On the Stability of Shear Flows

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Abstract
Actual fluid behavior depends on the flow instability. In this paper we tried to study and analyze the stability of shear flows, because only stable flows will be realized in nature or in an experiments and unstable flow is likely not to be observable in practice. Two matching conditions of hydrodynamics was derived through the Rayleigh's equation for stability and a dispersion relation was also obtained through this matching condition. It was later observed that the shear flow is unstable to all wavenumber between zero and the critical point, i.e. $0 < \alpha < \alpha_c = 0.63923$ and thereafter become stable after this critical point.

Keywords: *Shear flows, Rayleigh's equation, hydrodynamics, Inviscid stability theory, Newton-Raphson's method*

1. Introduction

The motion of fluid is naturally defined as either laminar or turbulent and stability is absolutely concerned with small perturbations to equilibrium solutions. Most of the observed fluid flow in nature or in an experiment must not only satisfy the equations of motion but most also be stable. To study the stability of this flow, we need the concept of hydrodynamics stability. Hydrodynamic stability theory is concerned with the response of a laminar flow to a disturbance of small or moderate amplitude, so that if the flow returns to its original laminar state, one defines the flow as stable and if the disturbance grows and cause the laminar flow to change into another different state, one defines the flow as unstable. However, the instabilities of a fluid flows has develop a lot of interest by many researchers as it was analyzed in [1-7]. There are different types of hydrodynamic stabilities, such as shear flow instability which arises due to velocity shear, viscous instability which occur due to viscosity, centrifugal instability which arises due to convection and so on. The most important out of these instabilities is the shear flow instability and it has a lot of importance in geophysical and astrophysical fluid dynamics [8-10]. The essence of shear flow instability was studied physically and mathematically by [12]. He extends the instability theory of a sheet vortex from the viewpoint of vortex dynamics.
Generalize Tollmien's solution of the Rayleigh problem of hydrodynamic stability to the case of arbitrary channel cross sections, known as the extended Rayleigh problem was also observed by [13].
In this paper work we will studied some of these instabilities through Rayleigh's equation for stability

2. Inviscid stability theory

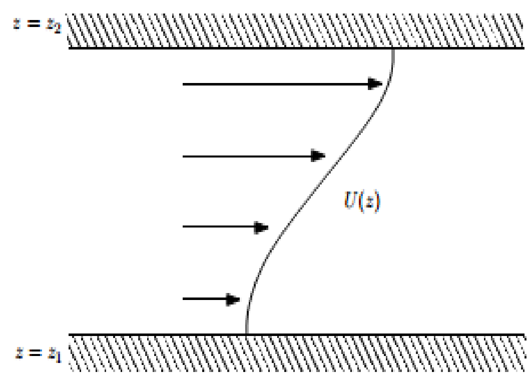

Fig. 1: Inviscid Shear Flow

If we consider z −dependence unidirectional basic state flow $\boldsymbol{U} = U(z)\hat{\boldsymbol{x}}$ between two planes z_1 and z_2 which is parallel along x-directionand vary along z-direction with constant density and uniform pressure p.If we also assumed that the flow is an incompressible and inviscid, then the motion of the fluid is governed by non-dimensional Euler equation (no buoyance) [6, 9,10].

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p, \qquad (1)$$

And the continuity equation

$$\nabla \cdot \mathbf{u} = 0, \qquad (2)$$

Where $\mathbf{u} = (u, v, w)$. clearly this governing equations satisfies the basic state, since

$\mathbf{U}.\nabla\mathbf{U} = 0$ and $\nabla \cdot \mathbf{U} = 0.$

3. Governing equations

Linear stability analysis involves perturbing the basic state (i.e. laminar flow for which its stability or instability is to be examined). So if we now perturb the basic state and the pressure such that

$$\mathbf{u}(x,t) = U(z)\hat{\mathbf{x}} + \tilde{u}(x,t), \qquad (3)$$
$$p(x,t) = p_* + \tilde{p}(x,t), \qquad (4)$$

Then by substituting eq. (3) and eq. (4) into eq. (1) – (2) and retains only the linear terms, we obtain an equation for disturbances quantities

$$\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \tilde{w}\frac{\partial U}{\partial z}\hat{\mathbf{x}} = -\nabla\tilde{p}. \qquad (5)$$

