e^{\mu t}$$

$$\int d(Ne^{\mu t}) = \int \Lambda e^{\mu t}\, dt$$

$$Ne^{\mu t} = \frac{\Lambda}{\mu} e^{\mu t} + C$$

$at\ t = 0, C = N_0 - \frac{\Lambda}{\mu}$,so we have

$$Ne^{\mu t} = \frac{\Lambda}{\mu} e^{\mu t} + N_0 - \frac{\Lambda}{\mu}$$

$$N = \frac{\Lambda}{\mu} + [N_0 - \frac{\Lambda}{\mu}]e^{-\mu t}$$

If $N_0 < \frac{\Lambda}{\mu}\ then\ N(t) \leq \frac{\Lambda}{\mu} for\ all\ t > 0$. If $N_0 > \frac{\Lambda}{\mu}$, then either the solution enters D_2 in finite time or $N(t) \to \frac{\Lambda}{\mu}\ as\ t \to \infty$. hence, D_2 attracts all solutions in $\mathbb{R}_+^8$.

Positivity of Solution

Lemma 2 *Let the initial data for the model* ***(1)****be* $S(0) > 0, E_S(0) > 0, I_P(0) > 0, I_S(0) > 0, L_{S1}(0) > 0, L_{S2}(0) > 0, I_T(0) > 0$ *and* $R_S(0) > 0$ *then the solution* $S(t), E_S(t), I_P(t), I_S(t), L_{S1}(t), L_{S2}(t), I_T(t)$ *and* $R_S(t)$ *with positive initial data will remain positive for all time* $t > 0$

Proof: Let $t_1 = sup\{t > 0 : S(t) > 0, E_S(t) > 0, I_P(t) > 0, I_S(t) > 0, L_{S1}(t) > 0, L_{S2}(t) > 0, I_T(t) > 0, R_S(t) > 0\} > 0$

$$\dot{S} = \Lambda - \lambda_S S - \mu S = \Lambda - (\lambda_S + \mu)S$$

To solve the ODE using the integrating factor method

$$I.F = exp\left[\mu t + \left\{\int_0^t \lambda_S(\tau)d(\tau)\right\}\right]$$

$$\frac{d}{dt}\left[S(t)exp\left\{\mu t + \int_0^t \lambda_S(\tau)d(\tau)\right\}\right] = \Lambda\left[exp\left\{\mu t + \int_0^t \lambda_S(\tau)d(\tau)\right\}\right]$$

$$S(t_1)exp\left\{\mu t_1 + \int_0^{t_1} \lambda_S(\tau)d(\tau)\right\} = S(0) + \int_0^{t_1} \Lambda\left[exp\left\{\mu y + \int_0^y \lambda_S(\tau)d(\tau)\right\}\right]dy$$

$$S(t_1) = S(0)exp\left\{-\mu t_1 - \int_0^{t_1} \lambda_S(\tau)d(\tau)\right\} + \left[exp\left\{-\mu t_1 - \int_0^{t_1} \lambda_S(\tau)d(\tau)\right\}\right]\int_0^{t_1} \Lambda\left[exp\left\{\mu y + \int_0^y \lambda_S(\tau)d(\tau)\right\}\right]dy > 0$$

For $\dot{E}_S = \lambda_S S - (\gamma_1 + \mu)E_S + \alpha_1\lambda_S R_S$ we have that $\dot{E}_S \geq -(\gamma_1 + \mu)E_S$

For $\dot{I}_P = \gamma_1 E_S - (\gamma_2 + r_1 + \mu)I_P$ we have that $\dot{I}_P \geq -(\gamma_2 + r_1 + \mu)I_P$

For $\dot{I}_S = \gamma_2 I_P - (\gamma_3 + r_2 + \mu)I_S$ we have that $\dot{I}_S \geq -(\gamma_3 + r_2 + \mu)I_S$

For $\dot{L}_{S1} = \gamma_3 I_S - (\gamma_4 + r_3 + \mu)L_{S1}$ we have that $\dot{L}_{S1} \geq -(\gamma_4 + r_3 + \mu)L_{S1}$

For $\dot{L}_{S2} = \gamma_4 L_{S1} - (\gamma_5 + r_4 + \mu)L_{S2}$ we have that $\dot{L}_{S2} \geq -(\gamma_5 + r_4 + \mu)L_{S2}$

For $\dot{I}_T = \gamma_5 L_{S2} - (r_5 + \mu)I_T$ we have that $\dot{I}_T \geq -(r_5 + \mu)I_T$

For $\dot{R}_S = r_1 I_P + r_2 I_S + r_3 L_{S1} + r_4 L_{S2} + r_5 I_T - \mu R_S - \alpha_1\lambda_S R_S$ we have that

$$\dot{R}_S \geq -(\mu + \alpha_1\lambda_S)R_S$$

Similarly, we can show that $E_S(t) > 0, I_P(t) > 0, I_S(t) > 0, L_{S1}(t) > 0, L_{S2}(t) > 0, I_T(t) > 0$ and $R_S(t) > 0$.

2.4 Local stability of disease-free equilibrium (DFE) of the Syphilis model

The model (1) has a disease-free equilibrium obtained by setting the right hand side of the model to zero given by

$$\xi_{(0,S)} = (S^*, E_S^*, I_P^*, I_S^*, L_{S1}^*, L_{S2}^*, I_T^*, R_S^*) = (\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0, 0, 0)$$

The stability of $\xi_{(0,S)}$ is established using the next generation operator method on the system (1). Using the notation in van den Driessche and Watmough (2002) the matrices F and V for the new infection terms and the remaining transfer terms respectively, are respectively given as

$$F = \begin{pmatrix} 0 & \beta_s & \theta_1\beta_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} \gamma_1 + \mu & 0 & 0 & 0 & 0 & 0 \\ -\gamma_1 & \mu + r_1 + \gamma_2 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_2 & \mu + r_2 + \gamma_3 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_3 & \mu + r_3 + \gamma_4 & 0 & 0 \\ 0 & 0 & 0 & -\gamma_4 & \mu + r_4 + \gamma_5 & 0 \\ 0 & 0 & 0 & 0 & -\gamma_5 & \mu + r_5 \end{pmatrix}$$

The spectral radius given by $\rho(FV^{-1}) = \frac{\beta_S\gamma_1(\mu+r_2+\gamma_3+\gamma_2\theta_1)}{(\mu+\gamma_1)(\mu+r_1+\gamma_2)(\mu+\gamma_3+r_2)} = R_T$

The value R_T is the effective reproduction number since there is the presence of control strategies.

Lemma 3 *The DFE of the system (1) is locally asymptotically stable if $R_T < 1$ and unstable if $R_T > 1$.*

The threshold quantity R_T is the effective or control reproduction number for the syphilis only sub-model. By lemma 3, biologically speaking, syphilis is eliminated from the population when $R_T < 1$ if the initial sizes of the subpopulations of the sub-model are in the region of attraction of $\xi_{(0,S)}$.

However, the disease free equilibrium may not be globally asymptotically stable even if $R_T < 1$ in the case when a backward bifurcation occurs. That is, there is the presence of a stable EEP co-existing with the DFE.

2.5 Analysis of the Control Reproduction Number R_T

Using the threshold parameter R_T, we wish to determine the effect of various treatment rates in both the primary and secondary stage of syphilis.

$$\lim_{r_1 \to \infty} R_T = 0 \qquad (1.1)$$

$$\lim_{r_2 \to \infty} R_T = \frac{\beta_S \gamma_1}{(\mu + \gamma_1)(\mu + r_1 + \gamma_2)} > 0 \qquad (1.2)$$

It therefore follows that a control strategy that allows for the high treatment rates r_1 and r_2 can lead to effective syphilis control if it results in making the right hand side of equation (1.2) less than unity. It is important to note that from (1.1) a near total eradication of syphilis is possible. Focusing on treating syphilis in its primary stage will obviously be effective as disease progressions proceeds from this stage.

