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SOME RESULTS ON GENERALIZED (k,μ) SPACE-FORMS

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Abstract: *In this paper we study η-recurrent and ϕ-recurrent generalized k,μ)- space forms. We prove that generalized (k,μ)-space-form has η-parallel Ricci tensor then it is constant. Also we study in a ϕ-recurrent generalized (k,μ)-space-form, characteristic vector field and the vector associated with 1-form are co-directional.*

Keywords: *generalized (k,μ) space forms, η-recurrent,. η-parallel , ϕ-recurrent.*

1. INTRODUCTION

A generalized Sasakian space form was defined by Carriazo et al. in [1], as an almost contact metric manifold (M,ϕ,ξ,η,g) whose curvature tensor R is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3, \tag{1.1}$$

where f_1, f_2, f_3 are some differentiable functions on M and

$$\begin{aligned} R_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y \\ R_2(X,Y)Z &= g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z \\ R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \end{aligned} \tag{1.2}$$

for any vector fields X,Y,Z on M . In [2] , the authors defined a generalized (k,μ) space form as an almost contact metric manifold (M,ϕ,ξ,η,g) whose curvature tensor can be written as

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \tag{1.3}$$

where $f_1, f_2, f_3, f_4, f_5, f_6$ are differentiable functions on M and R_1,R_2,R_3 are tensors defined above and

$$\begin{aligned} R_4(X,Y)Z &= g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y \\ R_5(X,Y)Z &= g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX \\ R_6(X,Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi, \end{aligned}$$

for any vector fields $X,Y,Z,$ where $2h = L_\xi\phi$ and L is the usual Lie derivative.This manifold was denoted by $M(f_1, f_2, f_3, f_4, f_5, f_6)$.

Natural examples of generalized (k,μ) space forms are (k,μ) space forms and generalized Sasakian space forms. The authors in[1] proved that contact metric generalized (k,μ) space forms are generalized (k,μ) spaces and if dimension is greater than or equal to 5, then they are (k,μ) spaces with constant $\phi-$ sectional curvature $2f_6 - 1$. They gave a method of constructing examples of generalized (k,μ) space forms and proved that generalized (k,μ) space forms with trans-Sasakian structure reduces to generalized Sasakian space forms. Further in [2], it is proved that under $D_a -$homothetic deformation generalized (k,μ) space form structure is preserved for dimension 3, but not in general. Another interesting and important class of manifolds is a class of manifolds of constant curvature. T.Takahashi [6] introduced the notion of locally ϕ-symmetry on a Sasakian manifold. Generalizing the notion of ϕ-symmetry , De and co-authors [7] introduced the notion of ϕ-recurrent Sasakian manifold. We use to study Ricci tensor of the space-form and characterize such space-forms to have η−recurrent and η-parallel Ricci tensor. Motivated by the above studies, In this paper we study η-recurrent and ϕ-recurrent generalized (k,μ)- space forms. We prove that in an η-recurrent generalized (k,μ)-space-form the 1-form A is closed and the generalized (k,μ)-space-form has η-parallel Ricci tensor then it is constant. Also we study in a ϕ-recurrent generalized (k,μ)-space-form, characteristic vector field and the vector associated with 1-form are co-directional.

2. PRELIMINARIES

A (2n+1)-dimensional Riemannian manifold (M,g) is said to be an almost contact metric manifold if it admits a tensor field ϕ of type (1,1), a vector field ξ, and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \phi\xi = 0, \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), g(X, \phi X) = 0, g(X, \xi) = \eta(X). \tag{2.3}$$

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$,

where $\Phi(X,Y) = g(X, \phi Y)$ is the fundamental 2-form of M.

It is well known that on a contact metric manifold (M, ϕ, ξ, η, g), the tensor h is defined by $2h = L_\xi \phi$ which is symmetric and satisfies the following relations.

$$h\xi = 0, h\phi = -\phi h, trh = 0, \eta \circ h = 0, \tag{2.4}$$

$$\nabla_X \xi = -\phi X - \phi h X, (\nabla_X \eta)Y = g(X + hX, \phi Y). \tag{2.5}$$

In a $(2n+1)$-dimensional (k,μ)-contact metric manifold, we have [5]

$$h^2 = (k-1)\phi^2, k \le 1, \tag{2.6}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.7}$$

$$\begin{aligned}(\nabla_X h)(Y) = \; & [(1-k)g(X, \phi Y) + g(X, h\phi Y)]\xi + \eta(Y)h(\phi X + \phi h X) \\ & - \mu \eta(X) \phi h Y.\end{aligned} \tag{2.8}$$

Definition 2.1: *A contact metric manifold M is said to be*

(i) Einstein if $S(X,Y)=\lambda g(X,Y)$, where λ is a constant and S is the Ricci tensor,

(ii) η-Einstein if $S(X,Y)=\alpha g(X,Y)+\beta\eta(X)\eta(Y)$, where α and β are smooth functions on M.

