International Journal of Scientific Engineering and Applied Science (IJSEAS) – Volume-7, Issue-11, November 2021
ISSN: 2395-3470
www.ijseas.com

Micro-Macro Mechanics Methods for Prediction of Effective Engineering Constants of Fibre-Reinforced, Variable Ply-Thickness Laminated Composite Panels

Umar Farooq[*1], Peter Myler[2], Mamoona Siddique[3]

[1, 2]Faculty of Engineering, Support and Advanced Sciences, University of Bolton BL3 5AB, United Kingdom

Email: U.Farooq@bolton.ac.uk, pm8@bolton.ac.uk

[3]Lecturer, Department of Mathematics, Government Degree College for Women, Sargodha 4000 Pakistan.

Email: monamaths0@gmail.com

Corresponding author: Umar Farooq[*1], Email: adalzai3@yahoo.co.uk

Abstract

This paper is concerned with prediction of effective engineering constants of variable ply-thickness laminated composite panels using micro-macro mechanics methods. The composite panels are being extensively used as building blocks of aerospace industry. The attaches industry attaches prime importance to screening engineering constants of structural elements at pre-design level. Composites are heterogeneous materials, thus full characterisation of their properties is difficult. Misaligned and damage fibres, non-uniform curing, cracks, voids and residual stresses are pre-assumed negligible that can severely influence the properties. Furthermore, standard test methods and setups lack in data logging: ply-level, non-symmetric laminates, variable ply-thickness, effects of mutual influence coefficients, and coupling deformations. Micro-macro mechanics methods are need to be applied to supplement the previous studies. Current study applies micro-macro mechanics methods laws: constitutive, kinematics, equilibrium, and strain-displacement compatibility conditions to supplement previous efforts. Response of isotropic, orthotropic, and anisotropic materials were formulated at ply and laminate levels against axial, off-axis, and coupled loading. The formulations were coded into computer programs using MATLAB™ software to predict the engineering constants. Quantities of the predicted engineering constants were plotted using MS-Excel™ 2020 software. Selected results were compared against the data results available in the literature and found to be within acceptable deviations (±10%). Comparisons of the results confirmed that proposed micro-macro mechanics methods could also be useful to reliably predict effective engineering constants for the other similar cases.

Keywords: A. Polymer Matrix Composites; B. Orthotropic Materials; C. Engineering Constants; D. Micro-Macro Mechanics.

1 Introduction

Fibre-reinforced composite materials are combination of fibres and resin rich matrix that exhibit superior properties in a specific application [1] while constituents keep their original properties. Fibres are oriented at different angles within the matrix to achieve uniform directional stiffness and optimal performance [2]. The efficient configuration of fibres in the direction of loading path to transfer unidirectional load [3]. However, multi-directional panels are used where unidirectional panels are inadequate. Due to superior performance and fabrication flexibilities, the composite materials are being widely utilized in civil and military, mercantile, and offshore structures as beams, pipes, sheets and plates, cylinders, and many other part shapes **Figure 1** [4]. The composites are becoming possible alternatives to steel due to high corrosion resistance, and specific strength properties: 20-40% weight savings, high stiffness, and reduced maintenance.

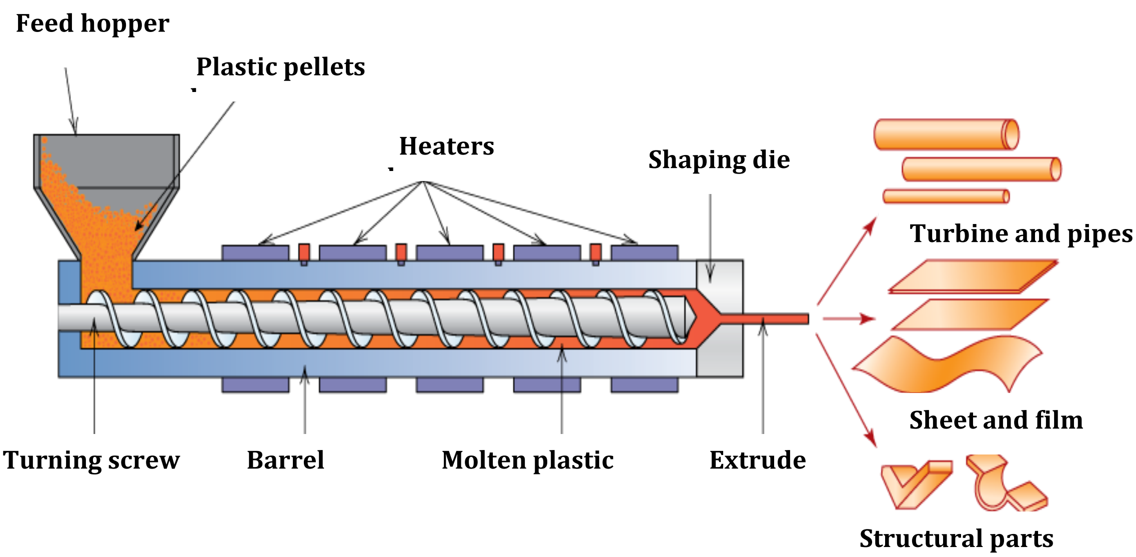


Figure 1: Schematic illustration fibrous composite parts

Structural elements have certain characteristics of shape, rigidity, stiffness, and strength for specific application. Thus comprehensive knowledge of engineering constants is important before putting them into load bearing services [5]. Therefore, extensive research is being carried out on various aspects of composites. For further references, the selected ones are being presented below.

One of the basic test method is 'Ignition loss' method used to determine quantities of volume fractions by weight in a composite sample. The method identifies volume fractions of quantities corresponding to response parameters of fibre, matrix types, and interfaces that influence strengths and stiffness [6]. The volume fractions of quantities are utilised in Rule of Mixture (ROM), Halping-Tsai relations, and physical testing to formulate relationships to evaluate performance of the composite panels [7]. Similarly, dispersion or distribution of the filler in the matrix, interfacial structure and morphology affect the modulus [8] and [9]. Influence of shear effects in the displacements is another important factor, larger span-to-depth ratios are used to reduce the influence are detailed in [10]. Physical test methods consist of tensile, compression, flexural, shear modulus, Iosipescu, and v-notch-rail detailed in [11]**.** However, factors affecting engineering property determination are complicated: nature of matrix and filler, compatibility, and material processing technology [12] and [13]. Young's modulus is one of the important characteristic, but it cannot be convenient procedure if the stacking sequence contains large number of plies [14]. They are solved by relating face conditions with symmetry conditions along the mid-plane in one half of the laminate through equations governing common variables between adjacent plies in layer-wise theory [15]. The deficiencies in the layer-wise theories are avoided using the author's extended Poisson's theory [16]. Use of such a theory in the preliminary analysis of extension and associated torsion problems is to be explored. As novel procedure based on this theory is envisaged in the analysis of unsymmetrical laminates. To the authors' knowledge, no proper procedure exists in the literature [17].

Since fibrous composites are inhomogeneous and anisotropic, their characterisation is complex. Laminates aligned reinforcement are stiff along the fibres, but weak in transverse to the fibre direction, reported in [18]. In order to obtain equal stiffness in all off-axis loading systems to present balanced angle plies were investigated in [19]. To obtain equal stiffness in all directions quasi-isotropic lay-up configurations were used in [20]. A composite laminate subjected to off-axis loading system presents tensile-shear interactions in its plies that leads to distortions and local micro-structural damage hence their testing can produce unreliable results. Thus, unidirectional lamina was tested at different fibre volume fractions to predict elastic constants using the finite element method [21]. However, use of the method is reported to be limited for cases of tensile-shear interaction if the off-axis loading system does not coincide with the main axes of a single lamina or if the panel is not balanced [22], [23]. Instead of such testing, the simplified property prediction schemes based on mathematical formulations are being preferred [24]. The study paved the way for solution to obtain equal stiffness of panels subjected in all directions within a plane. Characterization of in-plane mechanical properties of laminated hybrid composites, and mechanics-of-materials model for predicting Young's modulus can be found in [25]. Further complications exist as composite are anisotropic, thus they exhibit characteristics where normal loading induce normal and shear strains. Thus relationships between forces and deformations exhibit much more complications than conventional materials. The coupled complications have severe implications during service life and unexpected behaviour of structural components. Effective elastic moduli and associated Poisson's ratios for materials based on the theory of micro-macro mechanics along with linear elasticity and their limitations have been discussed in [26]and [27].

The literature review reveals that majority of the existing studies are experimental, resource and time consuming. Many test methods use different geometries for panels and holding-fixtures that produce different data. Researcher has to undergo series of experiments to obtain desired properties [28]. Moreover, anisotropic nature and characteristics of shape, rigidity, and strength make physical testing complicated. Furthermore, analytical studies based on neglecting deformation effects could not be relied to predict optimal mechanical properties. Micro-macro mechanics methods found in the literature consist of uniform ply thickness, balanced, and symmetric. Micro-mechanics applied to variable ply thickness, non-symmetric laminates, and influence of mutual influence coefficients are required to approximate at pre-design stage are desired [29] and [30].

Current study is based on micro-macro mechanics of fibrous composites. Stiffness matrices and invariants were formulated to include stress-strain effects and implemented in MATLAB™ code to approximate the effective engineering constants. Comparison and validation were carried out against intra-simulation and data results available in the literature and found to be within acceptable agreement. The study proposed that utilising the micro-macro mechanics laws the effective engineering can be reliably determined from computer codes.

2 Materials and methods

2.1 Fibre-reinforced panel and material properties

The schematic illustration of fibres, lamina, and fibre-reinforced laminated composite panel are shown in **Figure 2**.

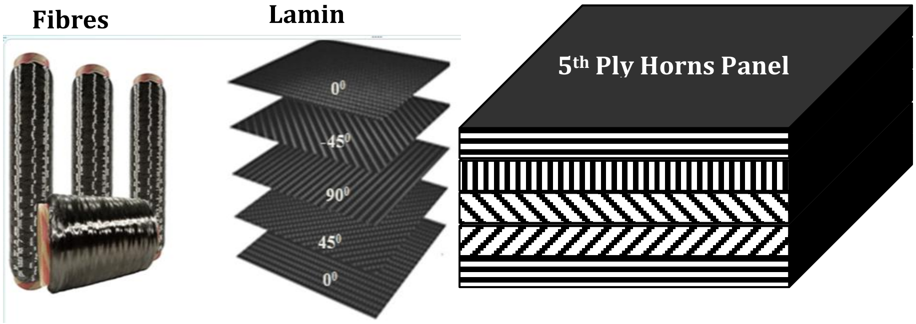


Figure 2: a) fibres, b) 5th harness satin weave, and c) laminated panel

The engineering constants calculated for a symmetric consisted of 4-Ply panel of AS4/3501-6 laid up in a $[0,+45]_S$ stacking sequence, and for a non-symmetric 2-Ply panel of the same material code AS4/3501-6 laid up in a $[0,+45]_S$ stacking sequence $[0,+45]_T$. Material properties for both the panels are given in **Table 1**.

Table 1: Properties of symmetric and non-symmetric panel

Ply thickness (in)	Young's modulus (Ib/in^2)		Poisson's ratio	
h	E_1	E_2	ν_{12}	ν_{21}
0.005	20010000	1301000	0.3	0.02

The engineering constants calculated for panel consisted of a 60-mm cube made up of graphite reinforced fibre polymer matrix composite material subjected to a tensile force of 100 kN perpendicular to a the fibre direction, directed along the 2-direction. The cube is free to expand or contract and the changes need to be determined in the 60-mm dimensions of the cube. The material constants for the graphite-reinforced polymer composite material are given **Table 2**.

Table 2: Properties of graphite fibre-reinforced panel

Ply thickness (mm)	Young's, Shear moduli (GPa)				Poisson's ratio	
h	E_1	$E_2 = E_3$	G_{23}	$G_{12} = G_{12}$	ν_{23}	$\nu_{12} = \nu_{13}$
0.15	155	12	3.2	4.4	0.458	0.28

The engineering constants calculated for panels consisted of fourth layer void- free, linear elastic plane of dimensions 150mm x 120mm laid in fibre-horns technique. Average thicknesses of laminates consist of eight-, sixteen-, and twenty-four plies with layup sequence codes and properties given **Table 3**.

Table 3: Properties of variable ply-thickness carbon fibre-reinforced panel

Panel Code: Fibredux 914C-833-40				
Panel	Lay-up code	Thickness mm	Property parameter	Unit MPa
8-Ply	$[0/90/45/-45]_S$	2.4	E_{xx} (0^0), E_{yy} (90^0) &	230
16-Ply	$[0/90/45/-45]2_S$	4.8	45^0, -45^0	23
24-Ply	$[0/90/45/-45]3_S$	7.2	Gxy	88
			Poisson's ratio	0.21

2.2 Micro-mechanics of a lamina

An isotropic lamina exhibits same behaviour in all three material 1-2-3 directions. Its engineering constants can be described by one value: Young's modulus, E(= $(E_1 = E_2 = E_3)$, Poisson's ratio, $\nu(\nu_{12} = \nu_{23} = \nu_{13})$, and shear moduli, $G(= G_{12} = G_{23} = G_{13})$ regardless of the direction of the applied load. Two independent material constants (E, ν) can characterize isotropic material; as shear modulus can be found from $G = \frac{E}{2(1+\nu)}$. Relationship between stress and strain is independent of the direction of force shown in **(a)** while especially orthotropic lamina described by two values shown in **Figure 3 (b)**.

$$\sigma = E\varepsilon \quad \textbf{(1)}$$

One along the longitudinal direction of the fibres, E_L, and one transverse to the direction of fibres, L_T. Subscripts 1 and 2 are used such as $E_L = E_1$ and $E_T = E_2$ at a direction with applied force stresses coincide with the principal material axes. Thus indices are added to the stress, strain, and modulus values to describe the direction of the applied force:

$$\sigma_1 = E_1\varepsilon_1 \text{ and } \sigma_2 = E_2\varepsilon_2 \quad \textbf{(2)}$$

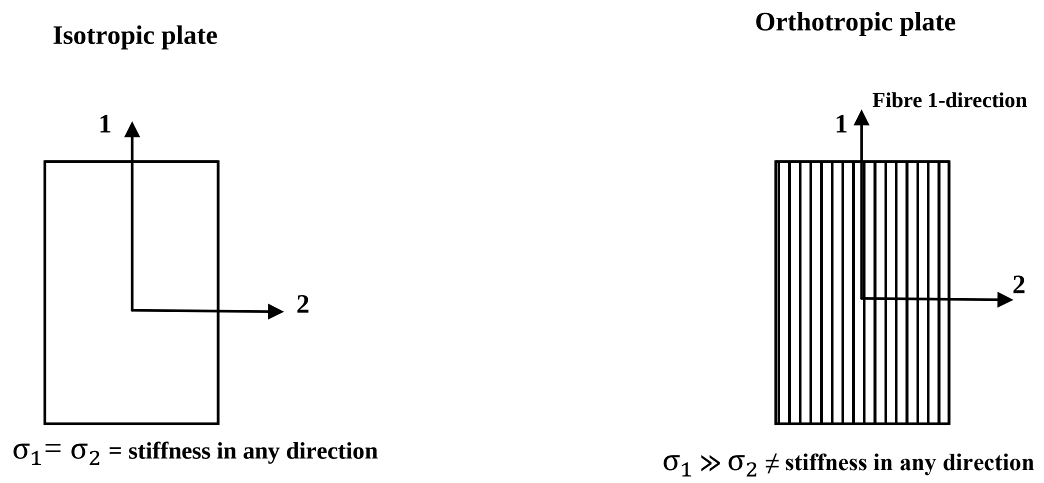


Figure 3: a) an isotropic and b) an orthotropic lamina

Poisson's ratio to the given loading direction is,

$$\nu_{12} = \frac{\varepsilon_T}{\varepsilon_L} = \frac{\varepsilon_2}{\varepsilon_1} \text{ or } \nu_{21} = \frac{\varepsilon_L}{\varepsilon_T} = \frac{\varepsilon_1}{\varepsilon_2} \quad \textbf{(3)}$$