Eq. (5) is a governing equations for three-dimensional disturbance quantities and is also called the *perturbation equation*. We can therefore enforce impermeable boundary conditions on the solid boundaries i.e.

$$\tilde{w}(z) = 0 \text{ on } z = z_1 \text{ and } z = z_2. \qquad (6)$$

3.1 Normal Mode Solution

Since the basic state is a function of z, then we can assume normal mode solution that exponentially depends on x, y and t which is of the form,

$$\tilde{u}(x,t) = Re[\hat{u}(z)e^{i(\alpha x+\beta y-\alpha ct)}], \qquad (7)$$

$$\tilde{p}(x,t) = Re[\hat{p}e^{i(\alpha x+\beta y-\alpha ct)}], \qquad (8)$$

Where $\hat{u}(z)$ is the complex amplitude, α and β are both real wave numbers, $c = c_r + ic_i$ is the complex wave speed. This wave is therefore stable whenever $\alpha c_i < 0$, unstable if $\alpha c_i > 0$ and neutrally stable if $\alpha c_i = 0$ [15, 16]. Now substituting eq. (7) and eq. (8) into eq. (5) to obtain,

$$i\alpha(U-c)\hat{u} + \frac{dU}{dz}\hat{w} = -i\alpha\hat{p}, \qquad (9)$$

$$i\alpha(U-c)\hat{v} = -i\beta\hat{p}, \qquad (10)$$

$$i\alpha(U-c)\hat{w} = -\frac{d\hat{p}}{dz}, \qquad (11)$$

$$i\alpha\hat{u} + i\beta\hat{v} + \frac{d\hat{w}}{dz} = 0, \qquad (12)$$

With an impermeable boundary, i.e. conditions (6). Eq. (9) – eq. (12) yield an eigenvalue problem of three dimensional disturbances which can be used to obtain an Eigen function $\hat{u}$ and $\hat{p}$ for particular value c with given $U, z_1, z_2, \alpha, \beta$. the flow is therefore linearly stable if $\alpha c_i \leq 0$ for all wave number α, β (both real) and unstable whenever $\alpha c_i > 0$ for at least one value of the wave numbers.

4. Rayleigh's Equation for Stability

Consider a unidirectional incompressible fluid flow with a fluid velocity

$$\mathbf{u}(x,z,t) = \big(u(x,z,t), 0, w(x,z,t)\big), \qquad (13)$$

Since the flow is incompressible, then $\nabla \cdot \mathbf{u} = 0$ and therefore we can introduce a fluid velocity for this flow in terms of a stream function $\varphi(x,z,t)$ for the perturbation by

$$\tilde{u} = \frac{\partial\varphi}{\partial z}, \qquad (14)$$

$$\tilde{w} = -\frac{\partial\varphi}{\partial x}, \qquad (15)$$

Let us also assume normal mode solution for this stream function as

$$\varphi(x,z,t) = \phi(z)e^{i(\alpha x-\alpha ct)}, \qquad (16)$$

Where $\phi(z)$ stands for a function of complex amplitude. Since three-dimensional disturbances can be transformed into two-dimensional disturbances through Squire's transformation, then we can solve an eigenvalue problem for the two-dimensional disturbance. Substituting eq. (16) into eq. (14) and eq. (15) we have,

$$\tilde{u} = \frac{\partial \phi}{\partial z} e^{i(\alpha x - \alpha c t)}, \qquad (17)$$

$$\tilde{W} = -i\alpha\phi e^{i(\alpha x - \alpha c t)}. \qquad (18)$$

If we again substitute eq. (17) and eq. (18) into eq. (7), taking $\beta = \theta = 0$,dropping the common exponential terms, taking the derivatives of the new equations and simplify it to get the following sets of equations

$$\frac{d\beta}{dz} = \frac{d^2 U}{dz^2}\phi - (U - c)\frac{d^2\phi}{dz^2}, \qquad (19)$$

$$\frac{d\beta}{dz} = -\alpha^2\phi(U - c). \qquad (20)$$

Equating and simplifying eq. (19) and eq. (20) to *Rayleigh's equation for stability*,