Computing the partial derivatives with respect to r_1 and r_2 further reveals the effect of these parameters on syphilis control in the community. Doing this, we have

$$\frac{\partial R_T}{\partial r_1} = -\frac{\beta_S \gamma_1 (\mu + \gamma_3 + r_2 + \gamma_2 \theta_1)}{(\mu + \gamma_1)(\mu + r_1 + \gamma_2)^2 (\mu + \gamma_3 + r_2)} < 0$$

$$\frac{\partial R_T}{\partial r_2} = -\frac{\beta_S \gamma_1 \gamma_2 \theta_1}{(\mu + \gamma_1)(\mu + r_1 + \gamma_2)(\mu + \gamma_3 + r_2)^2} < 0$$

Since the derivatives with respect to r_1 and $r_2$are less than zero unconditionally, we can say that effective treatment for those with syphilis at the primary and secondary stages will reduce the burden of syphilis in the population.

Without treatment or control, the reproduction number without control R_0 is given as

$$R_0 = \frac{\beta_S \gamma_1 (\mu + \gamma_3 + \gamma_2 \theta_1)}{(\mu + \gamma_1)(\mu + \gamma_2)(\mu + \gamma_3)}$$

With $R_T = \frac{(\mu+\gamma_2)(\mu+\gamma_3)(\mu+\gamma_3+r_2+\gamma_2\theta_1)}{(\mu+r_1+\gamma_2)(\mu+\gamma_3+r_2)(\mu+\gamma_3+\gamma_2\theta_1)} R_0 = A_1 R_0$, A_1 compares a population with treatment of syphilis with one without treatment of syphilis.

If $R_0 < 1$, then syphilis cannot develop into an epidemic in the absence of treatment.

2.6 Existence of Endemic Equilibrium Point (EEP)

Let the EEP of model (1) be denoted by$\xi_{(1,S)} = (S^{**}, E_S^{**}, I_P^{**}, I_S^{**}, L_{S1}^{**}, L_{S2}^{**}, I_T^{**}, R_S^{**})$. The equations in (1) are solved in terms of the force of infection at steady state and they are given as

$$S^{**} = \frac{\Lambda}{\lambda_S^{**} + \mu}$$

$$E_S^{**} = \frac{\Lambda g_2 g_3 g_4 g_5 g_6 \lambda_S^{**}(\mu + \alpha_1 \lambda_S^{**})}{(\mu + \lambda_S^{**})[g_1 g_2 g_3 g_4 g_5 g_6(\mu + \alpha_1 \lambda_S^{**}) - \alpha_1 \lambda_S^{**} \gamma_1 (r_1 g_3 g_4 g_5 g_6 + r_2 \gamma_2 g_4 g_5 g_6 + r_3 \gamma_2 \gamma_3 g_5 g_6 + r_4 \gamma_2 \gamma_3 \gamma_4 g_6 + r_5 \gamma_2 \gamma_3 \gamma_4 \gamma_5)]}$$

$$I_P^{**} = \frac{\Lambda g_3 g_4 g_5 g_6 \gamma_1 \lambda_S^{**}(\mu + \alpha_1 \lambda_S^{**})}{(\mu + \lambda_S^{**})[g_1 g_2 g_3 g_4 g_5 g_6(\mu + \alpha_1 \lambda_S^{**}) - \alpha_1 \lambda_S^{**} \gamma_1 (r_1 g_3 g_4 g_5 g_6 + r_2 \gamma_2 g_4 g_5 g_6 + r_3 \gamma_2 \gamma_3 g_5 g_6 + r_4 \gamma_2 \gamma_3 \gamma_4 g_6 + r_5 \gamma_2 \gamma_3 \gamma_4 \gamma_5)]}$$

$$I_S^{**} = \frac{\Lambda g_4 g_5 g_6 \gamma_1 \gamma_2 \lambda_S^{**}(\mu + \alpha_1 \lambda_S^{**})}{(\mu + \lambda_S^{**})[g_1 g_2 g_3 g_4 g_5 g_6(\mu + \alpha_1 \lambda_S^{**}) - \alpha_1 \lambda_S^{**} \gamma_1 (r_1 g_3 g_4 g_5 g_6 + r_2 \gamma_2 g_4 g_5 g_6 + r_3 \gamma_2 \gamma_3 g_5 g_6 + r_4 \gamma_2 \gamma_3 \gamma_4 g_6 + r_5 \gamma_2 \gamma_3 \gamma_4 \gamma_5)]}$$

$$L_{S1}^{**} = \frac{\Lambda g_5 g_6 \gamma_1 \gamma_2 \gamma_3 \lambda_S^{**}(\mu + \alpha_1 \lambda_S^{**})}{(\mu + \lambda_S^{**})[g_1 g_2 g_3 g_4 g_5 g_6(\mu + \alpha_1 \lambda_S^{**}) - \alpha_1 \lambda_S^{**} \gamma_1 (r_1 g_3 g_4 g_5 g_6 + r_2 \gamma_2 g_4 g_5 g_6 + r_3 \gamma_2 \gamma_3 g_5 g_6 + r_4 \gamma_2 \gamma_3 \gamma_4 g_6 + r_5 \gamma_2 \gamma_3 \gamma_4 \gamma_5)]}$$

$$L_{S2}^{**} = \frac{\Lambda g_6 \gamma_1 \gamma_2 \gamma_3 \gamma_4 \lambda_S^{**}(\mu + \alpha_1 \lambda_S^{**})}{(\mu + \lambda_S^{**})[g_1 g_2 g_3 g_4 g_5 g_6(\mu + \alpha_1 \lambda_S^{**}) - \alpha_1 \lambda_S^{**} \gamma_1 (r_1 g_3 g_4 g_5 g_6 + r_2 \gamma_2 g_4 g_5 g_6 + r_3 \gamma_2 \gamma_3 g_5 g_6 + r_4 \gamma_2 \gamma_3 \gamma_4 g_6 + r_5 \gamma_2 \gamma_3 \gamma_4 \gamma_5)]}$$

$$I_T^{**} = \frac{\Lambda \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \lambda_S^{**}(\mu + \alpha_1 \lambda_S^{**})}{(\mu + \lambda_S^{**})[g_1 g_2 g_3 g_4 g_5 g_6(\mu + \alpha_1 \lambda_S^{**}) - \alpha_1 \lambda_S^{**} \gamma_1 (r_1 g_3 g_4 g_5 g_6 + r_2 \gamma_2 g_4 g_5 g_6 + r_3 \gamma_2 \gamma_3 g_5 g_6 + r_4 \gamma_2 \gamma_3 \gamma_4 g_6 + r_5 \gamma_2 \gamma_3 \gamma_4 \gamma_5)]}$$

$$R_S^{**} = \frac{\Lambda \gamma_1 \lambda_S^{**}(r_1 g_3 g_4 g_5 g_6 + r_2 \gamma_2 g_4 g_5 g_6 + r_3 \gamma_2 \gamma_3 g_5 g_6 + r_4 \gamma_2 \gamma_3 \gamma_4 g_6 + r_5 \gamma_2 \gamma_3 \gamma_4 \gamma_5)}{(\mu + \lambda_S^{**})[g_1 g_2 g_3 g_4 g_5 g_6(\mu + \alpha_1 \lambda_S^{**}) - \alpha_1 \lambda_S^{**} \gamma_1 (r_1 g_3 g_4 g_5 g_6 + r_2 \gamma_2 g_4 g_5 g_6 + r_3 \gamma_2 \gamma_3 g_5 g_6 + r_4 \gamma_2 \gamma_3 \gamma_4 g_6 + r_5 \gamma_2 \gamma_3 \gamma_4 \gamma_5)]}$$