The strain components stretched due to an applied force, minus the contraction of Poisson's effect due to another force perpendicular to the applied force:

$$\varepsilon_1 = \frac{\sigma_1}{E_1} - \nu_{12}\varepsilon_2 \text{and } \varepsilon_2 = \frac{\sigma_2}{E_2} - \nu_{12}\varepsilon_1 \quad \textbf{(4)a}$$

Using equation (2) gives,

$$\varepsilon_1 = \frac{\sigma_1}{E_1} - \nu_{12}\frac{\sigma_2}{E_2} \text{ and } \varepsilon_2 = \frac{\sigma_2}{E_2} - \nu_{12}\frac{\sigma_1}{E_1} \quad \textbf{(4b)}$$

The shear stress related by the shear modulus,

$$\tau_{12} = \gamma_{12}G_{12} \quad \textbf{(5)}$$

Where $\tau_{12}$is the shear stress (indices (12) indicate shear in the 1-2 plane) and $\gamma_{12}$is the shear strain. A relationship between Poisson's ratio and the elastic moduli exists as:

$$\nu_{21}E_1 = \nu_{12}E_2 \quad \textbf{(6)}$$

Equations (4a) and (5) can be written in matrix form:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \quad \textbf{(7)}$$

Where, $S_{11} = \frac{1}{E_1}, S_{22} = \frac{1}{E_2}, S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2}, S_{66} = \frac{1}{G_{12}}$ **(8)**

Stress as a function of strain can be obtained by inversion of 3x3 compliance matrix,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{\gamma_{12}}{2} \end{Bmatrix} \quad \textbf{(9)}$$

Where:

$$Q_{11} = \frac{E_1}{1-\upsilon_{12}\upsilon_{21}}, Q_{22} = \frac{E_2}{1-\upsilon_{12}\upsilon_{21}}, Q_{12} = \frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}}, Q_{66} = G_{12} \quad \textbf{(10)}$$

The components of $[Q]$ matrix are referred as reduced stiffness matrix, and its inversion $[Q]^{-1}$as the constitutive equations.

The lamina is called generally orthotropic if it is loaded at some angle other than 0^0 or 90^0. In general, the loading direction does not coincide with the principal material direction, thus stresses and strains are transformed into coordinates for the wedged-shape differential element that to coincide with the principal material directions using free-body diagrams in **Figure 4(a)**:

International Journal of Scientific Engineering and Applied Science (IJSEAS) – Volume-7, Issue-11, November 2021
ISSN: 2395-3470
www.ijseas.com

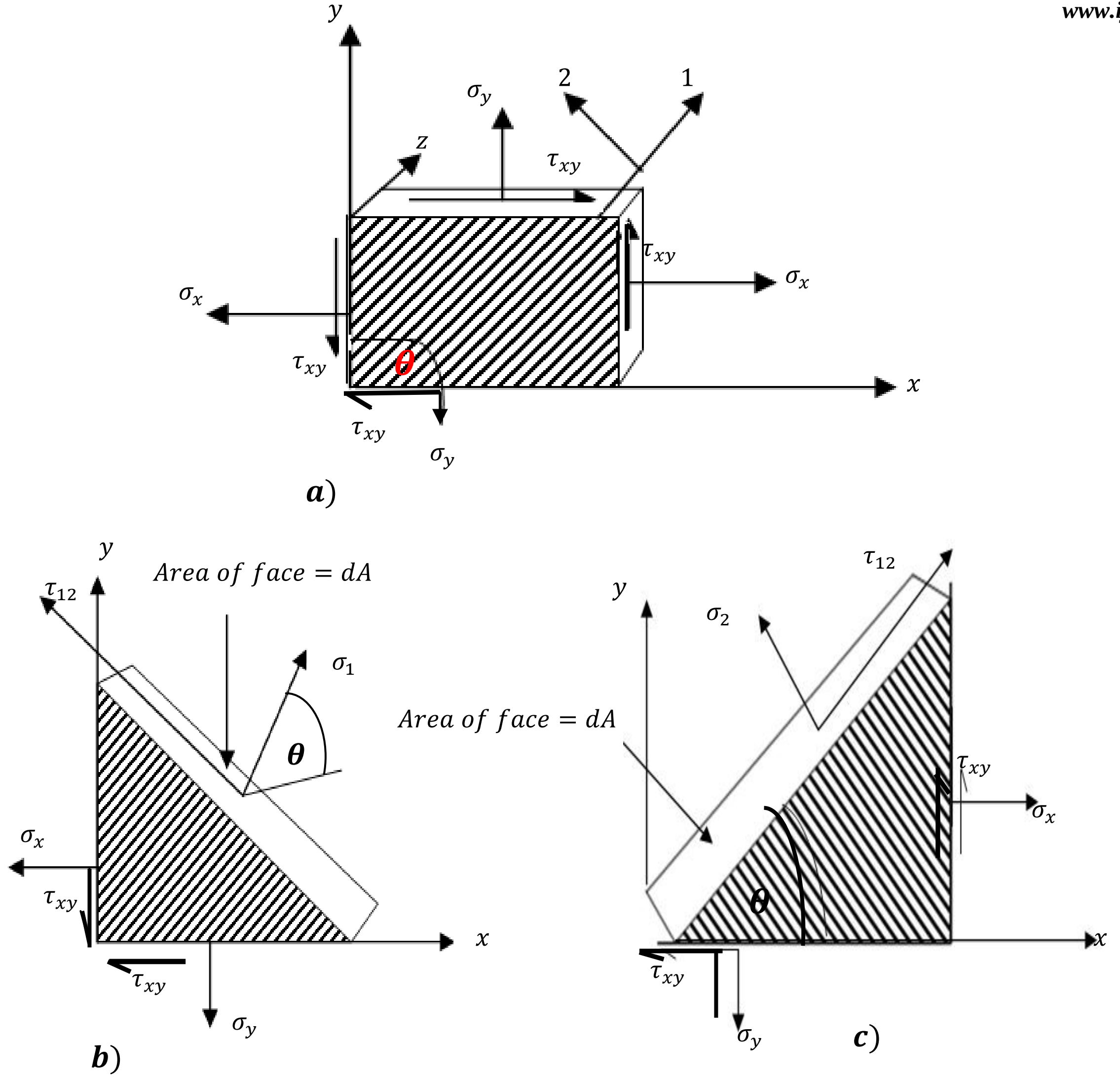


Figure 4: a) Generally orthotropic lamina, b) and c) wedge-shape elements

Static equilibrium of force by letting $m = cos\theta$ and $n = sin\theta$ can be written below.

Summing forces in the 1-direction in free body diagram **Figure 4(b)**:

$$\sum F_1 = 0 = \sigma_1 dA - \sigma_x(dAm)m - \sigma_y(dAn)n - \tau_{xy}(dAm)n - \tau_{xy}(dAn)m \quad \textbf{(11)}$$

And summing forces in the 2-direction of the free diagram **Figure 4(c)**:

$$\sum F_2 = 0 = \sigma_2 dA - \sigma_x(dAn)n - \sigma_y(dA)m - \tau_{xy}(dAm)n + \tau_{xy}(dAn)m \quad \textbf{(12)}$$

Summing forces in the 1-direction in the free diagram **Figure 4**(c):

$$\sum F_2 = 0 = \tau_{12} dA + \sigma_x(dAn)m - \sigma_y(dAm)n - \tau_{xy}(dAm)m + \tau_{xy}(dAn)n \quad \textbf{(13)}$$

Simplifying equations (11), (12), and (13),

$$\sigma_1 = \sigma_x m^2 + \sigma_y n^2 + 2\tau_{xy} mn$$

$$\sigma_2 = \sigma_x n^2 + \sigma_y m^2 - 2\tau_{xy} mn$$

$$\tau_{12} = -\sigma_x mn + \sigma_y nm + \tau_{xy}(m^2 - n^2) \tag{14}$$

Equation (14) may be written in matrix form as,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2nm \\ n^2 & m^2 & -2nm \\ -nm & 2nm & (m^2 - n^2) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{15}$$

The 3x3 matrix in equation (15) is the transformation matrix denoted by [T] that can also be used to transform strains. However, the tensorial shear strain is used not the engineering shear strain. Since the amount of shear strain must be equivalent to both the x-y axes transformed new 1-2 coordinate systems, while inverse of [T] is,

$$[T]^{-1} = \begin{bmatrix} m^2 & n^2 & -2nm \\ n^2 & m^2 & 2nm \\ -nm & -nm & (m^2 - n^2) \end{bmatrix} \tag{16}$$

Thus;

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \; and \; \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \tag{17}$$

Similarly for strains:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \; and \; \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{18}$$

Putting equation (9) into the second part of equation in (17):

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1}[Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [T]^{-1}[Q] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{19}$$

Now putting the first equation of equation (18) into equation (19):

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1}[Q] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{20}$$

Defining a new matrix called the lamina stiffness matrix $\bar{Q}$ as:

$$[\bar{Q}] = [T]^{-1}[Q] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} [T] \tag{21}$$

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}m^4 + Q_{22}n^4 + 2(Q_{12} + 2Q_{66})n^2m^2 \\
\bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})n^2m^2 + Q_{12}(m^4 + n^4) \\
\bar{Q}_{22} &= Q_{11}m^4 + Q_{22}n^4 + 2(Q_{12} + 2Q_{66})n^2m^2 \\
\bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})nm^3 - (Q_{22} - Q_{12} - 2Q_{66})mn^3 \\
\bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})mn^3 - (Q_{22} - Q_{12} - 2Q_{66})nm^3 \\
\bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})n^2m^2 + Q_{66}(m^4 + n^4)
\end{aligned} \tag{22}$$

The equation (22) is referred to as extension shear coupling that takes place when a lamina is loaded at an angle to the fibres ($\theta \neq 0, 90$), which generates non-zero $\bar{Q}_{16}$and$\bar{Q}_{26}$ terms. Putting into equation (20), the stress-strain equation becomes,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \textbf{(23)}$$

Since six constants in 3x3 matrix equation (23) govern stress-strain behaviour of a lamina are not independent, the elements in stiffness matrices can be expresses in terms of five invariant properties using trigonometric identities:

$$\bar{Q}_{11} = U_1 + U_2 cos2\theta + U_3 cos4\theta, \bar{Q}_{12} = U_4 - U_3 cos4\theta, \bar{Q}_{22} = U_1 - U_2 cos2\theta + U_3 cos4\theta$$

$$\bar{Q}_{16} = \frac{U_2}{2} sin2\theta + U_3 sin4\theta, \bar{Q}_{26} = \frac{U_2}{2} sin2\theta - U_3 sin4\theta, \qquad \bar{Q}_{66} = \frac{1}{2}(U_1 - U_4) - U_3 \cos 4\theta$$

(24a)

Where the set of invariant stiffness is defined as follows:

$$U_1 = \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}), U_2 = \frac{1}{2}(Q_{11} - Q_{22}), U_3 = \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})$$

$$U_4 = \frac{1}{8}(3Q_{11} + 3Q_{22} + 6Q_{12} - 4Q_{66}) \quad \textbf{(24b)}$$

The invariants due to rotations in equation (24) are simply four independent invariants as there are four independent elastic constants: easier to compute stiffness matrices. Likewise, the element oriented along fibre angle exhibits a shear strain when subjected to a normal stress exhibiting an extensional strain when subjected to a shear stress. Thus, $U_2$and U_3 are coefficients of sine or cosine (at 0^0 and 90^0) terms become zero in equation (24) when calculating $\bar{Q}$ values making the independent invariants $U_{1,} U_4, and\ U_5$ calculations along ply orientation (θ) easier. For the strain-displacement relationship, first assumption of the classical plate theory says that the lamina only deflects, it is un-damaged, and there is no strain in the thickness direction. The small deflection of the lamina in the x-direction is designated as u in **Figure 5**. Displacement along z-direction due to bending is $sin\theta$ times z. Since θ is small so $sin\theta = \theta$. Therefore, the displacement is $-z\theta$ where negative is compression and positive is tension. For the y-direction, it is designated as v and for the z-direction w. Strains can be obtained:

$$\varepsilon_x = \frac{\partial u}{\partial x};\ \varepsilon_y = \frac{\partial v}{\partial y};\ \gamma_{xy} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \quad \textbf{(25)}$$

The slope of the lamina if it is bending can be formulated:

$\frac{\partial w}{\partial x}$ along the x-direction, $\frac{\partial w}{\partial y}$ along the y-direction.

The total in-plane displacement at any point in the ply is the sum of the normal displacements plus the displacement introduced by bending. Denoting the displacements of the mid-plane of the plate for the x and y directions as u_0 and v_0 respectively. The total displacements become:

$$u = u_0 - z\frac{\partial w}{\partial x};\ v = v_0 - z\frac{\partial w}{\partial y}. \quad \textbf{(26)}$$

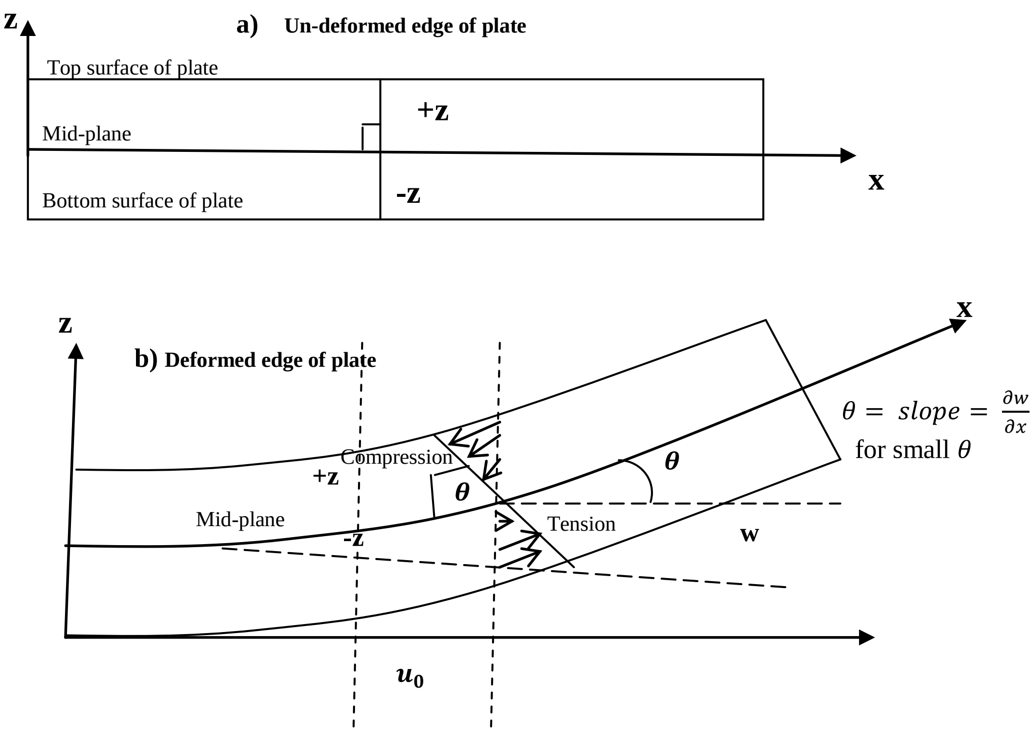


Figure 5: Total displacements in a) un-deformed and b) deformed plate

Strains-displacement relations gives,

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2},\ \varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2},\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y}, \quad \textbf{(27)}$$

Writing the mid-plane strains:

$$\frac{\partial u_0}{\partial x_0}\ as\ \varepsilon_x^0;\ \frac{\partial v_0}{\partial y_0}\ as\ \varepsilon_y^0;\ \text{and}\ \frac{\partial u_0}{\partial y_0} + \frac{\partial v_0}{\partial x_0} as\ \gamma_{xy}^0 \quad \textbf{(28)}$$

and curvature-displacement relation gives:

$$-\frac{\partial^2 w}{\partial x^2}\ as\ k_x;\ -\frac{\partial^2 w}{\partial y^2}\ as\ \ k_y;\ and - 2\frac{\partial^2 w}{\partial x \partial y}\ as\ k_{xy} \quad \textbf{(29)}$$

Thus strain-curvature equation gives,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad \textbf{(30)}$$

The lamina curvature k_x or k_y is the rate of change of slope of bending in either x- or y-direction, respectively. The curvature term k_{xy} is the amount of bending in the x-direction along the y-axis (i.e. twisting). The strains in equation can be expressed in terms of mid-plane$(\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0)$, and curvatures (k_x^0, k_x^0, k_{xy}^0)at reference surface. The equation (30) can be written in matrix form using equation (23) to determine stresses as,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{16} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 + zk_x^0 \\ \varepsilon_y^0 + zk_y^0 \\ \gamma_{xy}^0 + zk_{xy}^0 \end{Bmatrix} \tag{31}$$

The equation (31) formulates principal stresses transformations from isotropic material into the general orthotropic using the local-global coordinates.