$$(U - c)(\phi'' - \alpha^2\phi) - U''\phi = 0, \qquad (21)$$

And the impermeable boundary conditions (6) becomes

$$\phi(z_1) = \phi(z_2) = 0. \qquad (22)$$

If an Eigen function $\phi(z)$ and eigenvalue $c = c_r + ic_i$ are Eigen solution of the Rayleigh's equation for a given real wave number α, so also its complex conjugate ϕ^* and c^* with the same number α. Without loss of generality we could take $\alpha \geq 0$, since the system is term reversibility and the condition for instability is that $c_i > 0$, therefore to each and every growing disturbances there is an equivalent decaying disturbances, because to each positive c_i there exist a negative c_i. So, for a flow to be stable all disturbances must be neutrally stable for a real eigenvalue c (i.e. c_i must be zero) as it was elaborated in [15]. From Rayleigh's equation we can deduce two or three important results without specifying the velocity profile, i.e. Rayleigh's inflection point criterion and Fjortoft's theorem.

5.0 Matching/Jump Conditions

If $U(z)$ is considered to be a simple linear function, then the solution of Rayleigh's equation for stability (21) will either be a hyperbolic function or an exponential function that needs to satisfy some jump or matching conditions at the interface where $U(z)$ or $U'(z)$ is discontinuous [14].

The matching conditions for hydrodynamics can be derived by re-writing Rayleigh's equation (21) as

$$((U - c)\phi'' - U''\phi) = \alpha^2(U - c)\phi, \qquad (23)$$

Which is equivalent to,

$$\frac{d}{dz}((U - c)\frac{d\phi}{dz} - U'\phi = \alpha^2(U - c)\phi, \qquad (24)$$

Integrating eq. (24) over the edge from $z = z_s^{-\varepsilon}$ to $z = z_s^{+\varepsilon}$ gives,

$$[(U - c)\frac{d\phi}{dz} - U'\phi]_{z_s^-}^{z_s^+} = \alpha^2 \lim_{\varepsilon \to 0}\left(\int_{z_s - \varepsilon}^{z_s + \varepsilon}(U - c)\phi dz\right) \qquad (25)$$

If we, therefore assume that $\varepsilon \to 0$ leads to an important equation called *first matching/jump condition*

$$[(U - c) - U'\phi] = 0. \qquad (26)$$

This jump condition implies the continuity of pressure at the interface. For the *second matching/jump condition,* we have

$$F = (U - c)\phi' - U'\phi. \qquad (27)$$

If we also divide eq. (27) by $(U - c)^2$ and integrate the new equation over the edge from $z = z_s^{-\varepsilon}$ to $z = z_s^{+\varepsilon}$ with the assumption that $\varepsilon \to 0$, gives the *second matching/jump condition*

$$[\frac{\phi}{U - c}]_{z_s^{-\varepsilon}}^{z_s^{+\varepsilon}} = (\int_{z_s^{-\varepsilon}}^{z_s^{+\varepsilon}} \frac{F}{(U - c)^2} dz) \to 0. \qquad (28)$$

Eq. (28) is equivalent to material interface condition provided that $U(z) \neq 0$. The general solution for Rayleigh's equation (21) for a piecewise velocity profile gives,

$$\phi(z) - Ae^{-\alpha z} + Be^{\alpha z}, \qquad (29)$$

Where A and B are arbitrary constants. This eq. (29) can be solved for any piecewise linear problem with the aid of the matching conditions (27) and (28).

6.0 Velocity Shear Flow

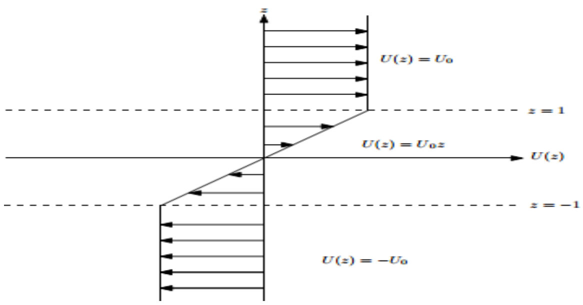

Fig. 2: Shear layer

If we now consider a shear flow profile

$$U(z) = \begin{cases} U_0, & z \geq 1 \\ U_0 z, & |z| < 1 \\ -U_0, & z \leq -1 \end{cases} \qquad (30)$$