$$N^{**}$$
$$g_1g_2g_3g_4g_5g_6(\mu+\alpha_1\lambda_S^{**}) - \Lambda\gamma_1\alpha_1\lambda_S^{**}(r_1g_3g_4g_5g_6 + r_2\gamma_2g_4g_5g_6 + r_3\gamma_2\gamma_3g_5g_6 + r_4\gamma_2\gamma_3\gamma_4g_6 + r_5\gamma_2\gamma_3\gamma_4\gamma_5) + \Lambda g_2g_3g_4g_5g_6\lambda_S^{**}(\mu+\alpha_1\lambda_S^{**}) + \Lambda g_3g_4g_5g_6\gamma_1\lambda_S^{**}(\mu+\alpha_1\lambda_S^{**}) + \Lambda g_4g_5g_6\gamma_1\gamma_2\lambda_S^{**}(\mu+\alpha_1\lambda_S^{**}) + \Lambda g_5g_6\gamma_1\gamma_2\gamma_3\lambda_S^{**}(\mu+\alpha_1\lambda_S^{**}) + \Lambda g_6\gamma_1\gamma_2\gamma_3\gamma_4\lambda_S^{**}(\mu+\alpha_1\lambda_S^{**}) + \Lambda\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\lambda_S^{**}(\mu+\alpha_1\lambda_S^{**}) + \Lambda\gamma_1\lambda_S^{**}(r_1g_3g_4g_5g_6 + r_2\gamma_2g_4g_5g_6 + r_3\gamma_2\gamma_3g_5g_6 + r_4\gamma_2\gamma_3\gamma_4g_6 + r_5\gamma_2\gamma_3\gamma_4\gamma_5)$$
$$= \frac{}{(\mu+\lambda_S^{**})[g_1g_2g_3g_4g_5g_6(\mu+\alpha_1\lambda_S^{**}) - \alpha_1\lambda_S^{**}\gamma_1(r_1g_3g_4g_5g_6 + r_2\gamma_2g_4g_5g_6 + r_3\gamma_2\gamma_3g_5g_6 + r_4\gamma_2\gamma_3\gamma_4g_6 + r_5\gamma_2\gamma_3\gamma_4\gamma_5)]}$$

With $g_1 = \mu + \gamma_1,\ g_2 = \mu + \gamma_2 + r_1, g_3 = \mu + \gamma_3 + r_2,\ g_4 = \mu + \gamma_4 + r_3,\ g_5 = \mu + \gamma_5 + r_4$

$$and\ g_6 = \mu + r_5$$

On expansion, it can be shown that

$$(\mu+\lambda_S^{**})[g_1g_2g_3g_4g_5g_6(\mu+\alpha_1\lambda_S^{**}) - \alpha_1\lambda_S^{**}\gamma_1(r_1g_3g_4g_5g_6 + r_2\gamma_2g_4g_5g_6 + r_3\gamma_2\gamma_3g_5g_6 + r_4\gamma_2\gamma_3\gamma_4g_6 + r_5\gamma_2\gamma_3\gamma_4\gamma_5)] > 0$$

And

$$g_1g_2g_3g_4g_5g_6(\mu+\alpha_1\lambda_S^{**}) - \Lambda\gamma_1\alpha_1\lambda_S^{**}(r_1g_3g_4g_5g_6 + r_2\gamma_2g_4g_5g_6 + r_3\gamma_2\gamma_3g_5g_6 + r_4\gamma_2\gamma_3\gamma_4g_6 + r_5\gamma_2\gamma_3\gamma_4\gamma_5) + \Lambda g_2g_3g_4g_5g_6\lambda_S^{**}(\mu+\alpha_1\lambda_S^{**}) + \Lambda g_3g_4g_5g_6\gamma_1\lambda_S^{**}(\mu+\alpha_1\lambda_S^{**}) + \Lambda g_4g_5g_6\gamma_1\gamma_2\lambda_S^{**}(\mu+\alpha_1\lambda_S^{**}) + \Lambda g_5g_6\gamma_1\gamma_2\gamma_3\lambda_S^{**}(\mu+\alpha_1\lambda_S^{**}) + \Lambda g_6\gamma_1\gamma_2\gamma_3\gamma_4\lambda_S^{**}(\mu+\alpha_1\lambda_S^{**}) + \Lambda\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\lambda_S^{**}(\mu+\alpha_1\lambda_S^{**}) + \Lambda\gamma_1\lambda_S^{**}(r_1g_3g_4g_5g_6 + r_2\gamma_2g_4g_5g_6 + r_3\gamma_2\gamma_3g_5g_6 + r_4\gamma_2\gamma_3\gamma_4g_6 + r_5\gamma_2\gamma_3\gamma_4\gamma_5) > 0$$

$$Now, \qquad \lambda_S^{**} = \beta_S\frac{(I_P^{**} + \theta_1 I_S^{**})}{N^{**}}$$

Substituting the values of $I_P^{**},\ I_S^{**}\ and\ N^{**}$ we have

$$A_0\lambda_S^{**^2} + A_1\lambda_S^{**} + A_2 = 0$$

$$A_0 = \alpha_1(g_2g_3g_4g_5g_6 + \gamma_1g_3g_4g_5g_6 + \gamma_1\gamma_2g_4g_5g_6 + \gamma_1\gamma_2\gamma_3g_5g_6 + \gamma_1\gamma_2\gamma_3\gamma_4g_6 + \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5)$$

$$A_1 = \alpha_1g_1g_2g_3g_4g_5g_6(1-R_T) - \alpha_1r_1\gamma_1g_3g_4g_5g_6 - \alpha_1r_2\gamma_1\gamma_2g_4g_5g_6 - \alpha_1r_3\gamma_1\gamma_2\gamma_3g_5g_6 - \alpha_1r_4\gamma_1\gamma_2\gamma_3\gamma_4g_6 - \alpha_1r_5\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5 + \mu g_2g_3g_4g_5g_6 + \mu\gamma_1g_3g_4g_5g_6 + \mu\gamma_1\gamma_2g_4g_5g_6 + \mu\gamma_1\gamma_2\gamma_3g_5g_6 + \mu\gamma_1\gamma_2\gamma_3\gamma_4g_6 + \mu\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5 + r_1\gamma_1g_3g_4g_5g_6 + r_2\gamma_1\gamma_2g_4g_5g_6 + r_3\gamma_1\gamma_2\gamma_3g_5g_6 + r_4\gamma_1\gamma_2\gamma_3\gamma_4g_6 + r_5\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5$$

$$A_2 = \mu g_1g_2g_3g_4g_5g_6(1-R_T)$$

By the Descartes rule of signs, there are few cases to be considered (depending on the signs of A_1 and A_2 since A_0 is nonnegative) to study the number of positive roots of $f(\lambda^{**}) = 0$.

Case 1: If $A_1 < 0$ and $A_2 = 0$ or $A_1^2 - 4A_1A_2 = 0$, the system has exactly one endemic equilibrium.