2.3 Macro-mechanics of laminated plates

Fibre-reinforced materials consist of multi-layers at different fibre orientations to form a laminate with stacking arrangements corresponding to structural response. **Figure 6** shows a global Cartesian system and a general laminate consisting of N layers. The laminate thickness is H and the thickness of an individual layer by h. Since the stress in each ply varies through the thickness of the laminate, stresses and strains in each ply need to be known in terms of equivalent forces and acting at the middle surface. Referring to figure 7, it can be seen that the stresses acting on an edge can be broken into increments and summed. The resulting integral is defined as the stress result (force per unit length and acts in the same direction).

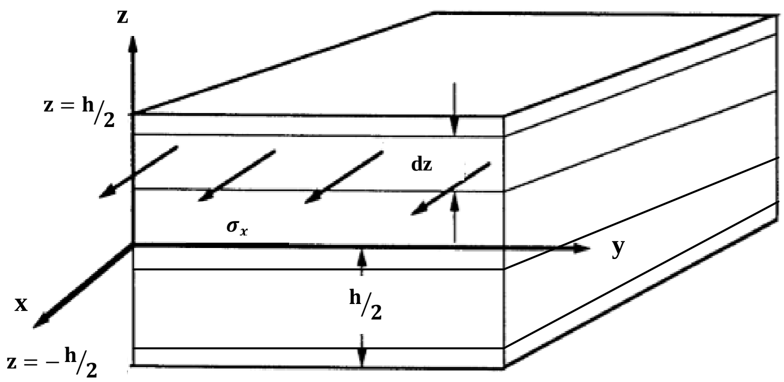


Figure 6: Schematic illustration of laminate with stress and moment resultants

The total force in x-direction = $\sum \sigma_x (dz)(y)$

As $dz \rightarrow 0, \sum \sigma_x (dz)\cancel{(y)} = \cancel{y} \int_{-h/2}^{h/2} \sigma_x dz = N_x = \int_{-h/2}^{h/2} \sigma_x dz$

Likewise, it can be drawn for the y-direction stress and shear stress. The stress resultants are therefore:

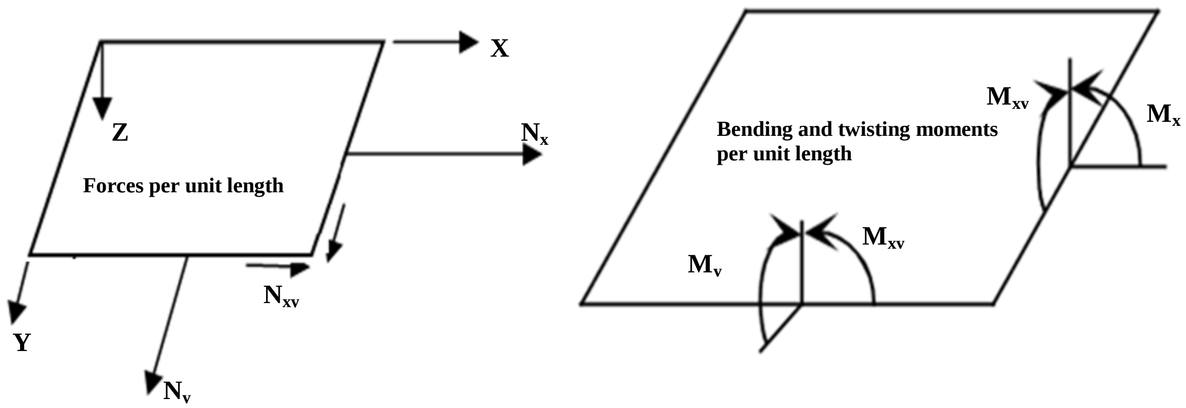
$$N_x = \int_{-t/2}^{t/2} \sigma_x dz \, , N_y = \int_{-t/2}^{t/2} \sigma_y dz, N_{xy} = \int_{-t/2}^{t/2} \tau_x y dz \tag{32}$$

The stress acting on an edge produces a moment (torque per unit length) about the mid-plane at a distance z from the mid-plane. Following the same procedure as for the stress resultants, the moment relations cane be defines as:

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz, M_y = \int_{-h/2}^{h/2} \sigma_y z dz, M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z dz \quad \textbf{(33)}$$

The moments M_x, M_y will cause the plate to bend and M_{xy} will cause the plate to twist. Once stresses are calculated for each lamina, the resulting forces and moments in the laminae can be determined.

Applying reverse process of the micro-mechanics the lamina to laminate stiffness matrix when force and moment resultants are known, stresses and strains through the thickness as well as the strains and curvatures on the reference surface can be determined. In Figures show the force and moment resultants, a small element of laminate surrounding a point (x, y) on the geometric surface is shown. The force resultants N_x, N_y, and N_{xy} can be related to strains and curvatures at the reference surface by equation (32). The positive forces are defined per unit length is shown in **Figure 7(a)**, and the positive bending and twisting moments per unit length are shown in **Figure 7(b)**.



Defined positive forces per   Defined positive bending and twisting moments

Figure 7: Schematic illustration of force and moment resultants

Integrating global stresses in each lamina gives the resultant forces and moments per unit length in xy-plane through-thickness. Putting equations (32) and (33) in matrix form gives,

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz \, , \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz \quad \textbf{(34)}$$

The integrals over equation (34) must be performed over each ply and then summed, since discontinuities in stresses can occur at ply interfaces, equations (35) and (36) can be written as:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz, \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z\, dz \quad \mathbf{(35)}$$

Now equation (30) can be substituted into equation (23), which can be substituted into equations into equations (34) and (35), the equations (12) and (13) simplifies as,

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \left\{ \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} dz + \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} k_x^0 \\ k_y^0 \\ k_{xy}^0 \end{Bmatrix} zdz \right\} \quad \mathbf{(36)}$$

and

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \left\{ \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} zdz + \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} k_x^0 \\ k_y^0 \\ k_{xy}^0 \end{Bmatrix} z^2 dz \right\}$$

(37)

Since the middle surface strains and curvatures are not a function of z (because these values are always at the middle surface z = 0), they need not be included in the integration. Also, the laminate stiffness matrix is constant for each ply, so it will be constant over the integration of a lamina thickness. Putting these constants to the front of the integral in equations (36) and (37) gives,

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \left\{ \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \int_{h_{k-1}}^{h_k} dz + \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \int_{h_{k-1}}^{h_k} zdz \right\} \quad \mathbf{(38)}$$

and

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \left\{ \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \int_{h_{k-1}}^{h_k} zdz + \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \int_{h_{k-1}}^{h_k} z^2 dz \right\}$$

(39)

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \left\{ \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} (h_k - h_{k-1}) + \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \frac{1}{2} (h_k^2 - h_{k-1}^2) \right\}$$

(40)

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \left\{ \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \frac{1}{2}\left(h_k^2 - h_{k-1}^2\right) + \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \frac{1}{3}\left(h_k^3 - h_{k-1}^3\right) \right\} \quad \textbf{(41)}$$

Since the middle surface strains and curvatures are not part of the summations, the laminate stiffness matrix and the h_k terms can be combined to form new matrices. From Equations (40) and (41) can be defined as:

$$A_{ij} = \sum_{k=1}^{n} \left[\bar{Q}_{ij}\right]_k (h_k - h_{k-1}), B_{ij} = \frac{1}{2}\sum_{k=1}^{n} \left[\bar{Q}_{ij}\right]_k \left(h_k^2 - h_{k-1}^2\right),$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n} \left[\bar{Q}_{ij}\right]_k \left(h_k^3 - h_{k-1}^3\right) \quad \textbf{(42)}$$

The constitutive equations can be written in ABD matrix form:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ - \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & | & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & | & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & | & B_{16} & B_{26} & B_{66} \\ - & - & - & | & - & - & - \\ B_{11} & B_{12} & B_{16} & | & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & | & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & | & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_y^0 \\ - \\ k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad \textbf{(43)}$$

Written in compact form, equation (43) becomes:

$$\begin{bmatrix} N \\ -- \\ M \end{bmatrix} = \begin{bmatrix} A & | & B \\ - & - & - \\ B & | & D \end{bmatrix} \begin{bmatrix} \varepsilon^0 \\ -- \\ K \end{bmatrix}, \text{ partial inversion gives: } \begin{bmatrix} \varepsilon^0 \\ -- \\ M \end{bmatrix} = \begin{bmatrix} A^* & | & B^* \\ - & - & - \\ C^* & | & D^* \end{bmatrix} \begin{bmatrix} N \\ -- \\ K \end{bmatrix} \quad \textbf{(44)}$$

Where, $[A^*] = [A]^{-1}$, $[B^*] = [A]^{-1}[B]$, $[C^*] = [B][A]^{-1}$, $[D^*] = [D] - [B][A]^{-1}[B]$ **(45)**

The fully inverted form is given by:

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ - \\ k_x \\ k_y \\ k_{xy} \end{bmatrix} = \begin{bmatrix} A'_{11} & A'_{12} & A'_{16} & | & B'_{11} & B'_{12} & B'_{16} \\ A'_{12} & A'_{22} & A'_{26} & | & B'_{12} & B'_{22} & B'_{26} \\ A'_{16} & A'_{26} & A'_{66} & | & B'_{16} & B'_{26} & B'_{66} \\ - & - & - & | & - & - & - \\ C'_{11} & C'_{12} & C'_{16} & | & D'_{11} & D'_{12} & D'_{16} \\ C'_{12} & C'_{22} & C'_{26} & | & D'_{12} & D'_{22} & D'_{26} \\ C'_{16} & C'_{26} & C'_{66} & | & D'_{16} & D'_{26} & D'_{66} \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ - \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} \quad \textbf{(46)}$$

Where,

$$[A'] = [A^*] - [B^*][D^*]^{-1}[C^*],\ [B'] = [B^*][D^*]^{-1},\ [C'] = [A^*] - [D^*]^{-1}[C^*]\ ,\ [D'] = [D^*]^{-1} \quad \textbf{(47)}$$

Symmetric laminates are configured such that the geometric mid-plane is mirror image of the ply configurations above and below the mid-plane, the geometric is also the neutral plane of the plate, and the $[B]$ matrix is equal to zero. However, if the laminate is un-symmetric as shown in **Figure 8**, then the plies near the bottom of the plate are much stiffer in the x-direction, then the geometric mid-plane will not be neutral plane of the plate; and the neutral plane will be closer to the bottom of the plate for x-direction bending. This is accounted for in the constitutive equations, since the $[B]$ matrix will have some nonzero elements, implying that a bending strain (plate curvature) will cause a mid-plane strain. Likewise, a mid-plane strain will cause a bending moment.

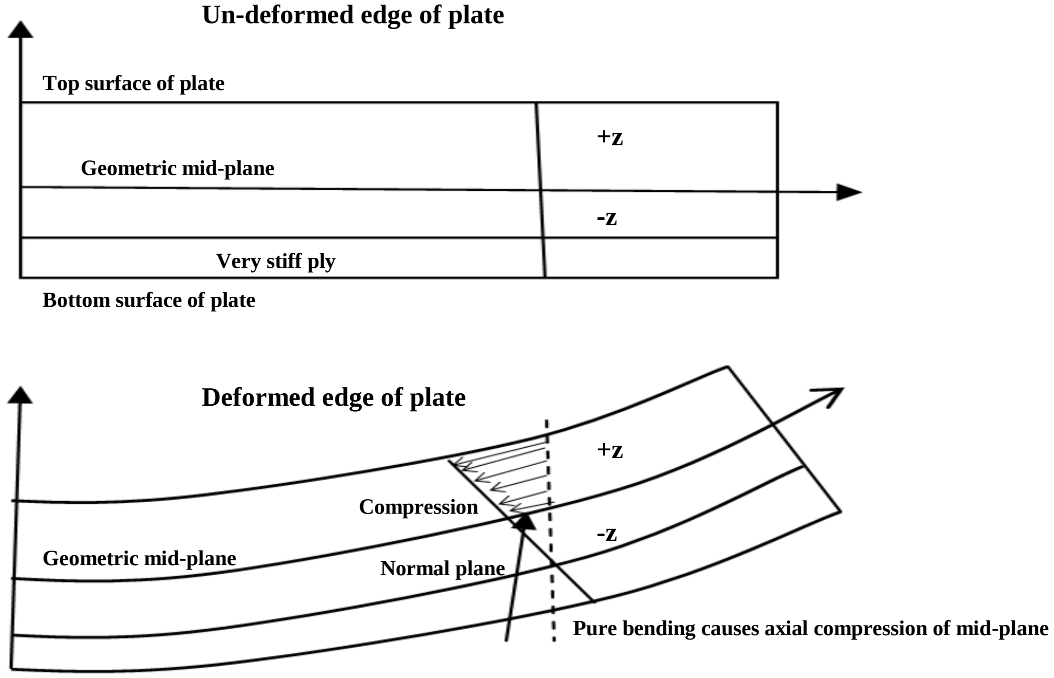


Figure 8: Displacement in an unsymmetrical plate

Referring to $[A]$ $[B]$, and $[D]$ matrices in equation (42), it can be seen that the last term in equation is the k[th] lamina thickness is denoted by t_k. Thus,

$$A_{ij} = \sum_{k=1}^{n} \left[\bar{Q}_{ij}\right]_k t_k \quad \textbf{(48)}$$

This is the extensional stiffness matrix, when A_{16} and $A_{26}$are nonzero and the laminate has a shear strain applied to it, normal stresses will result and vice versa. These terms are analogous to the Q_{16} and Q_{26} terms. Equation (42) can be written as:

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{n} \left[\bar{Q}_{ij}\right]_k (h_k - h_{k-1}) = \sum_{k=1}^{n} \left[\bar{Q}_{ij}\right]_k t_k \frac{(h_k + h_{k+1})}{2} \quad \textbf{(49)}$$

Where t_k is the thickness of k[th] ply, and $\frac{(h_k+h_{k+1})}{2}$ is the distance from the geometric mid-plane in the centre of the k[th] ply, the coupling stiffness matrix. The B_{16} and B_{26} terms relate twisting strains to normal stresses and shear strains to bending stresses. If the laminate is symmetric, then the B_{ij} terms will be the same for each mirrored ply above and below the mid-plane (-z) and positive if it is above the mid-plane (+z). Thus, when summed, the result will be zero for all B_{ij}. Now define: $\frac{(h_k+h_{k+1})}{2} = \bar{z}_k$. Part of equation (42) can be written as:

$$\begin{aligned}(\mathrm{h}_k^3-\mathrm{h}_{k-1}^3) &= [(\mathrm{h}_k^2-\mathrm{h}_{k-1}^2)(h_k-h_{k-1})+\mathrm{h}_k^2h_{k-1}-h_k\mathrm{h}_{k-1}^2]\\ &= [(h_k-h_{k-1})^3+3\mathrm{h}_k^2h_{k-1}-3h_k\mathrm{h}_{k-1}^2]\\ &= [(h_k-h_{k-1})^3+3(h_k-h_{k-1})(h_k+h_{k-1})^2-3(\mathrm{h}_k^3-\mathrm{h}_{k-1}^3)]\\ &= 4(\mathrm{h}_k^3-\mathrm{h}_{k-1}^3) = 12\,\mathrm{t}_k^3\,t\bar{z}_k^2 \qquad \textbf{(50)}\end{aligned}$$

Therefore, the equation can be written as

$$\mathrm{D_{ij}} = \frac{1}{3}\sum_{k=1}^{n}[\overline{\mathrm{Q}}_{ij}]_k\left(\frac{\mathrm{t}_k^3}{12}+t\bar{z}_k^2\right) \qquad \textbf{(51)}$$

It can be seen that the last term in the second moment of the k^{th} ply with respect to the geometric mid-plane $\mathrm{D_{ij}}$ is called the bending stiffness matrix and relates the amount of bending curvatures with the bending moments.