In a $(2n+1)$-dimensional generalized (k,μ) space-form, the following relations hold.

$$R(X,Y)\xi=(f_1-f_3)[\eta(Y)X-\eta(X)Y]+(f_4-f_6)[\eta(Y)hX-\eta(X)hY], \tag{2.9}$$

$$\begin{aligned} QX = \ & [2nf_1+3f_2-f_3]X+[(2n-1)f_4-f_6]hX \\ & -[3f_2+(2n-1)f_3]\eta(X)\xi, \end{aligned} \tag{2.10}$$

$$\begin{aligned} S(X,Y) = \ & [2nf_1+3f_2-f_3]g(X,Y)+[(2n-1)f_4-f_6]g(hX,Y) \\ & -[3f_2+(2n-1)f_3]\eta(X)\eta(Y), \end{aligned} \tag{2.11}$$

$$S(X,\xi)=2n(f_1-f_3)\eta(X), \tag{2.12}$$

$$r=2n[(2n+1)f_1+3f_2-2f_3], \tag{2.13}$$

$$\begin{aligned} \eta(R(X,Y)Z) = \ & (f_1-f_3)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)] \\ & +(f_4-f_6)[g(hY,Z)\eta(X)-g(hX,Z)\eta(Y)] \end{aligned} \tag{2.14}$$

for any vector fields X,Y,Z where Q is the Ricci operator, S is the Ricci tensor and r is the scalar curvature of $M(f_1,\ldots,f_6)$.

The relation between the associated functions $f_i, i=1,\ldots,6$ of $M(f_1,\ldots,f_6)$ was recently discussed by Carriazo et al. [5].

3. η -RECURRENT GENERALIZED (k,μ)-SPACE-FORM

Definition 3.1: A (2n+1)-dimensional generalized (k,μ)-space form is said to have η-recurrent Ricci tensor if there exists a non-zero 1-form A(X) such that

$$(\nabla_X S)(\phi Y,\phi Z)=A(X)S(Y,Z) \tag{3.1}$$

If the 1-form vanishes on M then the space-form is said to have η-parallel Ricci tensor. The notion of η-parallel Ricci tensor was introduced by Kon in the context of Sasakian geometry [4].

From (2.11) we have

$$\begin{aligned}&(\nabla_W S)(\phi X,\phi Y)=d(2nf_1+3f_2-f_3)(W)[g(X,Y)-\eta(X)\eta(Y)]\\&-(2nf_1+3f_2-f_3)[(\nabla_W\eta)(X)\eta(Y)+\eta(X)(\nabla_W\eta)(Y)]\\&-d[(2n-1)f_4-f_6](W)g(hX,Y).\end{aligned} \tag{3.2}$$

Suppose that the space-form has η- recurrent Ricci tensor. Then in view of (3.1) and (3.2) it follows that

$$\begin{aligned}&d(2nf_1+3f_2-f_3)(W)[g(X,Y)-\eta(X)\eta(Y)]\\&-(2nf_1+3f_2-f_3)[(\nabla_W\eta)(X)\eta(Y)+\eta(X)(\nabla_W\eta)(Y)]\\&-d[(2n-1)f_4-f_6](W)g(hX,Y)\\&=A(W)\begin{bmatrix}(2nf_1+3f_2-f_3)g(X,Y)+((2n-1)f_4-f_6)g(hX,Y)\\-(3f_2+(2n-1)f_3)\eta(X)\eta(Y)\end{bmatrix}.\end{aligned} \tag{3.3}$$

Replace X by ϕ X and Y by ϕ Y in (3.3) , we have

$$\begin{aligned}&d(2nf_1+3f_2-f_3)(W)[g(X,Y)-\eta(X)\eta(Y)]+d((2n-1)f_4-f_6)(W)g(hX,Y)\\&=A(W)[(2nf_1+3f_2-f_3)(g(X,Y)-\eta(X)\eta(Y))-((2n-1)f_4-f_6)g(hX,Y)].\end{aligned} \tag{3.4}$$

By taking $X=Y=e_i$ in (3.4) and summing over i = 1, 2,…..2n + 1, we obtain

$$d(2nf_1+3f_2-f_3)(W)=A(W)(2nf_1+3f_2-f_3)\ . \qquad . \tag{3.5}$$

(3.5)

Let $2nf_1+3f_2-f_3=f$ then (3.5) reduces to

$$fA(W)=df(W). \tag{3.6}$$

(3.6)

Then from (3.6) we obtain

$$df(Y)A(W)+(\nabla_Y A)(W)f=d^2f(W,Y). \tag{3.7}$$

(3.7)

Interchanging Y and W we get

$$df(W)A(Y)+(\nabla_W A)(Y)f = d^2 f(Y,W). \qquad (3.8) \qquad (3.8)$$

Subtracting (3.8) from (3.7) we get

$$(\nabla_W A)(Y)-(\nabla_Y A)(W)=0. \qquad (3.9) \qquad (3.9)$$

Hence the 1-form A(W) is closed.

Thus we have the following

Theorem 3.1: In an η-recurrent generalized (k,μ)-space-form the 1-form A is closed.

Since A(W) is non-zero, equation (3.5) leads us to state the following:

Theorem 3.2: If a (2n+1)-dimensional generalized (k,μ)-space-form has η-recurrent Ricci tensor then $2nf_1+3f_2-f_3$ can never be a non-zero constant.