Case 2: If $A_2 > 0$, then $R_S < 1$. $A_1 < 0, A_1^2 - 4A_1A_2 > 0$ then the system has two positive roots and so has two endemic equilibria. Hence there is an existence of multiple endemic equilibria co-existing with the DFE when $R_S < 1$; this is the basic evidence of the backward bifurcation phenomena.

2.7 Bifurcation Analysis of the Syphilis

Bifurcations are changes in the behavior or response of a dynamical system due to changes in the initial conditions or parameter values in the model. It is important to investigate the presence of these phenomena as this will go a long way to determine if the disease can be eradicated if the reproduction number is less than one.

To investigate the possibility of the existence of a backward bifurcation, at $R_T = 1$ we use the Center Manifold Theory as presented by Chavez and Song (2004)

For the system (1), let $X_1=S$, $X_2= E_S$, $X_3= I_P$, $X_4= I_S$, $X_5= L_{S1}$, $X_6= L_{S2}$, $X_7=I_T$ and $X_8=R_S$. so the system of equations can be written as

$$\dot{x}_1 = \Lambda - \beta_s \frac{(x_3 + \theta_1 x_4)}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8} x_1 - \mu x_1 = f_1$$

$$\dot{x}_2 = \beta_s \frac{(x_3 + \theta_1 x_4)}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8} x_1 - g_1 x_2 + \beta_s \frac{\alpha_1 (x_3 + \theta_1 x_4)}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8} x_8 = f_2$$

$$\dot{x}_3 = \gamma_1 x_2 - g_2 x_3 = f_3$$

$$\dot{x}_4 = \gamma_2 x_3 - g_3 x_4 = f_4$$

$$\dot{x}_5 = \gamma_3 x_4 - g_4 x_5 = f_5$$

$$\dot{x}_6 = \gamma_4 x_5 - g_5 x_6 = f_6$$

$$\dot{x}_7 = \gamma_5 x_6 - g_6 x_7 = f_7$$

$$\dot{x}_8 = r_1 x_3 + r_2 x_4 + r_3 x_5 + r_4 x_6 + r_5 x_7 - \beta_s \frac{\alpha_1 (x_3 + \theta_1 x_4)}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8} x_8 = f_8$$

The Jacobian evaluated at the disease free equilibrium for the system (1)

$$J(\xi_{(0,S)}) = \begin{pmatrix} -\mu & 0 & -\beta_S & -\theta_1\beta_S & 0 & 0 & 0 & 0 \\ 0 & -g_1 & \beta_S & \theta_1\beta_S & 0 & 0 & 0 & 0 \\ 0 & \gamma_1 & -g_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_2 & -g_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_3 & -g_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_4 & -g_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_5 & -g_6 & 0 \\ 0 & 0 & r_1 & r_2 & r_3 & r_4 & r_5 & -\mu \end{pmatrix}$$

Suppose $\beta_S = \beta_S^*$ is chosen as the bifurcation parameter at $R_T = 1$, we have that

$$\beta_S^* = \frac{g_1 g_2 g_3}{\gamma_1(g_3 + \gamma_2\theta_1)}$$

Eigenvectors

We obtain the right eigenvectors associated with the zero eigenvalue which is given as

$w = [w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8]^T$. Solving for the eigenvectors from the following equations

$$-\mu w_1 - \beta_S^* w_3 - \theta_1\beta_S^* w_4 = 0$$

$$-g_1 w_2 + \beta_S^* w_3 + \theta_1\beta_S^* w_4 = 0$$

$$\gamma_1 w_2 - g_2 w_3 = 0$$

$$\gamma_2 w_3 - g_3 w_4 = 0$$

$$\gamma_3 w_4 - g_4 w_5 = 0$$

$$\gamma_4 w_5 - g_5 w_6 = 0$$

$$\gamma_5 w_6 - g_6 w_7 = 0$$

$$r_1 w_3 + r_2 w_4 + r_3 w_5 + r_4 w_6 + r_5 w_7 - \mu w_8 = 0$$

We have that

$$w_1 = -\frac{(\beta_S^* w_3 + \theta_1 w_4)}{\mu}$$

$$w_2 = w_2 > 0$$

$$w_3 = \frac{\gamma_1 w_2}{g_2}$$

$$w_4 = \frac{\gamma_2 w_3}{g_3} = \frac{\gamma_1\gamma_2 w_2}{g_2 g_3}$$

$$w_5 = \frac{\gamma_3 w_4}{g_4} = \frac{\gamma_1\gamma_2\gamma_3 w_2}{g_2 g_3 g_4}$$

$$w_6 = \frac{\gamma_4 w_5}{g_5} = \frac{\gamma_1\gamma_2\gamma_3\gamma_4 w_2}{g_2 g_3 g_4 g_5}$$

$$w_7 = \frac{\gamma_5 w_6}{g_6} = \frac{\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5 w_2}{g_2 g_3 g_4 g_5 g_6}$$

$$w_8 = \frac{r_1 w_3 + r_2 w_4 + r_3 w_5 + r_4 w_6 + r_5 w_7}{\mu}$$

We can also obtain the left eigenvectors associated with the zero eigenvalue which is given as $v = [v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]^T$. Solving for the eigenvectors from the following equations

$$-\mu v_1 = 0$$

$$-g_1 v_2 + \gamma_1 v_3 = 0$$

$$-\beta_S^* v_1 + \beta_S^* v_2 - g_2 v_3 + \gamma_2 v_4 + r_1 v_8 = 0$$

$$-\theta_1\beta_S^* v_1 + \theta_1\beta_S^* v_2 - g_3 v_4 + \gamma_3 v_5 + r_2 v_8 = 0$$

$$-g_4 v_5 + \gamma_4 v_6 + r_3 v_8 = 0$$

$$-g_5 v_6 + \gamma_5 v_7 + r_4 v_8 = 0$$

$$-g_6 v_7 + r_5 v_8 = 0$$

$$-\mu v_8 = 0$$

We obtain the following values for the left eigenvectors

$v_1 = v_5 = v_6 = v_7 = v_8 = 0$ and $v_2 = v_2 > 0$. Also $v_3 = \left(\frac{\gamma_1\theta_1\beta_S^*}{g_2 g_3} + \frac{\beta_S^*}{g_2}\right) v_2$ *and* $v_4 = \frac{\theta_1\beta_S^* v_2}{g_3}$

Computations of a and b

The non-zero partial derivatives of the system (1) for computing a are given as

$$\frac{\partial^2 f_2}{\partial x_2 \partial x_3} = \frac{\partial^2 f_2}{\partial x_3 \partial x_2} = \frac{\partial^2 f_2}{\partial x_3 \partial x_5} = \frac{\partial^2 f_2}{\partial x_3 \partial x_6} = \frac{\partial^2 f_2}{\partial x_3 \partial x_7} = \frac{\partial^2 f_2}{\partial x_5 \partial x_3} = \frac{\partial^2 f_2}{\partial x_6 \partial x_3} = \frac{\partial^2 f_2}{\partial x_7 \partial x_3} = -\beta_S^* \frac{\mu}{\Lambda}$$

$$\frac{\partial^2 f_2}{\partial x_2 \partial x_4} = \frac{\partial^2 f_2}{\partial x_4 \partial x_2} = \frac{\partial^2 f_2}{\partial x_4 \partial x_5} = \frac{\partial^2 f_2}{\partial x_4 \partial x_6} = \frac{\partial^2 f_2}{\partial x_4 \partial x_7} = \frac{\partial^2 f_2}{\partial x_5 \partial x_4} = \frac{\partial^2 f_2}{\partial x_6 \partial x_4} = \frac{\partial^2 f_2}{\partial x_7 \partial x_4} = -\theta_1\beta_S^* \frac{\mu}{\Lambda}$$