2.4 Engineering constants for symmetric and un-symmetric laminates

For a given stacking sequence of laminate whose engineering constants are known, it is possible to determine the in-plane engineering constants for symmetric laminate from the A_{ij}matrix. The orthotropic materials have symmetric elastic properties with respect to the chosen axis, which are called the 'principal material direction'. The 1-direction is angled with the fibre direction if fibre-matrix system is replaced with homogeneous material **Figure 9**. Elastic properties for a mechanical test in 2-direction or 3- direction will give the same result. However, the properties in longitudinal 1-direction are different.

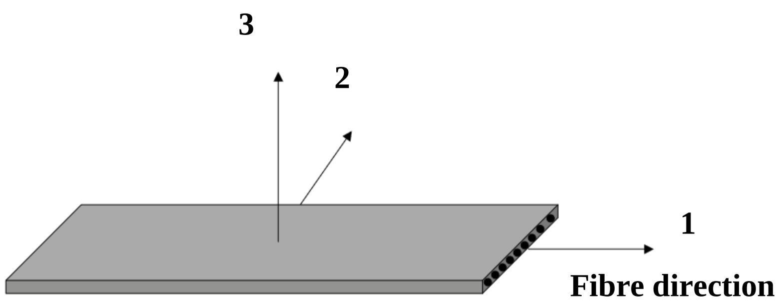


Figure 9: Lamina with different material properties

To find modulus in x-direction, values of stress and strain in the x-direction are calculated: $E_x = \frac{\sigma_x}{\varepsilon_x} = \frac{N_x/h}{\varepsilon_x}$ where h is lamina thickness. The matrix $B_{ij} = 0$, the constitutive equation is,

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \quad \textbf{(52)}$$

Relationship between N_x and ε_x^0 when load is applied in the x-direction,

$$N_x = A_{11}\varepsilon_x^0 + A_{12}\varepsilon_y^0 + A_{16}\gamma_{xy}^0$$

$$0 = A_{12}\varepsilon_x^0 + A_{22}\varepsilon_y^0 + A_{26}\gamma_{xy}^0$$

$$0 = A_{16}\varepsilon_x^0 + A_{26}\varepsilon_y^0 + A_{66}\gamma_{xy}^0 \quad \textbf{(53)}$$

Resolution of the equations gives,

$$\varepsilon_y^0 = \varepsilon_x^0 \left(\frac{A_{26}A_{16} - A_{12}A_{66}}{A_{22}A_{66} - A_{26}^2} \right) \quad \textbf{(54)}$$

and

$$\gamma_{xy}^0 = \varepsilon_x^0 \left(-\frac{A_{16}}{A_{66}} + \frac{A_{66}A_{26}A_{12} - A_{16}A_{26}^2}{A_{22}A_{66}^2 - A_{26}^2 A_{66}} \right) \quad \textbf{(55)}$$

$$\frac{N_x}{\varepsilon_x^0} = A_{11} + A_{12}\left(\frac{A_{26}A_{16} - A_{12}A_{66}}{A_{22}A_{66} - A_{26}^2} \right) + A_{16}\left(-\frac{A_{11}}{A_{66}} + \frac{A_{66}A_{26}A_{12} - A_{16}A_{26}^2}{A_{22}A_{66}^2 - A_{26}^2 A_{66}} \right) \quad \textbf{(56)}$$

The equations (53) and (54) can be substituted in equation (56). Thus, E_x can be as:

$$E_x = \frac{A_{11}}{H} + \frac{A_{12}}{H}\left(\frac{A_{26}A_{16} - A_{12}A_{66}}{A_{22}A_{66} - A_{26}^2} \right) + \frac{A_{16}}{H}\left(-\frac{A_{11}}{A_{66}} + \frac{A_{66}A_{26}A_{12} - A_{16}A_{26}^2}{A_{22}A_{66}^2 - A_{26}^2 A_{66}} \right) \quad \textbf{(57)}$$

The same process is followed to obtain E_y. The constitutive equation (53) changes for N_y from zero.

$$0 = A_{11}\varepsilon_x^0 + A_{12}\varepsilon_y^0 + A_{16}\gamma_{xy}^0$$

$$N_y = A_{11}\varepsilon_x^0 + A_{22}\varepsilon_y^0 + A_{26}\gamma_{xy}^0$$

$$0 = A_{16}\varepsilon_x^0 + A_{26}\varepsilon_y^0 + A_{66}\gamma_{xy}^0$$

$$\varepsilon_x^0 = \varepsilon_y^0 \left(\frac{A_{26}A_{16} - A_{12}A_{66}}{A_{11}A_{66} - A_{16}^2} \right) \quad \textbf{(58)}$$

and

$$\gamma_{xy}^0 = \varepsilon_y^0 \left(-\frac{A_{26}}{A_{66}} + \frac{A_{16}A_{12}A_{66} - A_{26}A_{16}^2}{A_{11}A_{66}^2 - A_{16}^2 A_{66}} \right) \quad \textbf{(59)}$$

$$\frac{N_y}{\varepsilon_y^0} = A_{12}\left(\frac{A_{26}A_{16} - A_{12}A_{66}}{A_{22}A_{66} - A_{26}^2} \right) + A_{22} + A_{26}\left(-\frac{A_{26}}{A_{66}} + \frac{A_{66}A_{16}A_{12} - A_{26}A_{16}^2}{A_{11}A_{66}^2 - A_{16}^2 A_{66}} \right) \text{ gives,}$$

$$E_y = \frac{A_{12}}{H}\left(\frac{A_{26}A_{16} - A_{12}A_{66}}{A_{11}A_{66} - A_{16}^2} \right) + \frac{A_{22}}{H} + \frac{A_{26}}{H}\left(-\frac{A_{26}}{A_{66}} + \frac{A_{66}A_{16}A_{12} - A_{26}A_{16}^2}{A_{11}A_{66}^2 - A_{16}^2 A_{66}} \right) \quad \textbf{(60)}$$

G_{xy} can be found in the same manner. In the constitutive replace zero by N_{xy}.

$$0 = A_{11}\varepsilon_x^0 + A_{12}\varepsilon_y^0 + A_{16}\gamma_{xy}^0$$

$$0 = A_{11}\varepsilon_x^0 + A_{22}\varepsilon_y^0 + A_{26}\gamma_{xy}^0$$

$$N_{xy} = A_{16}\varepsilon_x^0 + A_{26}\varepsilon_y^0 + A_{66}\gamma_{xy}^0 \quad \textbf{(61)}$$

$$\varepsilon_x^0 = \gamma_{xy}^0 \left(\frac{A_{12}A_{26} - A_{16}A_{22}}{A_{11}A_{22} - A_{12}^2}\right) \quad \mathbf{(62)}$$

$$\varepsilon_y^0 = \gamma_{xy}^0 \left(\frac{-A_{26}}{A_{22}} + \frac{A_{26}A_{12}A_{22} - A_{12}^2 A_{26}}{A_{11}A_{22}^2 - A_{12}^2 A_{22}}\right) \quad \mathbf{(63)}$$

Dividing equations above by the laminate thickness give,

$$G_{xy} = \frac{A_{66}}{H} - \frac{A_{26}^2}{HA_{22}} + \frac{2A_{12}A_{16}A_{26}A_{22} - A_{12}^2A_{26}^2 - A_{16}^2A_{22}^2}{H(A_{11}A_{22}^2 - A_{22}A_{12}^2)} \quad \mathbf{(64)}$$

To find Poisson's ratios of the laminate, equation (61) and (62) are utilised to obtain:

$0 = A_{12}\varepsilon_x^0 + A_{22}\varepsilon_y^0 + A_{26}\left(-\frac{A_{16}}{A_{66}}\varepsilon_x^0 - \frac{A_{26}}{A_{66}}\varepsilon_y^0\right)$. Re-arranging gives Poisson's ratios:

$$\nu_{xy} = -\frac{\varepsilon_y^0}{\varepsilon_x^0} = \frac{\left(A_{12} - \frac{A_{16}A_{26}}{A_{66}}\right)}{\left(A_{22} - \frac{A_{26}^2}{A_{66}}\right)} \quad \mathbf{(65)}$$

$$\nu_{yx} = -\frac{\varepsilon_x^0}{\varepsilon_y^0} = \frac{\left(\frac{A_{16}A_{26}}{A_{66}} - A_{12}\right)}{\left(\frac{A_{16}^2}{A_{66}} - A_{11}\right)} \quad \mathbf{(66)}$$

For a non-symmetric laminate, procedure is the same as for symmetric laminates. If matrix $B_{ij} \neq 0$, the constants for un-symmetric laminates can be determined from A_{ij}, B_{ij}, and D_{ij} matrices. The bottom half of a symmetric laminate does not consist of a negative mirror image of stresses and strains from the top half due to bending moments. The plane of zero strain (neutral plane) for any direction in an un-symmetric laminate can be calculated pure bending in the x-direction,

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} B'_{11} & M_x \\ B'_{12} & M_x \\ B'_{16} & M_x \end{bmatrix} + z\begin{bmatrix} D'_{11} & M_x \\ D'_{12} & M_x \\ D'_{16} & M_x \end{bmatrix} \quad \mathbf{(67)}$$

$z = \frac{-B'_{11}}{D'_{11}}$, for $\varepsilon_x = 0$plane, $z = \frac{-B'_{12}}{D'_{12}}$, for $\varepsilon_y = 0$ plane, and $z = \frac{-B'_{16}}{D'_{16}}$ for $\gamma_{xy} = 0$ plane. For $\varepsilon_x = 0$ plane: $0 = -B'_{11}M_x + B'_{12}M_x + B'_{16}M_x + z(D'_{11}M_x + D'_{12}M_x + D'_{16}M_{xy})$,

for $\varepsilon_y = 0$ plane:$0 = -B'_{12}M_x + B'_{22}M_x + B'_{26}M_x + z(D'_{12}M_x + D'_{22}M_x + D'_{26}M_{xy})$, and

for $\gamma_{xy} = 0$plane: $0 = B'_{16}M_x + B'_{26}M_x + B'_{66}M_x + z(D'_{16}M_x + D'_{26}M_x + D'_{66}M_{xy})$.

The constitutive equations to find E_x, only the x-direction from N_x and ϵ_x^0 relationship:

$$\begin{Bmatrix} N_x \\ 0 \\ 0 \\ - \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & | & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & | & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & | & B_{16} & B_{26} & B_{66} \\ - & - & - & | & - & - & - \\ B_{11} & B_{12} & B_{16} & | & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & | & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & | & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_y^0 \\ - \\ k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad \mathbf{(68)}$$

International Journal of Scientific Engineering and Applied Science (IJSEAS) – Volume-7, Issue-11, November 2021
ISSN: 2395-3470
www.ijseas.com

Using Cramer's rule to solve for $$\epsilon_x^0 = \frac{\begin{bmatrix} N_x & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ 0 & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ 0 & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ 0 & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ 0 & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ 0 & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}}{\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}} \quad \textbf{(69)}$$

Determinant of two 6x6 matrices are found from cofactor expression in the numerator:

$$\epsilon_x^0 = \frac{N_x \begin{bmatrix} A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{26} & A_{26} & B_{12} & B_{16} & B_{66} \\ B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}}{\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}} \frac{1}{h} \text{ gives } \frac{\frac{1}{h}N_x}{\epsilon_x^0} = E_x = \frac{\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \frac{1}{h}}{\begin{bmatrix} A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{26} & A_{26} & B_{12} & B_{16} & B_{66} \\ B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}} \quad \textbf{(70)}$$

E_y can be found in a similar manner, but denominator is different as solved for ϵ_x^0:

$$\frac{\frac{1}{h}N_y}{\epsilon_y^0} = E_y = \frac{\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \frac{1}{h}}{\begin{bmatrix} A_{11} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{16} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}} \quad \textbf{(71)}$$

G_{xy} is obtained as:

$$\frac{\frac{1}{h}N_{xy}}{\gamma_{xy}^0} = G_{xy} = \frac{\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \frac{1}{h}}{\begin{bmatrix} A_{11} & A_{12} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & B_{12} & B_{22} & B_{26} \\ B_{11} & B_{12} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & D_{16} & D_{26} & D_{66} \end{bmatrix}} \quad \textbf{(72)}$$

Poisson's ratio in x-direction for symmetric panels, ν_{xy}, contraction in y-direction,,

$$\nu_{xy} = \frac{-\varepsilon_y^0}{\varepsilon_x^0} = \frac{\begin{bmatrix} A_{12} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}}{\begin{bmatrix} A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}} \quad \textbf{(73)}$$

and ν_{yx}can be obtained as,

International Journal of Scientific Engineering and Applied Science (IJSEAS) – Volume-7, Issue-11, November 2021
ISSN: 2395-3470
www.ijseas.com

$$\nu_{yx} = \frac{-\varepsilon_x^0}{\varepsilon_y^0} = -\frac{\begin{bmatrix} A_{12} & A_{26} & B_{11} & B_{12} & B_{16} \\ A_{16} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}}{\begin{bmatrix} A_{11} & A_{16} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}} \quad (74)$$

2.5 Effective engineering constants of laminated composite panels

A laminate with 1-z axis drawn downward extending in the z-direction from H/2 to –H/2 is shown in **Figure 10**. The relationship of material properties, relative volume contents, and geometric arrangement of the constituent materials are computed from macro-mechanics laws. The quantities ε_y^0, k_y^0, γ_{xy}^0, and k_{xy}^0 referred to as the reference surface extensional strain in the y direction, the reference surface curvature in the y-direction, the reference surface in-plane shear strain, and the reference surface twisting curvature are also required.