If the upper region (i.e. $z \geq 1$) is designated as region I, the middle region (i.e. $|z| < 1$) as region II and the lower region (i.e. $z \leq -1$) as region III. In each of this region $U''(z) = 0$ and therefore the solution of the Rayleigh's equation (21) satisfying an unbounded boundary condition $\phi(z) \to 0$ as $z \to \pm\infty$, gives

$$\phi(z) = \begin{cases} Pe^{-\alpha z}, & z \geq 1 \\ Qe^{-\alpha z} + Re^{\alpha z}, & |z| < 1 \\ Se^{\alpha z}, & z \leq -1 \end{cases} \qquad (31)$$

Applying matching condition (26) to eq. (31) at $z = 1$ and $z = -1$, gives

$$\alpha(U_0 - c)P - (\alpha(U_0 - c) + U_0)Q + (\alpha(U_0 - c) - U_0)e^{2\alpha}R = 0 \qquad (32)$$

and

$$(\alpha(U_0 + c) - U_0)e^{2\alpha}Q - (\alpha(U_0 + c) + U_0)R + (\alpha(U_0 + c)S) = 0, \qquad (33)$$

respectively. Also applying matching condition (28) to eq. (31) at $z = 1$ and $z = -1$, gives

$$P - Q - Re^{2\alpha} = 0, \qquad (34)$$

and

$$Qe^{2\alpha} + R - S = 0, \qquad (35)$$

Respectively [14].

We have now obtained four different equations with four different unknowns P, Q, R and S. so to find the non-trivial solution, the determinants of the equations of these coefficients must be equal to zero. i.e.

$$\begin{vmatrix} 1 & -1 & -e^{2\alpha} & 0 \\ \alpha(U_0 - c) & -(\alpha(U_0 - c) + U_0) & (\alpha(U_0 - c) - U_0)e^{2\alpha} & 0 \\ 0 & e^{2\alpha} & 1 & -1 \\ 0 & (\alpha(U_0 + c) - U_0)e^{2\alpha} & -(\alpha(U_0 + c) + U_0) & \alpha(U_0 + c) \end{vmatrix} = 0. \qquad (36)$$

After rigorous simplification of eq. (36), we have

$$U_0^2 - (2\alpha(U_o + c) - U_o)[(\alpha(U_o - c) - U_o) + \alpha(U_o - c)]e^{4\alpha} \qquad (37)$$

Simplifying eq. (37) again yields the dispersion relation

$$c^2 = \frac{U_0^2}{4\alpha^2}[(1 - 2\alpha)^2 - e^{-4\alpha}], \qquad (38)$$

For the shear flow profile. So if the quantity on the right hand side of eq. (38) is less than zero, then the eigenvalue c is purely imaginary and therefore the shear flow is unstable but if it is greater than zero, then the eigenvalue c is purely real and therefore the system is said to be stable.

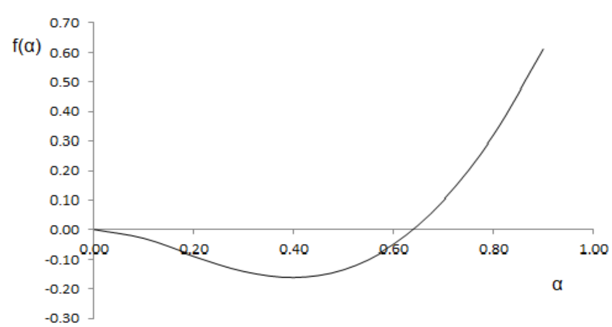

Fig. 3: Graph of $f(\alpha)$ versus α showing the critical point α_c of a shear layer

7. Conclusion

We derived Rayleigh's equation for stability and it was also used to develop two matching conditions for piecewise linear velocity profile, i.e. the dynamic boundary condition which corresponds to the continuity of pressure at the interface and the kinematic boundary condition which corresponds to the material interface condition. These matching condition were used to derive the dispersion relation of shear layer flow and studied its stability. The most important features of the result obtained from this dispersion relation is the existence of a neutral point also known as critical point, which occurred at $\alpha_c = 0.63923.$ This critical point is the point of transition between stable and unstable mode and it was obtained numerically through Newton-Raphson's method. So the shear layer flow is unstable to all wavenumber between zero and this critical point, i.e. $0 < \alpha < \alpha_c = 0.63923$ and later become stable after this critical point as shown in fig. 3. As discussed in [1, 3 and 12].

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