$$\frac{\partial^2 f_2}{\partial x_3 \partial x_3} = -2\beta_S^* \frac{\mu}{\Lambda}, \qquad \frac{\partial^2 f_2}{\partial x_4 \partial x_{34}} = -2\theta_1\beta_S^* \frac{\mu}{\Lambda}$$

$$\frac{\partial^2 f_2}{\partial x_3 \partial x_8} = \frac{\partial^2 f_2}{\partial x_8 \partial x_3} = -\beta_S^* \frac{\mu}{\Lambda} + \alpha_1 \beta_S^* \frac{\mu}{\Lambda}$$

$$\frac{\partial^2 f_2}{\partial x_3 \partial x_4} = \frac{\partial^2 f_2}{\partial x_4 \partial x_3} = -\beta_S^* \frac{\mu}{\Lambda} - \theta_1 \beta_S^* \frac{\mu}{\Lambda}$$

$$\frac{\partial^2 f_2}{\partial x_4 \partial x_8} = \frac{\partial^2 f_2}{\partial x_8 \partial x_4} = -\theta_1 \beta_S^* \frac{\mu}{\Lambda} + \alpha_1 \theta_1 \beta_S^* \frac{\mu}{\Lambda}$$

The non-zero partial derivatives of the system (1) for computing b are given as

$$\frac{\partial^2 f_2}{\partial x_3 \partial \beta_S^*} = 1, \qquad \frac{\partial^2 f_2}{\partial x_4 \partial \beta_S^*} = \theta_1$$

So we have that

$$\begin{aligned} a = {} & 2v_2\alpha_1\beta_S^* \frac{\mu}{\Lambda}(w_3w_8 + \theta_1 w_4 w_8) \\ & - 2v_2\beta_S^* \frac{\mu}{\Lambda}(w_2w_3 + w_3w_5 + w_3w_6 + w_3w_7 + w_3^2 + w_3w_8 + w_3w_4) \\ & - 2v_2\theta_1\beta_S^* \frac{\mu}{\Lambda}(w_2w_4 + w_4w_5 + w_3w_4 + w_4w_8 + w_4^2 + w_4w_6 + w_4w_7) \end{aligned}$$

$$b = v_2w_3 + \theta_1 v_2 w_4$$

Since b>0, the system (1) will undergo a backward bifurcation if a > 0. This is possible if

$$\alpha_1 > \frac{(w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8)(w_3 + \theta_1 w_4)}{w_8(w_3 + w_4)}$$

It therefore follows that the model (1) does not undergo a backward bifurcation if $\alpha_1 = 0$. This shows that the loss of transitory (natural) immunity after a successful treatment is the cause of backward bifurcation in the model.

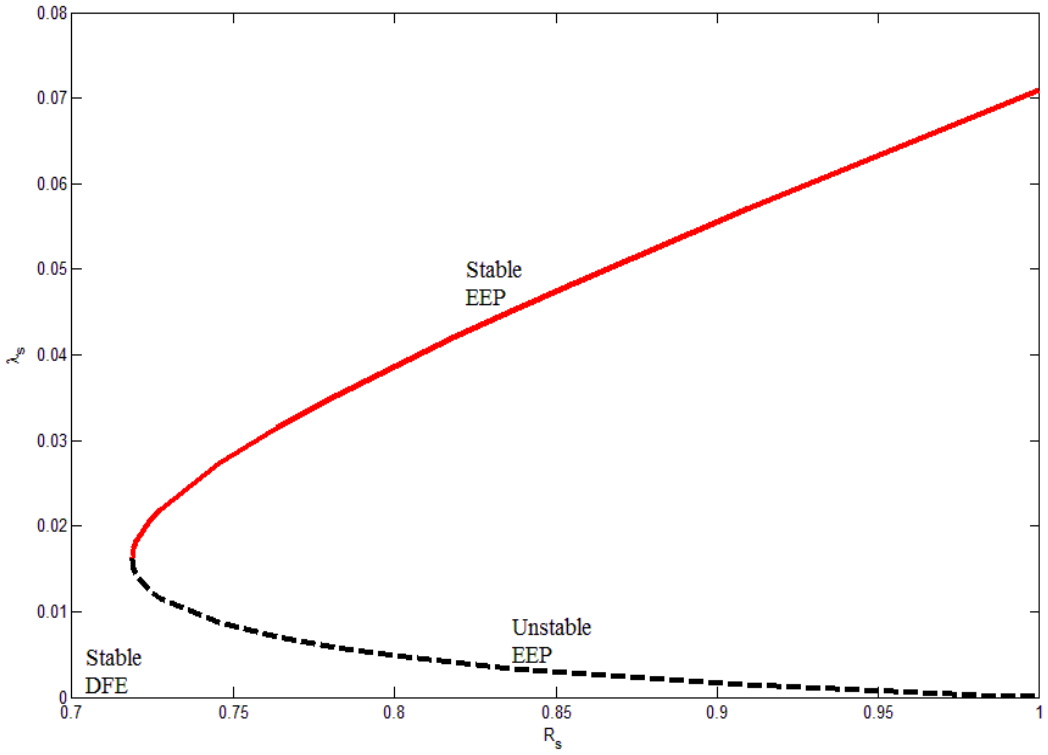


Figure 2: Backward bifurcation diagram for the model (1) showing the force of infection $\lambda_s$as a function of the control reproduction number R_S with all the parameters used as stated in Table (3) except $\alpha_1=3$, $\theta_1=1.7$ and $\beta_s=2$ so that $R_T< 1$

2.8 Global Stability of the DFE and EEP of Syphilis

Global Stability of DFE

With the cause of the backward bifurcation set to zero, we can prove the global stability of the DFE. We can use the Lyapunov function approach to provide a sufficient condition for the global stability of the DFE when $R_T \leq 1$. Considering the model (1) where there is no loss of transitory or natural immunity, we claim the following

Theorem 1: *The DFE of the system (1) with $\alpha_1= 0$ is globally asymptotically stable in D_2 whenever $R_T< 1$*

Proof: Consider the Lyapunov function

$$V = \gamma_1(g_3 + \gamma_2\theta_1)E_S + g_1(g_3 + \gamma_2\theta_1)I_P + g_1g_2\theta_1I_S$$

Clearly, $V>0$ except at the DFE. Differentiating V with respect to time, we have

$$\dot{V} = (I_P + \theta_1 I_S)\left(\frac{\beta_S S}{N}[\gamma_1(g_3 + \gamma_2\theta_1)] - (g_1 g_2 g_3)\right)$$

$$\dot{V} = g_1 g_2 g_3 (I_P + \theta_1 I_S)\left(\frac{S}{N}R_T - 1\right)$$

On D_2 , $S \leq N \leq \frac{\Lambda}{\mu}$ $hence, \frac{S}{N} \leq 1$. So we have

$$\dot{V} \leq g_1 g_2 g_3 (I_P + \theta_1 I_S)(R_T - 1)$$

With the equality only at the DFE. For $R_T \leq 1$, we have that $\dot{V} \leq 0$ with that equality sign only limit set of each solution is contained in the largest invariant set for which $I_P = I_S = 0$, which is the singleton DFE. Therefore, V is a Lyapunov function in D_2 and it follows from the LaSalle's Invariance Principle that every solution to the equation in (1) with $\alpha_1 = 0$ with initial conditions in D_2 converges to $\xi_{(0,S)}$ as $t \to \infty$

This means that

$$(E_S(t), I_P(t), I_S(t), L_{S1}(t), L_{S2}(t), I_T(t), R_S(t)) \to (0,0,0,0,0,0,0)\ as\ t \to \infty$$

Substituting $E_S = I_P = I_S = L_{S1} = L_{S2} = I_T = R_S = 0$ into the system (3.3) gives $S(t) \to \frac{\Lambda}{\mu}$ $as\ t \to \infty$

So that$(S, E_S, I_P, I_S, L_{S1}, L_{S2}, I_T, R_S) \to \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0, 0, 0, 0\right)$ $as\ t \to \infty\ for\ R_T \leq 1$, and so the DFE $\xi_{(0,S)}$ is Gas in D_2 for $\alpha_1 = 0$

Thee epidemiological significance of this is that in the absence of the loss of natural or transitory immunity, syphilis can be eliminated from the population if $R_T < 1$.