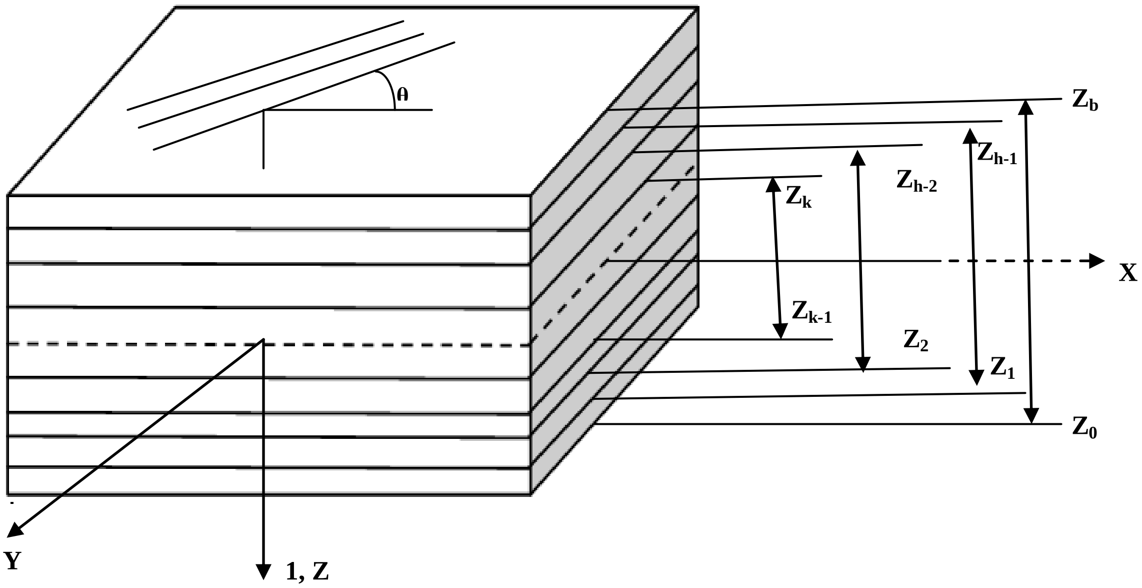


Figure 10: Schematic illustration of the geometry of laminated composite plate

Applying the second assumption of classical lamination theory, stresses, strains, and curvatures of the reference surface can be found if volume of the laminate is in a state of plane stress. The geometric mid-plane can be within a particular layer or at an interface between layers as shown in **Figure 11**. Referring to the ply at the most negative location as ply 1, the next layer in as layer 2, the ply at an arbitrary location as ply k, and the ply at the most positive z position as ply n. The locations of the ply interfaces are denoted by a subscribed z; the first ply is bound by locations h_0 and h_1, the second by h_1 and h_2, the kth ply by h_{k-1} and h_k, and the nth ply by h_{n-1} and

h_n. The thickness of the k[th] ply is denoted by h_k of the through-thickness coordinate, designated h, is located at the laminate geometric mid-plane. The geometric mid-plane may be within a particular ply or at an interface between plies. The ply k and ply k+1 are the same lamina but separated into two plies by the mid-plane:

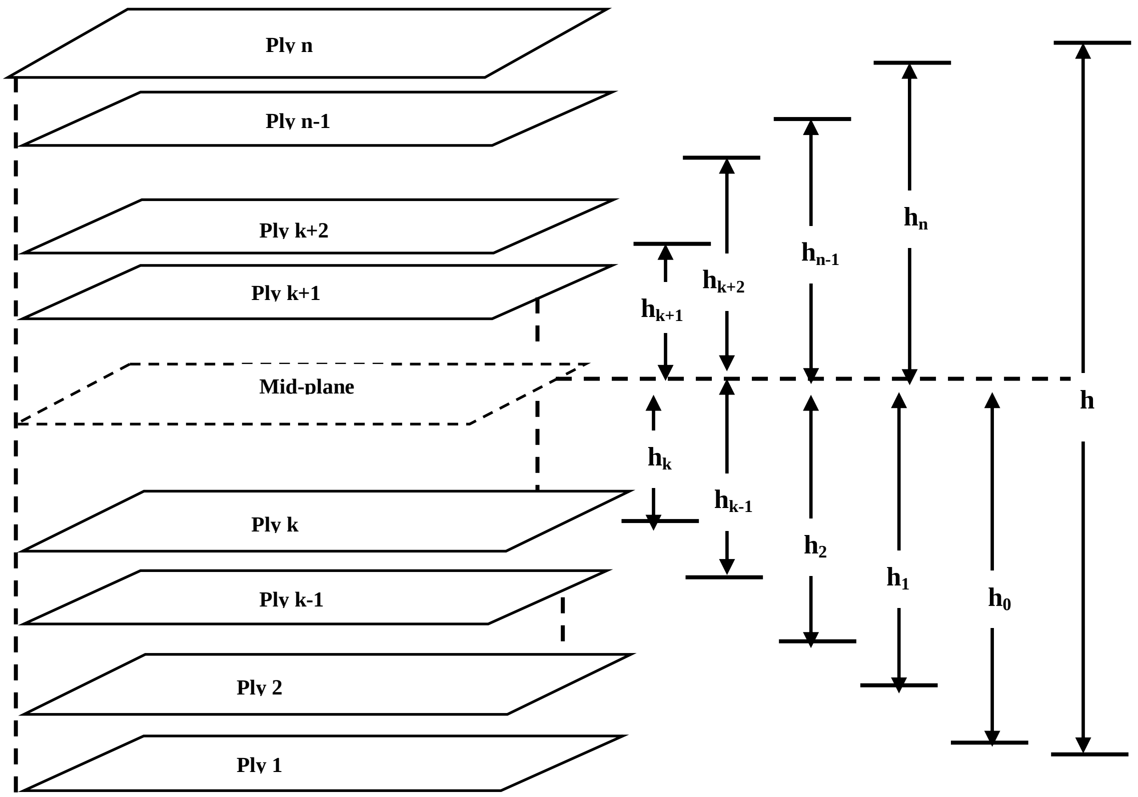


Figure 11: Ply-level schematic illustration of a laminate

The constitutive relations for an orthotropic material were writtenin terms of the stress and strain components that are referred to a coordinate system that coincides with the principal material coordinate system to transform constitutive equations from the material coordinates (1, 2, 3) of each layer to the coordinates (x, y, z) used to write the governing equations of a laminate Refer---related as follows:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = [T_{ij}] \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}, \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{12} \end{Bmatrix} = [T_{ij}] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix} \qquad \textbf{(75)}$$

Where transformation matrix is given as,

$$T_{ij}=\begin{bmatrix} m_1^2 & n_1^2 & p_1^2 & 2n_1p_1 & 2m_1p_1 & 2n_1m_1 \\ m_2^2 & n_2^2 & p_2^2 & 2n_2p_2 & 2m_2p_2 & 2n_2m_2 \\ m_3^2 & n_3^2 & p_3^2 & 2n_3p_3 & 2m_3p_3 & 2n_3m_3 \\ m_2m_3 & n_2n_3 & p_2p_3 & n_2p_3+p_2n_3 & p_2m_3+p_3m_2 & m_2n_3+n_2m_3 \\ m_1m_3 & n_1n_3 & p_1p_3 & n_3p_1+p_3n_1 & m_1p_3+p_1m_3 & n_1m_3+m_1n_3 \\ m_2m_1 & n_2n_1 & p_2p_1 & n_1p_2+p_1n_2 & m_2p_1+p_2m_1 & n_2m_1+m_2n_1 \end{bmatrix}$$

The transformation relations for angles ($\theta_{x1}\ \theta_{x2}\ \theta_{x3}$) measured from x-axis to axis 1-, 2-, 3-axes in terms of direction cosines m_i, n_i and p_i are given as:

$m_1 = \cos\theta_{x1}$; $m_2 = \cos\theta_{x2}$; $m_3 = \cos\theta_{x3}$

$n_1 = \cos\theta_{y1}$; $n_2 = \cos\theta_{y2}$; $n_3 = \cos\theta_{y3}$

$p_1 = \cos\theta_{z1}$; $p_2 = \cos\theta_{z2}$; $p_3 = \cos\theta_{z3}$

In an anisotropic material, properties are different in all directions so that the materials contain no planes of material property symmetry. Fibre-reinforced composites, in general, contain three orthogonal planes of material property symmetry, namely, the 1-2, 2-3, and 1-3 as shown in **Figure 12**. and are classified as orthogonal materials.

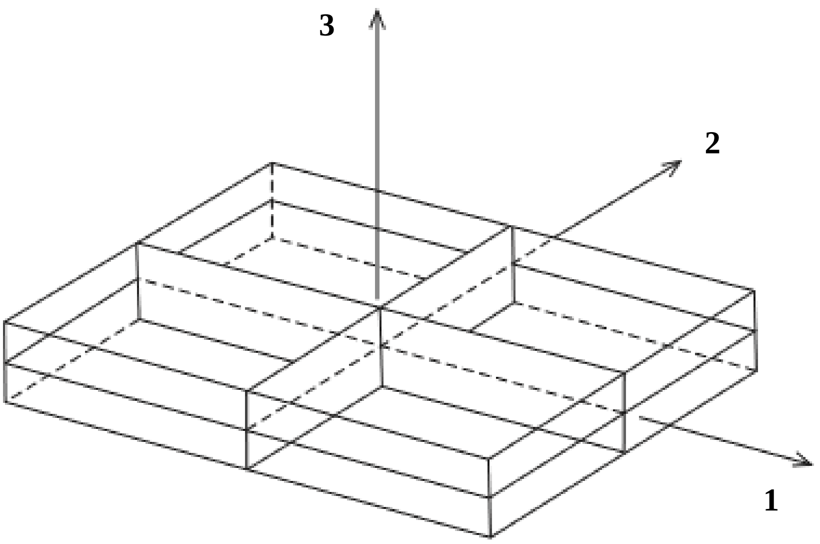


Figure 12: Three planes of symmetry

The stress-relation matrix reduces to,

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & Q_{16} \\ Q_{21} & Q_{22} & Q_{23} & 0 & 0 & Q_{26} \\ Q_{31} & Q_{32} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} & 0 \\ 0 & 0 & 0 & Q_{45} & Q_{55} & 0 \\ Q_{16} & Q_{26} & Q_{36} & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{Bmatrix} \quad (76)$$

Where $(\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy})$ are stress components, Q_{ij} are the reduced stiffness coefficients, and $(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy})$ are strain components. Strains are also second-order tensor quantities, transformation equations derived for stresses are also valid for tensor components of strains. However, the single-column formats for stress are not valid for single-column formats of strains because of the definitions: $2\varepsilon_{12} = \varepsilon_6$, $2\varepsilon_{13} = \varepsilon_5$, $2\varepsilon_{23} = \varepsilon_4$, thus modification yields relations for the engineering components of strains as presented for stress components.

$\bar{Q}_{13} = Q_{13}m^2 - 2Q_{36}mn + Q_{23}n^2$

$\bar{Q}_{23} = Q_{23}m^2 + 2Q_{36}mn + Q_{13}n^2$

$\bar{Q}_{33} = Q_{33}$

$\bar{Q}_{36} = (Q_{13} - Q_{23})mn + Q_{36}(m^2 - n^2)$

$\bar{Q}_{44} = Q_{44}m^2 + Q_{55}n^2 + 2Q_{45}mn$

$\bar{Q}_{45} = Q_{45}(m^2 - n^2) + (Q_{55} - Q_{44})mn$

$\bar{Q}_{55} = Q_{55}m^2 + Q_{44}n^2 - 2Q_{45}mn$

$\bar{Q}_{14} = Q_{14}m^3 + (Q_{15} - 2Q_{46})m^2n - (Q_{24} - 2Q_{56})mn^2 + Q_{25}n^3$

$\bar{Q}_{15} = Q_{15}m^3 - (Q_{45} + 2Q_{56})m^2n - (Q_{25} + 2Q_{46})n^2 - Q_{24}n^3$

$\bar{Q}_{24} = Q_{24}m^3 + (Q_{25} + 2Q_{46})m^2n + (Q_{14} + 2Q_{56})mn^2 + Q_{15}n^3$

$\bar{Q}_{25} = Q_{25}m^3 + (2Q_{56} - Q_{24})m^2n + (Q_{15} - 2Q_{46})mn^2 - Q_{14}n^3$

$\bar{Q}_{34} = Q_{34}m + Q_{35}n$

$\bar{Q}_{35} = Q_{35}m - Q_{34}n$

$\bar{Q}_{46} = Q_{46}m^3 + (Q_{56} + Q_{14} - Q_{24})m^2n + (Q_{15} - Q_{25} - Q_{46})mn^2 - Q_{56}n^3$

$\bar{Q}_{56} = Q_{56}m^3 + (Q_{15} - Q_{25} - Q_{46})m^2n + (Q_{24} - Q_{14} - Q_{56})mn^2 + Q_{46}n^3$

Where the $\bar{Q}_{ij}$ are the transformed elastic coefficients referred to the [x, y, z] coordinate system, which are related to the elastic coefficients in the material coordinates Q_{ij}. For an orthotropic material matrix [Q], the following additional stiffness components are required: $Q_{13} = \frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}}$, $Q_{44} = G_{23}$, $Q_{55} = G_{13} = G_{13} = Q_{66}$, $Q_{23} = \frac{\upsilon_{23}E_2}{1-\upsilon_{23}\upsilon_{23}} \frac{\upsilon_{32}E_3}{1-\upsilon_{33}\upsilon_{33}}$, while $Q_{14}, Q_{15}, Q_{16}, Q_{24}, Q_{25,} Q_{26,} Q_{34,} Q_{35,} Q_{36,} Q_{45}, Q_{46}$, and Q_{56} are equal to zero.