Suppose Ricci tensor is η-parallel, *ie.*, $(\nabla_W S)(\phi X,\phi Y)=0$ and taking $X=Y=e_i$ in (3.2) and summing over i = 1, 2, …..2n + 1, we obtain

$$d(2nf_1+3f_2-f_3)(W)=0. \qquad (3.10)$$

it implies that $2nf_1+3f_2-f_3$ is constant.
This leads the following:

Lemma 3.1: A (2n+1)-dimensional generalized (k,μ)-space-form has η-parallel Ricci tensor then $2nf_1+3f_2-f_3$ is constant.

Again from (2.11) we have

$$(\nabla_Z S)(X,Y) = d(2nf_1 + 3f_2 - f_3)(Z)g(X,Y) + d[(2n-1)f_4 - f_6](Z)g(hX,Y)$$
$$-(2nf_1 + 3f_2 - f_3)[(\nabla_W \eta)(X)\eta(Y) + \eta(X)(\nabla_W \eta)(Y)]$$
$$-d(3f_2 + (2n-1)f_3)(Z)\eta(X)\eta(Y)$$
$$-[3f_2 + (2n-1)f_3][(\nabla_Z \eta)(X)\eta(Y) + \eta(X)(\nabla_Z \eta)(Y)] \quad (3.11)$$

By using above lemma(3.1) and taking $X = Y = e_i$ in (3.11) and summing over i = 1,2,…..2n + 1, we obtain

$$dr(Z) = -d[3f_2 + (2n-1)f_3](Z), \quad (3.12)$$

which implies that,

$r = -[3f_2 + (2n-1)f_3]$.

Hence we can state that

Theorem 3.3: A (2n+1)-dimensional generalized (k,μ)-space-form with η-parallel Ricci tensor then $r = -[3f_2 + (2n-1)f_3]$

4. ϕ-RECURRENT GENERALIZED (k,μ)-SPACE-FORM

Definition 4.1: A (2n+1)-dimensional generalized (k,μ)-space form is said to have

ϕ-recurrent if there exists a non-zero 1-form A such that

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z, \quad (4.1)$$

For an arbitrary vector field X,Y,Z and W.

Two vector fields are said to be co-directional, if $X = fY$ where f is a non -zero scalar, *ie.*,

$$g(X,Z) = fg(Y,Z) \quad (4.2)$$

For all X.

Now from (2.1) and (4.1), we have

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z \quad . \tag{4.3}$$

From (4.3) and the Bianchy identity we get

$$A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) = 0 \,. \tag{4.4}$$

By virtue of (2.14) , we obtain from (4.4) that

$$A(W)\begin{bmatrix}(f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ +(f_4 - f_6)[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)]\end{bmatrix}$$

$$+ A(W)\begin{bmatrix}(f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ +(f_4 - f_6)[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)]\end{bmatrix}$$

$$+ A(W)\begin{bmatrix}(f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ +(f_4 - f_6)[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)]\end{bmatrix} = 0 \,. \tag{4.5}$$

Putting $Y = Z = e_i$ in (4.4) and taking summation over i, $1 \le i \le 2n+1$, we get

$$2n(f_1 - f_3)[A(W)\eta(X) - A(X)\eta(W)] = 0 \,. \tag{4.6}$$

If $(f_1 - f_3) \neq 0$, then

$$A(W)\eta(X) = A(X)\eta(W) \tag{4.7}$$

For all vector fields X,W.

Replacing X by ξ in (4.7) we get

$$A(W) = \eta(\rho)\eta(W), \tag{4.8}$$

where $A(X) = g(X,\rho)$ and ρ is the vector field associated to the 1-form A .
Thus we can state the following

Theorem 4.1:

In a ϕ-recurrent generalized (k,μ)-space form, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by (4.8), provided $(f_1 - f_3) \neq 0$.

References

[1] P. Alegre, D. E. Blair and A. Carriazo, Generalized Sasakian space-forms, Israel J. Math. 141(2004), 157-183.

[2] Alfonso Carriazo and Veronica Martin-Molina, Generalized (k,μ)-space forms and D_a-homothetic deformations. Balkan Journal of Geometry and its Applications, Vol.16, No.1, 2011, 37-47.

[3] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, Contact metric manifolds satisfying a nullity condition, Israel Journal of Mathematics, Vol.91, No.1-3, 1995, 189-214.

[4] Kon.M, Invariant submanifoldsin Sasakian manifolds, Math.Ann.,219(1976),277-290.

[5] A. Carriazo, V. Martin-Molina and M. M. Tripathi, Generalized (k,μ)-space forms, Mediterr. J. Math. DOI 10.1007/s00009-012-0196-2, April(2012).

[6] Takahashi, Sasakian ∅-symmetric space, Tohoku Math. J., 29(1977), 91-113.

[7] U. C. De, A. A. Shaikh, Sudipta Biswas, On ∅-recurrent Sasakian manifold, Novi. Sad. J. Math., 32(2)(2003), 43-48.