Since the global stability of $\xi_{(0,S)}$ follows if $R_T < 1$we can find the range of values for which effective treatment rates r_1 and r_2 can lead to $R_T < 1$. By setting $R_T < 1$ and making r_1 and r_2 the subject formula we have

$$r_1 > \frac{\beta_S \gamma_1(\mu + r_2 + \gamma_3 + \gamma_2\theta_1)}{(\mu + \gamma_1)(\mu + \gamma_3 + r_2)} - (\mu + \gamma_2)$$

So if the above inequality holds, $R_T < 1$ holds thus syphilis will be eventually eradicated. However, if $\frac{\beta_S \gamma_1(\mu + r_2 + \gamma_3 + \gamma_2\theta_1)}{(\mu + \gamma_1)(\mu + \gamma_3 + r_2)} < (\mu + \gamma_2)$ then treatment of individuals in I_P is not necessary as $r_1 = 0$ results in $R_T < 1$.

$$r_2 < \frac{(\mu + \gamma_1)(\mu + r_1 + \gamma_2)(\mu + \gamma_3) - \beta_S \gamma_1(\mu + \gamma_3 + \gamma_2\theta_1)}{\beta_S \gamma_1 - (\mu + \gamma_1)(\mu + r_1 + \gamma_2)}$$

Also, if the inequality above holds for r_2 syphilis will be eventually eradicated

Global Stability of EEP of Syphilis

Consider a special case of model (1) where $\alpha_1=0$. Let

$$D_0 = \{(S, E_S, I_P, I_S, L_{S1}, L_{S2}, I_T, R_S) \in D_2 : E_S = I_P = I_S = L_{S1} = L_{S2} = I_T = R_S = 0\}$$

Be the stable manifold of the DFE (ξ') we claim the following

Theorem 2 *The unique endemic equilibrium $\xi_{(1,S)}$ of the model (1) with $\alpha_1=0$ is GAS in $D_2 \backslash D_0$ whenever $R_T > 1$*

Proof: Consider the model (1) with $\alpha_1=0$ and $R_T > 1$, so that the associated unique endemic equilibrium exists. Also, let the non-linear Lyapunov function (of the Goh-Volterra type) be

$$F = S - S^{**} - S^{**}\ln\frac{S}{S^{**}} + (E_S - E_S^{**} - E_S^{**}\ln\frac{E_S}{E_S^{**}}) + \frac{\beta_S S^{**}}{(\mu + r_1 + \gamma_2)}(1 + \frac{\gamma_2\theta_1}{(\mu + r_2 + \gamma_3)})(I_P - I_P^{**} - I_P^{**}\ln\frac{I_P}{I_P^{**}}) + \frac{\theta_1\beta_S S^{**}}{(\mu + r_2 + \gamma_3)}(I_S - I_S^{**} - I_S^{**}\ln\frac{I_S}{I_S^{**}})$$

$$\dot{F} = \dot{S} - \frac{S^{**}}{S}\dot{S} + \dot{E}_S - \frac{E_S^{**}}{E_S}\dot{E}_S + \frac{\beta_S S^{**}}{(\mu + r_1 + \gamma_2)}\left(1 + \frac{\gamma_2\theta_1}{(\mu + r_2 + \gamma_3)}\right)\left(\dot{I}_P - \frac{I_P^{**}}{I_P}\dot{I}_P\right) + \frac{\theta_1\beta_S S^{**}}{(\mu + r_2 + \gamma_3)}(\dot{I}_S - \frac{I_S^{**}}{I_S}\dot{I}_S)$$

With $N = \frac{\Lambda}{\mu}, \lambda_S = \beta_S\frac{(I_P+\theta_1 I_S)}{N}$ *becomes* $\tilde{\lambda}_S = \beta_S\frac{\Lambda}{\mu}(I_P + \theta_1 I_S) = \tilde{\beta}_S(I_P + \theta_1 I_S)$ *where*

$\tilde{\beta}_S = \beta_S\frac{\Lambda}{\mu}$ We have

$$\dot{F} = \Lambda - \tilde{\beta}_S(I_P + \theta_1 I_S)S - \mu S - \frac{S^{**}}{S}\left(\Lambda - \tilde{\beta}_S(I_P + \theta_1 I_S)S - \mu S\right) + \tilde{\beta}_S(I_P + \theta_1 I_S)S - (\gamma_1 + \mu)E_S - \frac{E_S^{**}}{E_S}\left(\tilde{\beta}_S(I_P + \theta_1 I_S)S - (\gamma_1 + \mu)E_S\right) + \frac{\beta_S S^{**}}{(\mu + r_1 + \gamma_2)}\left(1 + \frac{\gamma_2\theta_1}{(\mu + r_2 + \gamma_3)}\right)\left(\gamma_1 E_S - (\gamma_2 + r_1 + \mu)I_P - \frac{I_P^{**}}{I_P}(\gamma_1 E_S - (\gamma_2 + r_1 + \mu)I_P)\right) + \frac{\theta_1\beta_S S^{**}}{(\mu + r_2 + \gamma_3)}\left(\gamma_2 I_P - (\gamma_3 + r_2 + \mu)I_S - \frac{I_S^{**}}{I_S}(\gamma_2 I_P - (\gamma_3 + r_2 + \mu)I_S)\right)$$

It can be shown from (1) that at steady state,

$$\Lambda = \tilde{\beta}_S(I_P^{**} + \theta_1 I_S^{**})S^{**} + \mu S^{**},\ \gamma_1 + \mu = \frac{\tilde{\beta}_S(I_P^{**} + \theta_1 I_S^{**})S^{**}}{E_S^{**}}, \gamma_2 + r_1 + \mu = \frac{\gamma_1 E_S^{**}}{I_P^{**}},$$
$$\mu + r_2 + \gamma_3 = \frac{\gamma_2 I_P^{**}}{I_S^{**}}$$

Using the above relations, we obtain

$$\dot{F} = \mu S^{**}\left(2 - \frac{S^{**}}{S} - \frac{S}{S^{**}}\right) + \tilde{\beta}_S S^{**} I_P^{**}\left(3 - \frac{S^{**}}{S} - \frac{I_P^{**} E_S}{I_P E_S^{**}} - \frac{E_S^{**} I_P S}{S^{**} I_P^{**} E_S}\right)$$
$$+ \tilde{\beta}_S S^{**} I_S^{**}\theta_1\left(4 - \frac{S^{**}}{S} - \frac{E_S^{**} I_S S}{S^{**} I_S^{**} E_S} - \frac{I_P^{**} E_S}{I_P E_S^{**}} - \frac{I_S^{**} I_P}{I_P^{**} I_S}\right)$$