The transformed matrices provide a mean to convert stress components referred to the problem (laminate) coordinate system to the material (lamina) coordinate system. Thus, stress equation can be written as,

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & -2n \\ n^2 & m^2 & 0 & 0 & 0 & 2n \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & n & 0 \\ 0 & 0 & 0 & -n & m & 0 \\ nm & -nm & 0 & 0 & 0 & m^2-n^2 \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \quad \textbf{(77)}$$

The inverse relationships transformed reduced stiffness matrix can be used to compute ply level stresses. If nonlinear terms are considered, use the following stress equations:

$$\sigma_x = \bar{Q}_{11}\epsilon_x^0 + \bar{Q}_{12}\epsilon_y^0 + \bar{Q}_{16}\gamma_y^0 + \left(\bar{Q}_{11}k_x^1 + \bar{Q}_{12}k_y^1 + \bar{Q}_{16}k_{xy}^1\right)z$$

$$+\left(\bar{Q}_{11}k_x^2 + \bar{Q}_{12}k_y^2 + \bar{Q}_{16}k_{xy}^2\right)z^3 + L_n\left(\bar{Q}_{11}\epsilon_x^n + \bar{Q}_{12}\epsilon_y^n + \bar{Q}_{16}\gamma_{xy}^n\right)$$

$$\sigma_y = \bar{Q}_{12}\epsilon_x^0 + \bar{Q}_{22}\epsilon_y^0 + \bar{Q}_{26}\gamma_y^0 + \left(\bar{Q}_{12}k_x^1 + \bar{Q}_{22}k_y^1 + \bar{Q}_{26}k_{xy}^1\right)z$$

$$+\left(\bar{Q}_{12}k_x^2 + \bar{Q}_{22}k_y^2 + \bar{Q}_{26}k_{xy}^2\right)z^3 + L_n\left(\bar{Q}_{12}\epsilon_x^n + \bar{Q}_{22}\epsilon_y^n + \bar{Q}_{26}\gamma_{xy}^n\right)$$

$$\sigma_{xy} = \bar{Q}_{16}\epsilon_x^0 + \bar{Q}_{26}\epsilon_y^0 + \bar{Q}_{66}\gamma_{xy}^0 + \left(\bar{Q}_{16}k_x^1 + \bar{Q}_{26}k_y^1 + \bar{Q}_{66}k_{xy}^1\right)z +$$

$$\left(\bar{Q}_{16}k_x^2 + \bar{Q}_{26}k_y^2 + \bar{Q}_{66}k_{xy}^2\right)z^3 + L_n\left(\bar{Q}_{16}k_x^n + \bar{Q}_{26}k_y^n + \bar{Q}_{66}\gamma_{xy}^n\right)z^3$$

$$\sigma_{yz} = \bar{Q}_{44}\left(\gamma_{yz}^1 + z^2\gamma_{yz}^2\right) + \bar{Q}_{45}(\gamma_{xz}^1 + z^2\gamma_{xz}^2)$$

$$\sigma_{xz} = \bar{Q}_{45}\left(\gamma_{yz}^1 + z^2\gamma_{yz}^2\right) + \bar{Q}_{55}(\gamma_{xz}^1 + z^2\gamma_{xz}^2)$$

If nonlinear terms, L_n, and applied load $\left(P_x, P_y, P_{xy}\right)$ are considered, the constitutive equations becomes,

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ P_x \\ P_y \\ P_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} A_{12} A_{16} B_{11} B_{12} B_{16} E_{11}\, E_{12}\, E_{16} \\ A_{12} A_{22} A_{26} B_{12} B_{22} B_{26} E_{12}\, E_{22}\, E_{26} \\ A_{16} A_{26} A_{66} B_{16} B_{26} B_{66} E_{16}\, E_{26}\, E_{66} \\ B_{11} B_{12} B_{16} D_{11} D_{12} D_{16} F_{11}\, F_{12}\, F_{16} \\ B_{12} B_{22} B_{26} D_{12} D_{22} D_{26} F_{12}\, F_{22}\, F_{26} \\ B_{16} B_{26} B_{66} D_{16} D_{26} D_{66} F_{16}\, F_{26}\, F_{66} \\ E_{11}\, E_{12}\, E_{16}\, F_{11}\, F_{12}\, F_{12}\, H_{11} H_{12} H_{16} \\ E_{12}\, E_{22}\, E_{26}\, F_{12}\, F_{22}\, F_{26}\, H_{12} H_{22} H_{16} \\ E_{16}\, E_{26}\, E_{66}\, F_{16}\, F_{26}\, F_{66}\, H_{16} H_{26} H_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 + L_n\epsilon_x^n \\ \epsilon_y^0 + L_n\epsilon_y^n \\ \gamma_y^0 + L_n\gamma_y^n \\ k_x^1 \\ k_y^1 \\ k_{xy}^1 \\ k_x^2 \\ k_y^2 \\ k_{xy}^2 \end{Bmatrix} \quad \textbf{(78)}$$

If reduced transformed stiffness matrices and the ply thickness reference coordinate h_k can be combined, new stiffness matrices can be formed: [A], [B], [D], [E], [F], and [H],

$$\mathrm{A_{ij}} = \textstyle\sum_{k=1}^{n}\left[\bar{Q}_{ij}\right]_k(h_k - h_{k-1}) = \sum_{k=1}^{n}\left[\bar{Q}_{ij}\right]_k t_k,$$

$$\mathrm{B_{ij}} = \tfrac{1}{2}\textstyle\sum_{k=1}^{n}\left[\bar{Q}_{ij}\right]_k\left(h_k^2 - h_{k-1}^2\right), = \sum_{k=1}^{n}\left[\bar{Q}_{ij}\right]_k t_k \frac{(h_k+h_{k-1})}{2},$$

$$\mathrm{D_{ij}} = \tfrac{1}{3}\textstyle\sum_{k=1}^{n}\left[\bar{Q}_{ij}\right]_k\left(h_k^3 - h_{k-1}^3\right) = \tfrac{1}{3}\sum_{k=1}^{n}\left[\bar{Q}_{ij}\right]_k\left(t_k^3 + t\bar{z}_k{}^2\right),$$

$$\mathrm{E_{ij}} = \frac{1}{4}\sum_{k=1}^{n}\left[\bar{Q}_{ij}\right]_k\left(h_k^4 - h_{k-1}^4\right) = \frac{1}{4}\sum_{k=1}^{n}\left[\bar{Q}_{ij}\right]_k\left(t_k^4 + t\bar{z}_k{}^3\right)$$

International Journal of Scientific Engineering and Applied Science (IJSEAS) – Volume-7, Issue-11, November 2021
ISSN: 2395-3470
www.ijseas.com

$$F_{ij}=\frac{1}{5}\sum_{k=1}^{n}\left[\overline{Q}_{ij}\right]_k\left(Z_K^5-Z_{K-1}^5\right)=\frac{1}{5}\sum_{k=1}^{n}\left[\overline{Q}_{ij}\right]_k\left(t_k^5+t\bar{z}_k{}^4\right)$$

$$H_{ij}=\tfrac{1}{7}\textstyle\sum_{k=1}^{n}\left[\overline{Q}_{ij}\right]_k\left(Z_K^7-Z_{K-1}^7\right)=\tfrac{1}{7}\sum_{k=1}^{n}\left[\overline{Q}_{ij}\right]_k\left(t_k^7+t\bar{z}_k{}^6\right)\quad ; i, j =1, 2, 6; \tag{79}$$

The parameter, L_n, is solution character coefficient: $L_n = 0$ leaner and $L_n = 1$ non-linear solution. Applying unidirectional load in x-axis direction: the mid-plane strain $\{\varepsilon^0\}$; rotations of transverse normal curvatures $\{k^1\}$, and $\{k^2\}$; shear strains $\{\gamma^1\}, \{\gamma^2\}$; *and normal strain* $\{\varepsilon^n\}$; the equation (88) becomes,

$$\begin{Bmatrix} N_x\\0\\0\\0\\0\\0\\0\\0\\0 \end{Bmatrix}=\begin{bmatrix} A_{11}A_{12}A_{16}B_{11}B_{12}B_{16}E_{11}\,E_{12}\,E_{16}\\ A_{12}A_{22}A_{26}B_{12}B_{22}B_{26}E_{12}\,E_{22}\,E_{26}\\ A_{16}A_{26}A_{66}B_{16}B_{26}B_{66}E_{16}\,E_{26}\,E_{66}\\ B_{11}B_{12}B_{16}D_{11}D_{12}D_{16}F_{11}\,F_{12}\,F_{16}\\ B_{12}B_{22}B_{26}D_{12}D_{22}D_{26}F_{12}\,F_{22}\,F_{26}\\ B_{16}B_{26}B_{66}D_{16}D_{26}D_{66}F_{16}\,F_{26}\,F_{66}\\ E_{11}\,E_{12}\,E_{16}\,F_{11}\,F_{12}\,F_{12}\,H_{11}H_{12}H_{16}\\ E_{12}\,E_{22}\,E_{26}\,F_{12}\,F_{22}\,F_{26}\,H_{12}H_{22}H_{16}\\ E_{16}\,E_{26}\,E_{66}\,F_{16}\,F_{26}\,F_{66}\,H_{16}H_{26}H_{66} \end{bmatrix}\begin{Bmatrix} \epsilon_x^0+L_n\epsilon_x^n\\ \epsilon_y^0+L_n\epsilon_y^n\\ \gamma_y^0+L_n\gamma_y^n\\ k_x^1\\ k_y^1\\ k_{xy}^1\\ k_x^2\\ k_y^2\\ k_{xy}^2 \end{Bmatrix} \tag{80}$$

Using the relation

$$\left(h_k^3-h_{k-1}^3\right)=\left[(h_k-h_{k-1})^3+3(h_k-h_{k-1})(h_k+h_{k-1})^2-3\left(h_k^3-h_{k-1}^3\right)\right]$$
$$= t_k^3+12t\bar{z}_k{}^2$$

Where h_{j-1}; distance from the mid-plane to the top of the j^th lamina, h_j: distance from the mid-plane to the bottom of the j^th lamina, and thickness of the k^th lamina denoted by t_k and $\bar{z}_k = \frac{(h_k+h_{k-1})}{2}$.. Using symmetry of rotations and putting Ln =0, the components of the reduced stiffness matrix equate to the load-induced matrix as follows:

$$\begin{bmatrix} A_{11}A_{12}A_{16}B_{11}B_{12}B_{16}E_{11}\,E_{12}\,E_{16}\\ A_{12}A_{22}A_{26}B_{12}B_{22}B_{26}E_{12}\,E_{22}\,E_{26}\\ A_{16}A_{26}A_{66}B_{16}B_{26}B_{66}E_{16}\,E_{26}\,E_{66}\\ B_{11}B_{12}B_{16}D_{11}D_{12}D_{16}F_{11}\,F_{12}\,F_{16}\\ B_{12}B_{22}B_{26}D_{12}D_{22}D_{26}F_{12}\,F_{22}\,F_{26}\\ B_{16}B_{26}B_{66}D_{16}D_{26}D_{66}F_{16}\,F_{26}\,F_{66}\\ E_{11}\,E_{12}\,E_{16}\,F_{11}\,F_{12}\,F_{12}\,H_{11}H_{12}H_{16}\\ E_{12}\,E_{22}\,E_{26}\,F_{12}\,F_{22}\,F_{26}\,H_{12}H_{22}H_{16}\\ E_{16}\,E_{26}\,E_{66}\,F_{16}\,F_{26}\,F_{66}\,H_{16}H_{26}H_{66} \end{bmatrix}=\begin{bmatrix} N_xA_{12}A_{16}B_{11}B_{12}B_{16}E_{11}\,E_{12}\,E_{16}\\ 0\;A_{22}A_{26}B_{12}B_{22}B_{26}E_{12}\,E_{22}\,E_{26}\\ 0\;A_{26}A_{66}B_{16}B_{26}B_{66}E_{16}\,E_{26}\,E_{66}\\ 0\;B_{12}B_{16}D_{11}D_{12}D_{16}F_{11}\,F_{12}\,F_{16}\\ 0\;B_{22}B_{26}D_{12}D_{22}D_{26}F_{12}\,F_{22}\,F_{26}\\ 0\;B_{26}B_{66}D_{16}D_{26}D_{66}F_{16}\,F_{26}\,F_{66}\\ 0\;E_{12}\,E_{16}\,F_{11}\,F_{12}\,F_{12}\,H_{11}H_{12}H_{16}\\ 0\;E_{22}\,E_{26}\,F_{12}\,F_{22}\,F_{26}\,H_{12}H_{22}H_{16}\\ 0\;E_{26}\,E_{66}\,F_{16}\,F_{26}\,F_{66}\,H_{16}H_{26}H_{66} \end{bmatrix} \tag{81}$$

The equation of mid-plane strain is obtained as,

$$\epsilon_x^0 = \frac{\begin{bmatrix} N_x & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & E_{11} & E_{12} & E_{16} \\ 0 & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & E_{12} & E_{22} & E_{26} \\ 0 & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & E_{16} & E_{26} & E_{66} \\ 0 & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & F_{11} & F_{12} & F_{16} \\ 0 & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & F_{12} & F_{22} & F_{26} \\ 0 & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & F_{16} & F_{26} & F_{66} \\ 0 & E_{12} & E_{16} & F_{11} & F_{12} & F_{12} & H_{11} & H_{12} & H_{16} \\ 0 & E_{22} & E_{26} & F_{12} & F_{22} & F_{26} & H_{12} & H_{22} & H_{16} \\ 0 & E_{26} & E_{66} & F_{16} & F_{26} & F_{66} & H_{16} & H_{26} & H_{66} \end{bmatrix}}{\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & E_{11} & E_{12} & E_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & E_{12} & E_{22} & E_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & E_{16} & E_{26} & E_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & F_{11} & F_{12} & F_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & F_{12} & F_{22} & F_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & F_{16} & F_{26} & F_{66} \\ E_{11} & E_{12} & E_{16} & F_{11} & F_{12} & F_{12} & H_{11} & H_{12} & H_{16} \\ E_{12} & E_{22} & E_{26} & F_{12} & F_{22} & F_{26} & H_{12} & H_{22} & H_{16} \\ E_{16} & E_{26} & E_{66} & F_{16} & F_{26} & F_{66} & H_{16} & H_{26} & H_{66} \end{bmatrix}} \frac{1}{h}, E_x = \frac{N_x/h}{\epsilon_x^0} = \frac{\begin{bmatrix} A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & E_{12} & E_{22} & E_{26} \\ A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & E_{16} & E_{26} & E_{66} \\ B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & F_{11} & F_{12} & F_{16} \\ B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & F_{12} & F_{22} & F_{26} \\ B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & F_{16} & F_{26} & F_{66} \\ E_{12} & E_{16} & F_{11} & F_{12} & F_{16} & H_{11} & H_{12} & H_{16} \\ E_{22} & E_{26} & F_{12} & F_{22} & F_{26} & H_{12} & H_{22} & H_{26} \\ E_{26} & E_{66} & F_{16} & F_{26} & F_{66} & H_{16} & H_{26} & H_{66} \end{bmatrix}}{\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & E_{11} & E_{12} & E_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & E_{12} & E_{22} & E_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & E_{16} & E_{26} & E_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & F_{11} & F_{12} & F_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & F_{12} & F_{22} & F_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & F_{16} & F_{26} & F_{66} \\ E_{11} & E_{12} & E_{16} & F_{11} & F_{12} & F_{12} & H_{11} & H_{12} & H_{16} \\ E_{12} & E_{22} & E_{26} & F_{12} & F_{22} & F_{26} & H_{12} & H_{22} & H_{16} \\ E_{16} & E_{26} & E_{66} & F_{16} & F_{26} & F_{66} & H_{16} & H_{26} & H_{66} \end{bmatrix}} \quad \textbf{(82)}$$

For the effective engineering constants, the stress-stain in each ply can be calculated from equation (31), and transformed into the principal material direction, and integrated to obtain resultant effective stresses,

$$\sigma_x = \frac{1}{H}\int_{-1/H}^{1/H} \sigma_x \, dz, \quad \sigma_y = \frac{1}{H}\int_{-1/H}^{1/H} \sigma_y \, dz, \text{ and } \tau_{xy} = \frac{1}{H}\int_{-1/H}^{1/H} \tau_{xy} \, dz \quad \textbf{(83)}$$

Stresses can be obtained from the effective force resultants:

$$\sigma_x = \frac{1}{H} N_x, \quad \sigma_y = \frac{1}{H} N_y, \text{ and } \tau_{xy} = \frac{1}{H} N_{xy} \quad \textbf{(84)}$$

Solving and substituting the results,

$$\begin{Bmatrix} N_x \\ N_y \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \end{Bmatrix}, \; N_{xy} = A_{66}\gamma_x^0 \quad \textbf{(85)}$$

The effective mid-plane strains can be obtained from A_{ij} terms of the matrix [A],

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \tau_{xy}^0 \end{Bmatrix} = \begin{bmatrix} a_{11}H & a_{11}H & 0 \\ a_{11}H & a_{22}H & 0 \\ 0 & 0 & a_{66}H \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad \textbf{(86)}$$

The matrix 3x3 is defined as the laminate compliance matrix for symmetric balanced laminates. Therefore, by analogy constitutive equations (reduced compliances), the following effective engineering constants can be obtained:

$$E_x = \frac{1}{a_{11}H}, \quad E_y = \frac{1}{a_{22}H}, \; G_{xy} = \frac{1}{a_{66}H}, \; \nu_{xy} = \frac{a_{12}}{a_{11}}, \; \nu_{yx} = \frac{a_{12}}{a_{22}} \quad \textbf{(87)}$$

Coefficients of mutual influence are defined by analogy of Poisson's ratio if lamina is under coupled loading, the ratio of a shear strain to an extensional strain if ($\sigma_x \neq 0$) as the coefficient of mutual influence of the second kind $\eta_{xy,x} = \frac{\gamma_{xy}}{\varepsilon_x}$ and if ($\sigma_y \neq 0$) another as $\eta_{xy,y} = \frac{\gamma_{xy}}{\varepsilon_y}$.