Since arithmetic mean is greater than geometric mean, we have that

$$2 - \frac{S^{**}}{S} - \frac{S}{S^{**}} \leq 0,$$
$$3 - \frac{S^{**}}{S} - \frac{I_P^{**} E_S}{I_P E_S^{**}} - \frac{E_S^{**} I_P S}{S^{**} I_P^{**} E_S} \leq 0 \ \text{and}\ 4 - \frac{S^{**}}{S} - \frac{E_S^{**} I_S S}{S^{**} I_S^{**} E_S} - \frac{I_P^{**} E_S}{I_P E_S^{**}} - \frac{I_S^{**} I_P}{I_P^{**} I_S} \leq 0$$

Thus, we have that $\dot{F} \leq 0\ for\ R_T > 1$. Since the relevant variables in the equations for L_{S1}, L_{S2}, I_T and R_S are at the endemic steady state, it follows that these can be substituted into the equations for L_{S1}, L_{S2}, I_T and R_S so that

$$\left(L_{S1}(t), L_{S2}(t), I_T(t), R_S(t)\right) \to (L_{S1}^{**}, L_{S2}^{**}, I_T^{**}, R_S^{**})\ as\ t \to \infty.$$

Hence, F is a Lyapunov function in $D_2 \backslash D_0$

3 Numerical Simulations

The model (1) is now simulated using the parameter estimates in Table 3

(unless otherwise stated); to assess the potential impact of treatment strategies for syphilis. These simulations were carried out using the ODE45 Solver in MATLAB.

Figures 3 show the effect of the treatments rates $r_1$and $r_2$on the cumulative incidence of infected individuals who transmit syphilis. With

$$r_1 = r_2 = 0\ \ for\ low\ effectiveness\ level$$

$$r_1 = r_2 = 5\ \ for\ average\ effectiveness\ level$$

$$r_1 = r_2 = 10\ \ for\ high\ \ effectiveness\ level$$

It is seen that for 3 the average effectiveness level was effective for the first 10years and 15 years respectively. Simulation results therefore show that increasing treatment rates both for primary and secondary stages of syphilis will have a positive impact as this is seen to help reduce the burden of the diseases in the community.

Table 3: Parameter Values for model (1)

Parameter	Nominal Value	Reference
μ	0.02043	[14]
Λ	5000	[14]
β_S	(6,10)	[11]
r_1	3.422	[1]
r_2	1.469	[9]
r_3	0.036	[9]
r_4	0.036	[9]
r_5	0.021	[9]
γ_1	6	[9]
γ_2	4.18	[9]
γ_3	2.5	[9]
γ_4	3	[9]
γ_5	0.033	[9]
α_1	0.6	[9]

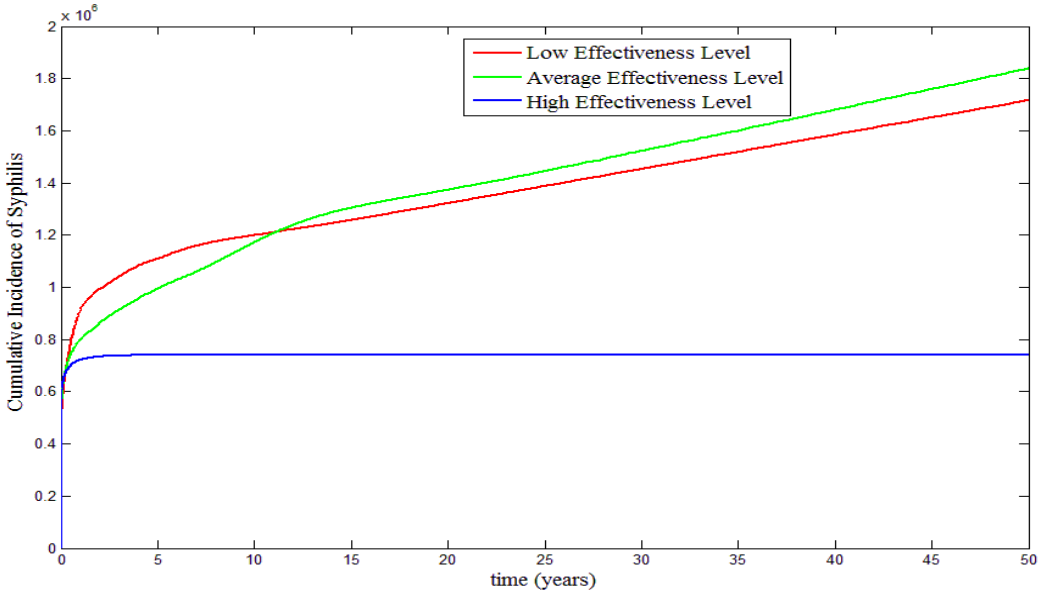


Figure 3: Simulations of the model (1), showing the cumulative incidence of syphilis as a function of time,

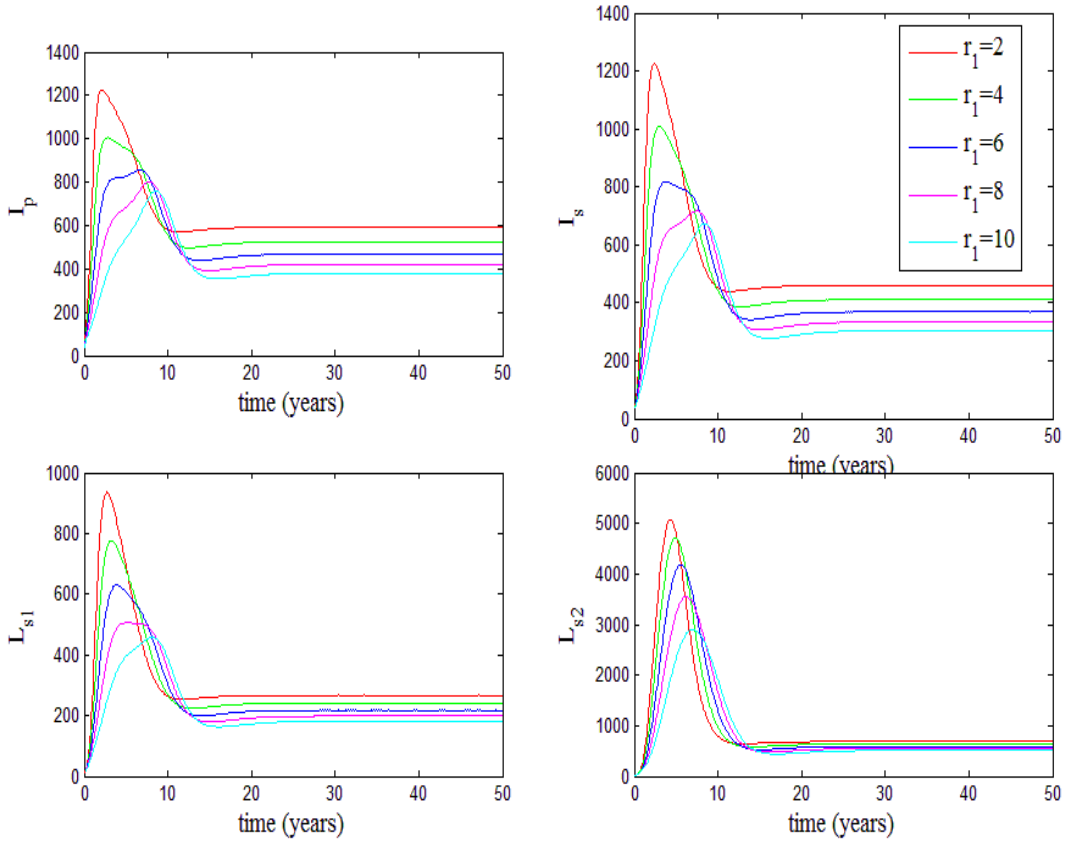


Figure 4: Simulation of model (1) showing the infected classes as a function of time with varying values of the treatment rate r_1.

In Figure 4, we see the proportion of individuals in the various classes reduce as we vary treatment rate r_1. However, this reduction is not very significant in the L_{S2} class.

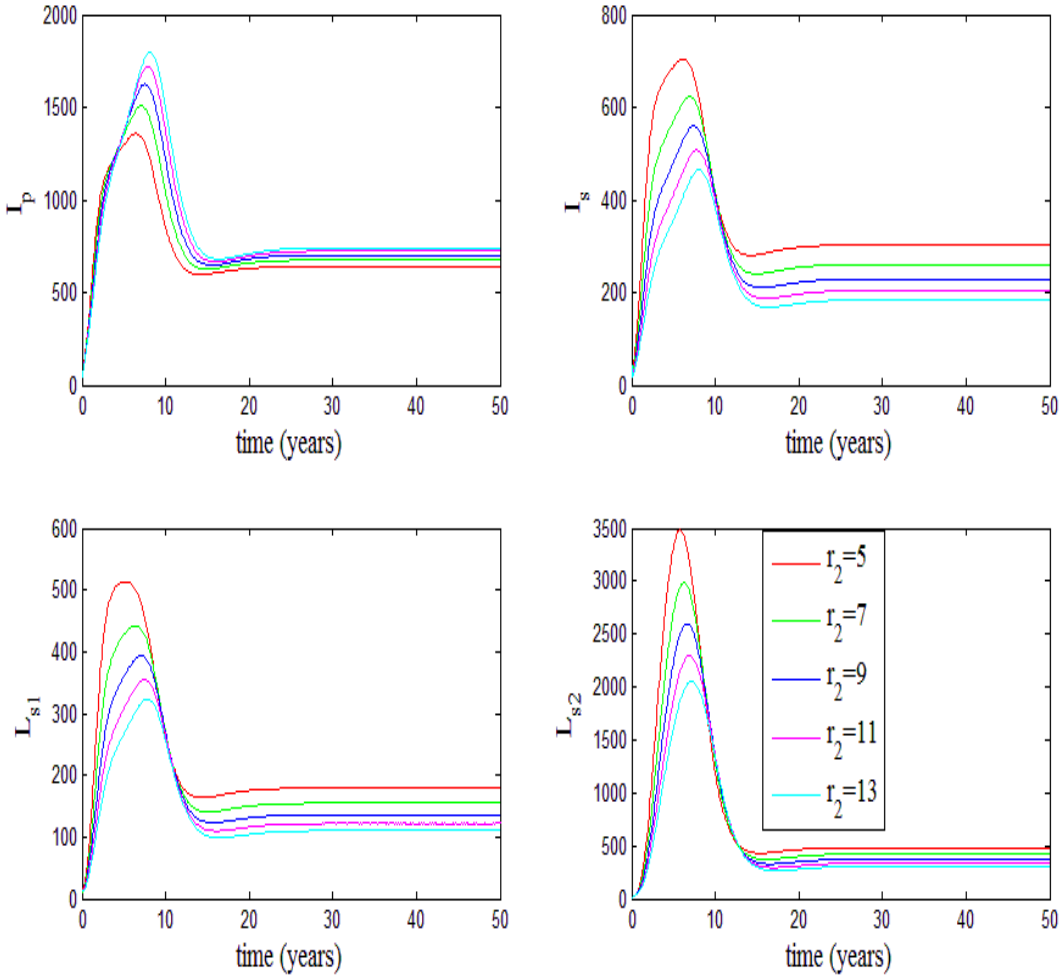


Figure 5: Simulation of model (1) showing the infected classes as a function of time with varying values of the treatment rate r_2

The treatment rate r_2 has more positive impact on the proportion of individuals in the I_S and L_{S1} classes than in the I_P and L_{S2} classes as seen in Figure 5

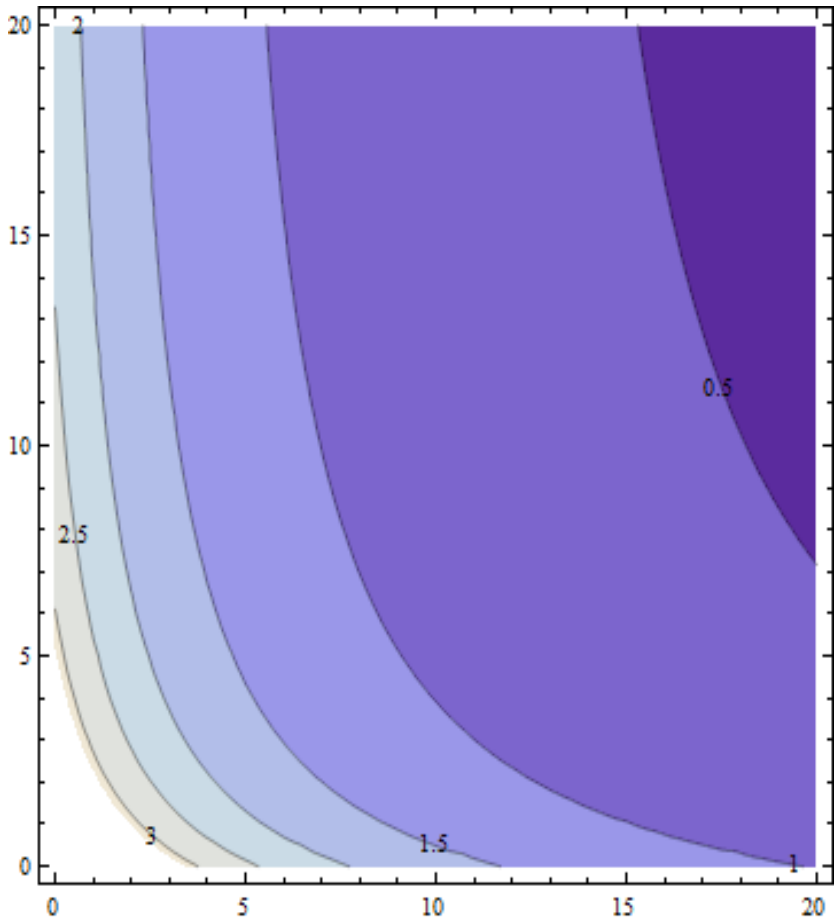


Figure 6: A contour plot of R_T as a function of r_1 and r_2. The horizontal axis is r_1 while the vertical axis is r_2.

As seen from Figure 6, increasing the values of r_1 and r_2 results in the decrease of R_T. Its value decreased from $R_T = 3$ to 0.5. Since $RT < 1$, we can see that syphilis can be eradicated from the population.

4.1 Summary

A mathematical model for the transmission dynamics of Syphilis in a human population is designed and used to assess the impact of control strategies on co-infection cases. The findings are summarized as follows:

- The Disease free equilibrium and Endemic equilibrium point of the Syphillis model (1) was seen to be both locally and globally asymptotically stable.
- The syphilis model (1) undergoes a backward bifurcation when the transitory immunity is nonzero.

4.2 Conclusion

The world is concerned about the rising prevalence of Syphillis which causes a lot of

inconveniences in different communities. Almost all countries have increasingly recognized the need to find effective prevention and control strategies for the disease and this is especially true in developing countries where treatment of Syphillis is almost always not available. This research work informs how the use of treatment strategies of syphilis in its primary and secondary infected individuals will help reduce the cases.

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