The coefficients of mutual influence of the first kind are defined if ($\gamma_{xy} \neq 0$)as $\eta_{x,xy} = \frac{\varepsilon_x}{\gamma_{xy}}$ and

$\eta_{y,xy} = \frac{\varepsilon_y}{\gamma_{xy}}$. Mutual influence coefficients can be found from superposition of loading, stress-strain relations in terms of elastic constants,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_s \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{sx}}{G_{xy}} \\ -\frac{\nu_{yx}}{E_{xx}} & \frac{1}{E_{yy}} & \frac{\eta_{sy}}{G_{xy}} \\ \frac{\eta_{sx}}{G_{xy}} & \frac{\eta_{sy}}{G_{xy}} & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{Bmatrix} \quad \textbf{(88)}$$

Elastic constants derived from equation (88) in global coordinates become:

$$E_x = 1 \Big/ \left[\frac{m^2}{E_{11}}(m^2 - n^2\nu_{12}) + \frac{n^2}{E_{22}}(n^2 - m^2\nu_{12}) + \frac{m^2n^2}{G_{12}}\right] \quad \textbf{(89)}$$

$$E_y = 1 \Big/ \left[\frac{n^2}{E_{11}}(n^2 - m^2\nu_{12}) + \frac{m^2}{E_{22}}(m^2 - n^2\nu_{21}) + \frac{m^2n^2}{G_{12}}\right] \quad \textbf{(90)}$$

$$G_{xy} = 1 \Big/ \left[\frac{4m^2n^2}{E_{11}}(1 + \nu_{12}) + \frac{4m^2n^2}{E_{22}}(1 + \nu_{21}) + \frac{(m^2-n^2)^2}{G_{12}}\right] \quad \textbf{(91)}$$

$$\nu_{xy} = E_{xx} \Big/ \left[\frac{m^2}{E_{11}}(m^2\nu_{12} - n^2) + \frac{n^2}{E_{22}}(n^2\nu_{12} - m^2) + \frac{m^2n^2}{G_{12}}\right] \quad \textbf{(92)}$$

$$\nu_{yx} = E_{yy}\nu_{xy} \Big/ E_{xx} \quad \textbf{(93)}$$

$$\eta_{xs} \big/ E_{xx} = \eta_{sx} \big/ G_{xy} = \left[\frac{2m^3n}{E_{11}}(1 + \nu_{12}) - \frac{2mn^3}{E_{22}}(1 + \nu_{21}) - \frac{mn(m^2-n^2)}{G_{12}}\right] \quad \textbf{(94)}$$

$$\eta_{ys} \big/ E_{yy} = \eta_{sy} \big/ G_{xy} = \left[\frac{2mn^3}{E_{11}}(1 + \nu_{12}) - \frac{2nm^3}{E_{22}}(1 + \nu_{21}) + \frac{mn(m^2-n^2)}{G_{12}}\right] \quad \textbf{(95)}$$

The effective engineering constants of laminated structural elements beam/plate panels of thickness H consisting of N plies rotated at angles θ_i can be determined as follows:

$$\bar{E}_1 = \frac{1}{HN}\sum_{i=0}^{N} E_1(\theta_i),\ \bar{E}_2 = \frac{1}{HN}\sum_{i=0}^{N} E_2(\theta_i),\ \bar{G}_{12} = \frac{1}{HN}\sum_{i=0}^{N} G_{12}(\theta_i),$$

$$\bar{\nu}_{12} = \frac{1}{HN}\sum_{i=0}^{N} \nu_{12}(\theta_i)\ \bar{\nu}_{21} = \frac{1}{HN}\sum_{i=0}^{N} \nu_{21}(\theta_i) \quad \textbf{(96)}$$

3 Results and discussions

Computer programs were developed in commercial software MATLAB™ software to approximate the engineering constants for symmetric, non-symmetric, and effective panels. The programs are capable to execute and predict engineering constants for laminated structural elements consisting of plies of variable ply-thickness and number of plies for given material properties. Selected cases are being presented with brief discussions.

3.1 Engineering constants for symmetric and non-symmetric panels

Since elastic constants are interdependent, elastic constants only in the fibres parallel directions are shown in all cases. The following engineering constant, E_x, were predicted by computer simulations for symmetric and non-symmetric panels:

a) for symmetric panel, $E_x = 11333000\,\frac{lb}{in^2}$ and

b) for non-symmetric panel, $E_x = 5839000\,\frac{lb}{in^2}$.

The other engineering constants can be predicted executing the same program. As expected, the predicted quantities of the engineering constants exactly agreed to the result data available in the literature.

3.2 Effective engineering constants

3.2.1 Engineering constants of uniform ply thickness

Computer programs were also executed to predict the effective engineering constants: Young's and shear moduli, Poisson's ratios, and coefficients of mutual influence. The selected engineering constants were plotted as functions against ply orientation angles range: $-{}^{\pi}/_{2} \leq \theta \leq {}^{\pi}/_{2}$ at the step difference of 10^0. All plots exhibit mirror images of quantities about mid-planes zero-position through thick: $-90^0 < 0^0$ and $0^0 < 90^0$. Thus, variations of plotted quantities against $0^0 < 90^0$ angles are being briefly discussed below.

Young's modulus parallel to x-direction, E_x, is depicted in **Figure 13**.The quantities make bell type curve with maximum value at 0^0 angle along the fibre direction. However, it abruptly decreases as angle increases within the 0^0-20^0 range, and proceeds almost smoothly along increase in fibre orientations up to 90^0. Plot of modulus, E_y, quantities in y-direction, the perpendicular to fibre direction is depicted in **Figure 14**. The plot illustrates almost no change and smoothly continues within the range 0-70^0. However, its behaviour abruptly changes afterward following steep increase up to 90^0.

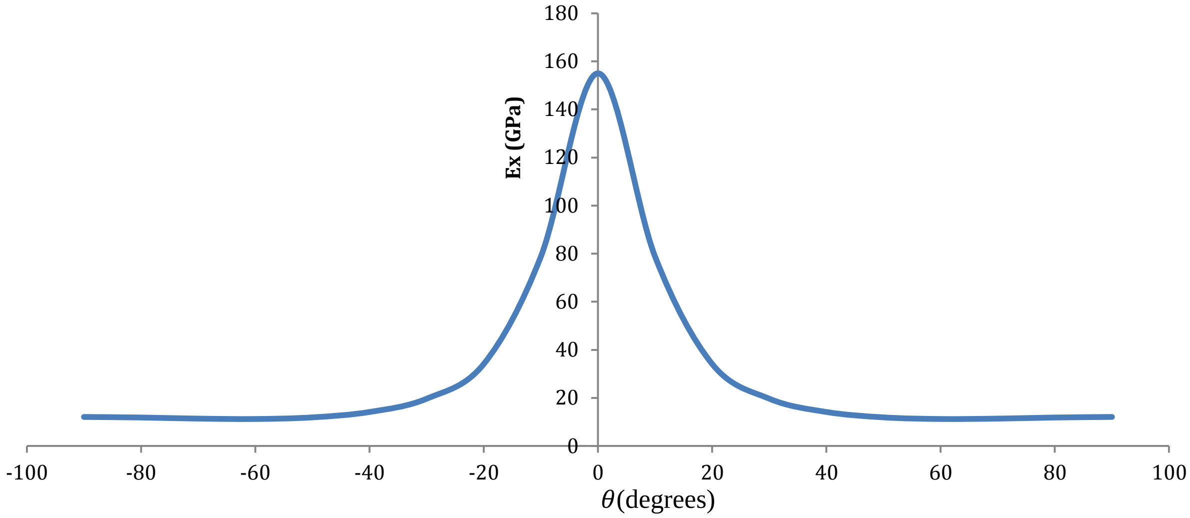


Figure 13: Variation of E_x versus angle

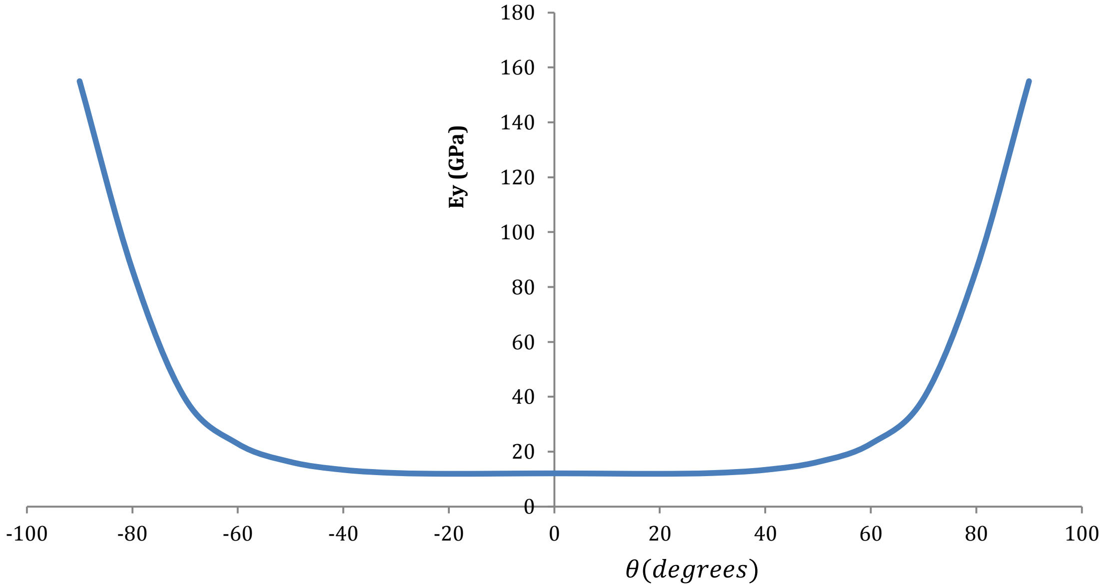


Figure 14: Variation of E_y versus angle

Plot of Poisson's ratios, v_{xy}, is depicted in **Figure 15**.The a significant drop can be seen in the curve of minimal quantity at 0^0 and abrupt increase within the range 0^0-20^0. Rest of the curve shows steady downward trends until it reaches 90^0. Plot of Poisson's ratios, v_{yx}, is depicted in **Figure 16**. The curve shows a continuous slight increase within the range 0^0 -80^0. After that point, rest of the curve shows significant increasing trends until it reaches 90^0.

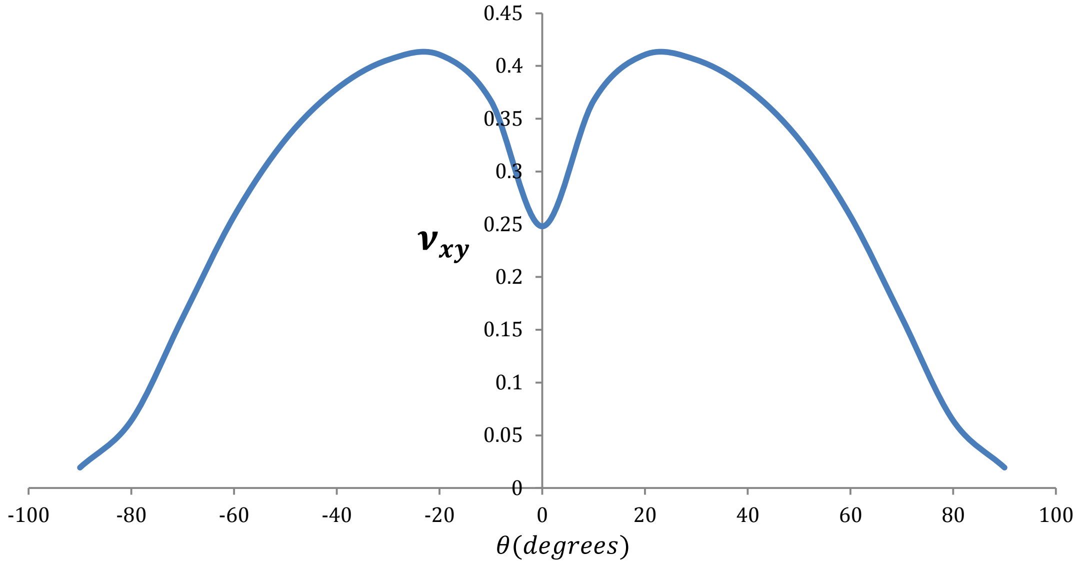


Figure 15: Variation of ν_{xy}versus angle

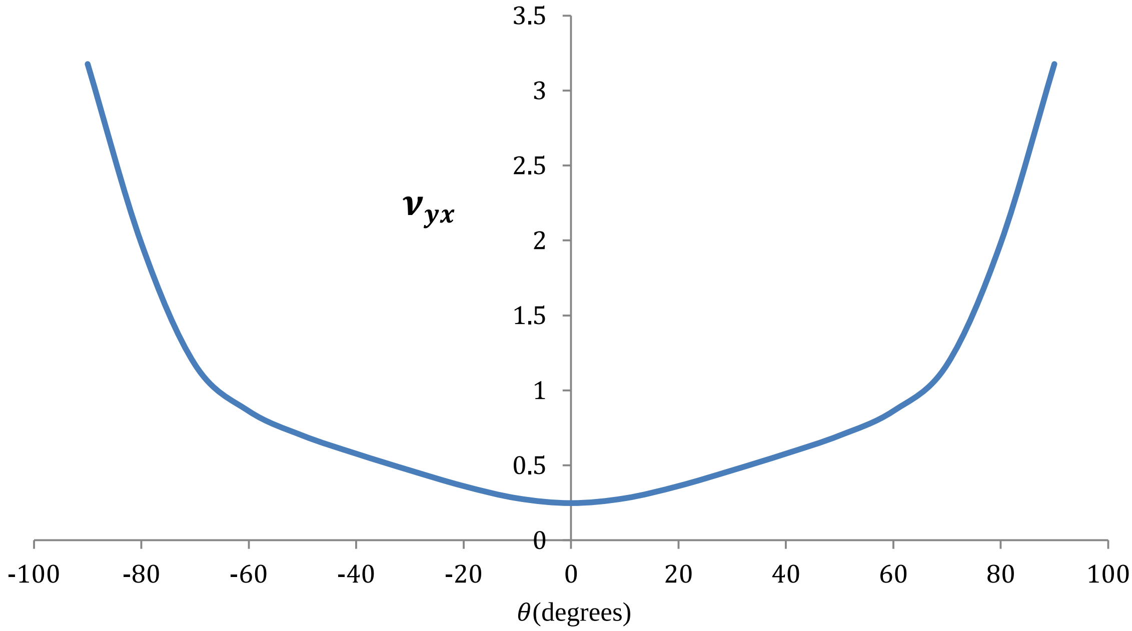


Figure 16: Variation of ν_{yx}versus angle

Plot of shear modulus (G_{xy}) shows a wave like curve as depicted in **Figure 17**. The quantities at 0^0 show almost a minimum, and curve shows increasing trends up to 50^0, after that point the curve shows steady downward trends until it reaches 90^0.

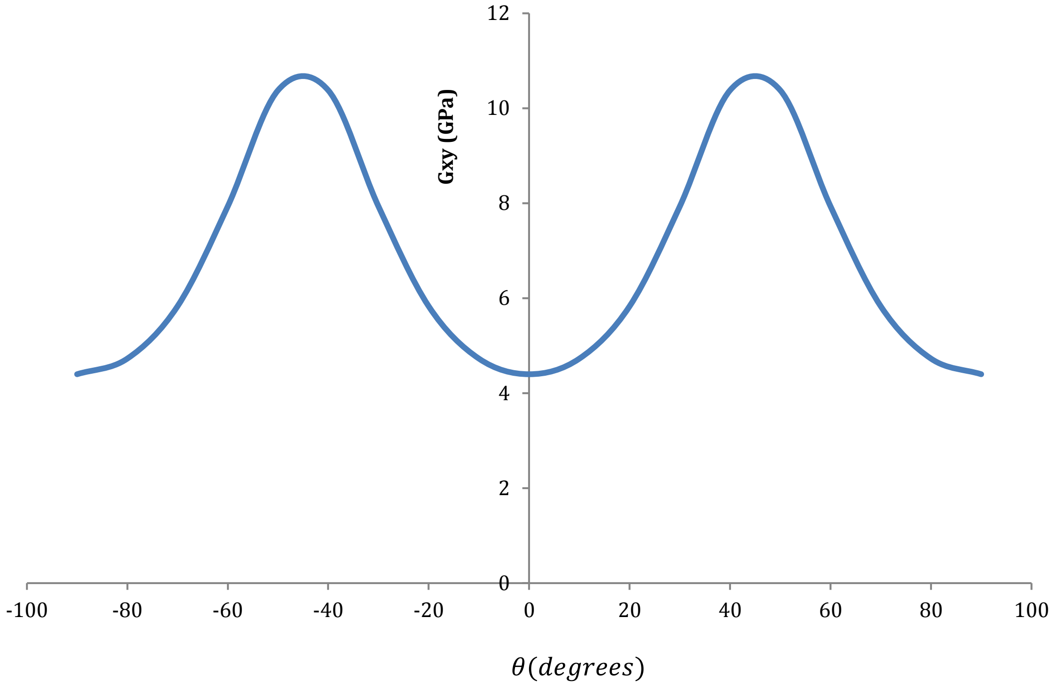


Figure 17: Variation of G_{xy} versus angle

Plot of mutual influence coefficient $\eta_{xy,x}$ depicted in **Figure 18**. The quantities at 0^0 shows almost zero value. The quantities make decreasing trends with increasing angles up to $\pm 20^0$, after that point the curve shows reverse trends, and quantities show increase with increasing angles until reaching again zero at 50^0. At that point, curve shows increasing trends up to 750^0 and afterwards decreasing trends until it ends at 90^0.

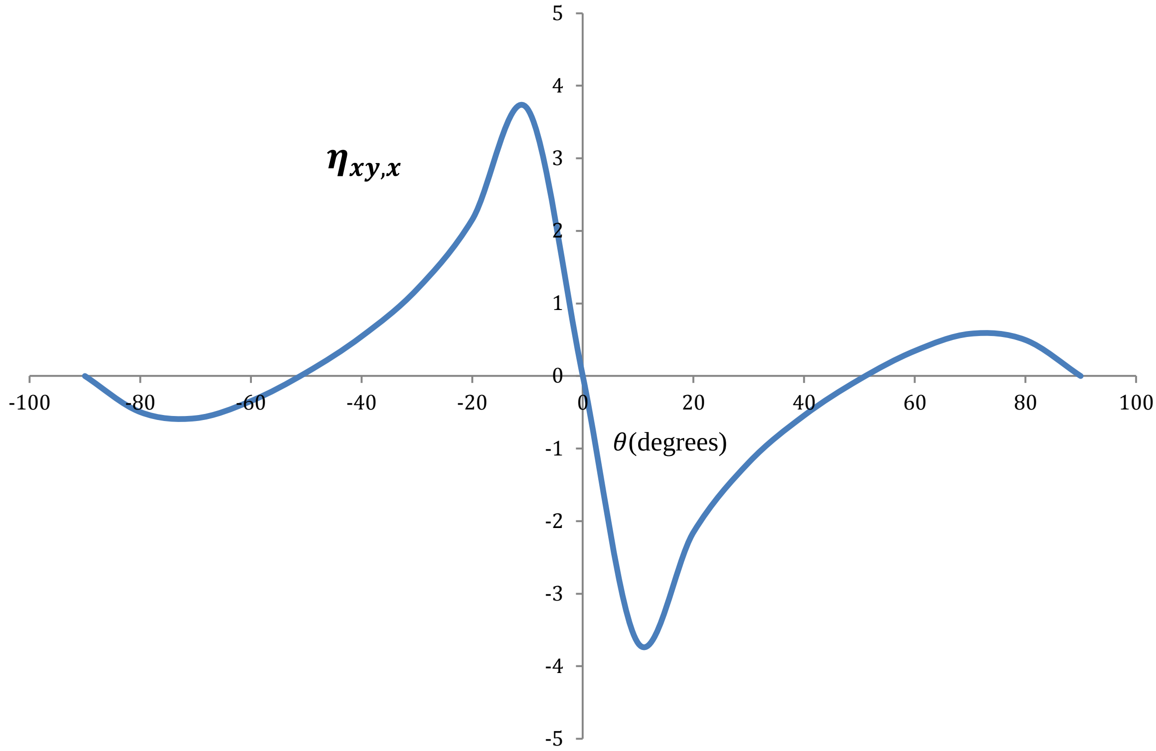


Figure 18: Variation of $\eta_{xy,x}$ versus angle

Plot of mutual influence coefficient $\eta_{xy,y}$ is depicted in **Figure 19**. The curve shows increase in quantities with increased in angle up to 20^0, and decreasing trends against increase in angles within the range 40^0-75^0. After that point curve shows abrupt increase as angle increases until it reaches zero value at 90^0.

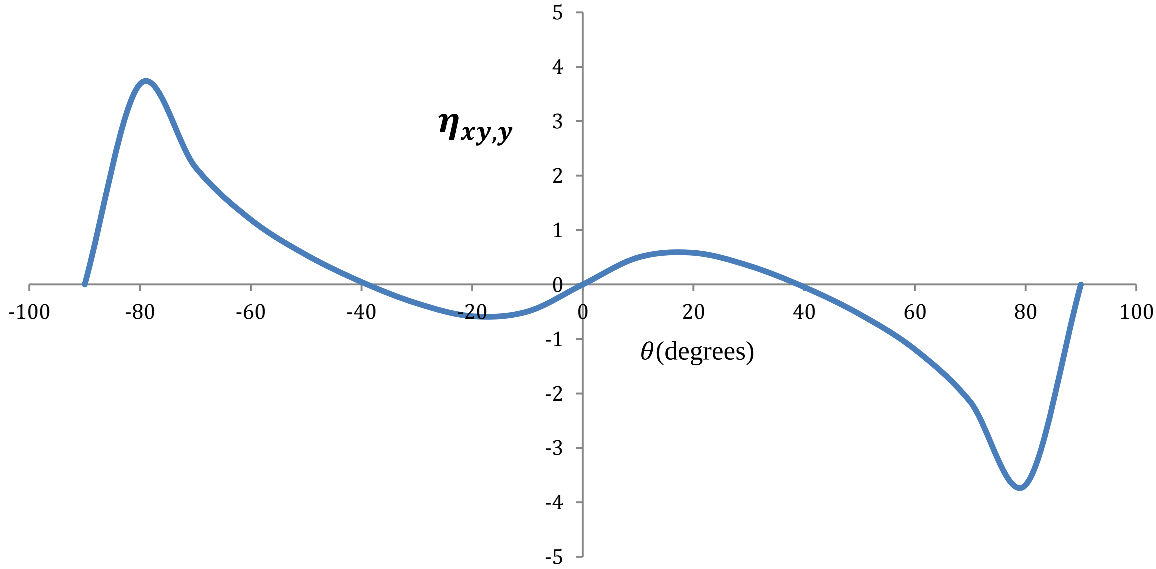


Figure 19: Variation of $\eta_{xy,y}$ versus angle

3.2.2 Effective engineering constants for variable ply thickness

A six-layer $[\pm 30/0]_S$ graphite composite polymer matrix laminate was considered with mechanical properties in **Table 2**. The laminate has total thickness of 0.9 mm. The four layers are of equal thickness. MATLAB code was executed to determine the five effective constants for the laminate. The distances $z_k (k = 1, 2, 3, 4, 5, 6)$ are calculated as follows: $z_1 = -0.45, z_2 = -0.3, z_3 = -0.15, z_0 = 0, z_4 = 0.15, z_5 = 0.3, z_6 = 0.45$. And five calls were made to the five MATLAB™ functions to calculate the five effective constants:

$\bar{E}_x = 83.8, \bar{E}_y = 12.5, \bar{v}_{xy} = 1.3, \bar{v}_{yx} = 0.193, \bar{G}_x y = 42.8$

Predicted values of elastic constants were compared with equivalent quasi-isotropic beam and plate panels [29]. The flexural modulus of the laminated beam depending on the ply stacking sequence and moduli in **Table 3** were utilised. The ply index lay-up, angles, and multiples for selected 8-Ply panel are shown in **Table 4**.

Table 4: Lay-up, angle, and modulus of 8-Ply laminate

Ply index, t t:N-n, n=0:N-1	$t^3 - (t-1)^3$	Ply angle	Modulus, $E^{(t)}$
8	[169]	0	59
7	[127]	90	59
6	[91]	45	6
5	[61]	-45	6 Neutral axis
4	[37]	0	59
3	[19]	90	59
2	[7]	45	6
1	[1]	-45	6

Utilising the equation, $\boldsymbol{E_x = \frac{8}{N^3}\sum_{t=c+1}^{\frac{N}{2}} E^{(t)}(t^3 - (t-1)^3)}$, where subscript c = 0 being the starting at lower-half of the neutral axis. Progressions of the values:

6[1]+6[7]+59[19]+59[37] equate to 3352/64 = 52.4**.** From ply-laminate ratios the predicted elastic constant in x-direction gives, **76.2 GPa.** The engineering constants for the other panels can be determined following the same procedure.

4 Conclusions

In this work, mathematical formulations have been described in detail to calculate effective engineering constants for isotropic, orthotropic, and three-dimensional panels using micro-mechanics methods. Computer programs were developed into MATLAB™ software to predict the effective engineering constants. The following conclusions can be extracted:

- Mathematical formulations of the fibrous composite panels were described in detail from ply-level to stack of plies using micro-macro mechanics methods. Relationships of loading directions for isotropic, orthotropic, and anisotropic materials to fibres aligned directions in local-global coordinates were also presented.
- Effects of coefficients of mutual influences and coupled loading were also considered in formulations of effective engineering constants.
- Simulation produced results were compared and validated against the data results available in the literature and found to be within acceptable range of (±10%) deviations.

Based on comparison of the results, the present study proposed efficient and reliable simulation model to predict engineering constants for variable-ply thickness panel based on micro-macro mechanics methods. The study could be modified and extended to study similar materials and cases coupling hygro-thermal environmental loadings.

References

[1] Morgan P. Carbon Fibres and their composites. Taylor & Francis, Boca Raton, USA (2005).

[2] Naik, N. S., and Sayyad, A. S. 1D analysis of laminated composite and sandwich plates using a new fifth order plate theory. Latin American Journal of Solids and Structures 2018; 15:1-17.

[3] Vijayakumar K. On Uniform Approximate Solutions in Bending of Symmetric Laminated Plates". CMC: Computers, Materials & Continua 2013; 34 (1):1-25.

[4] John EL, XiaowenY, Mark IJ. Characterisation of voids in fibre reinforced composite materials. NDT&E International 2012;46:122–127.

[5] Khayal Mohammed Elmardi. Buckling Analysis of Thin Laminated Composite Plates using Finite Element Method', International Journal of Engineering Research And Advanced Technology 2017;volume 3, issue 3:1-18.

[6] Hussain SA, Reddy SB and V, and Reddy VN. Prediction of elastic properties of FRP composite lamina for longitudinal loading. Asian Research Publishing Network (ARPN) 2008;3(6):70-75.

[7] Keshavamurthy YC, Nanjundaradhya NV, Ramesh SS, Kulkarni RS. Investigation of tensile properties of fibre reinforced angle ply laminated composites. Int J Emerg Technol Adv Eng 2012;2(4):700-703.

[8] Naika N S. Sayyada, A. S.. 2D analysis of laminated composite and sandwich plates using a new fifth-order plate theory. Latin American Journal of Solids and Structures, 2018, 15(9):1-27.

[9] Altaf HS, PandurangaduV, Amba PRG. Prediction of elastic constants of carbon T300/ epoxy composite using soft computing. Int J Inn Research in Sci Eng Technol 2013;2(7): 2762-2770.

[10] Satish KG, Siddeswarappa B, Mohamed KK. Characterization of in-plane mechanical properties of laminated hybrid composites. J Miner Mater Character Eng 2010;9(2):105-114.

[11] Irina P, Cristina M, Constant I. The determination of Young modulus for cfrp using three point bending tests at different span lengths. Scientific Bulletin-University Politehnica of Bucharest 2013;75:121-128.

[12] By Angelo R, Maligno L. Finite element investigations on the microstructure of composite materials. PhD thesis The University of Nottingham, UK (2007).

[13] Khayal Mohammed Elmardi, Mahdi IE. Vibration of laminated composite decks beams. International Journal of Engineering Applied Sciences and Technology 2018; volume 3, issue 1, ISSN No. 2455-2143; 61-66.

[14] Konrad Gliesche, Tamara Hubner, Holger Orawetz. Investigations of in-plane shear properties of $\pm 45^0$ carbon/epoxy composites using tensile testing and optical deformation analysis. Compos Sci Technol 2005;65:163-171.

[15] Liang JZ. Predictions of Young's modulus of short inorganic fibre reinforced polymer composites. Composites: Part B 2012;2001- 2012.

[16] Kaza Vijayakumar. Extended Poisson Theory For Analysis Of Laminated Plates Running Title: Extended Poisson Theory of plates. Journal of Multidisciplinary Engineering Science and Technology 2016; Vol. 3 Issue 2, ISSN: 3159-0040.

[17] Eric Jason Booth. Progress and challenges in non-linear constitutive modelling for composite materials. MSc thesis, Montana State University, USA (2007).

[18] Ali S. Humod and Muhannad Al_Waily. Experimental and Analytical Study of Tensile Properties for Hybrid Composite Material. The Iraqi Journal for Mechanical and Material Engineering 2010; Vol. 10, no. 1: 101-14.

[19] Shaofan L, Gang W. Introduction to micromechanics and nano-mechanics, World Scientific Pub., ISBN 978-981-281-413-2; (2008).

[20] Abdalla F. H., Megat M. H., Sapuan M. S. and B. B. Sahari 'Determination of Volume Fraction Values of filament Wound Glass and Carbon Fiber Reinforced Composites' ARPN Journal of Engineering and Applied Sciences 2008; Vol. 3, no. 4: 7-11.

[21] Suleiman Mohammed, Osman Mahmoud, and Hassan Tagelsir. Biaxial Buckling of Thin Laminated Composite Plates, International Journal of Engineering Applied Sciences and Technology 2016; volume 1, issue 8:1- 6.

[22] Laszlo P. Kollar and George S. Springer 'Mechanics of Composite Structures' Book, Cambridge University Press, 2003.

[23] Khayal Mohammed Elmardi. Hassan T. Mathematical modelling of laminated plates on buckling. International Journal of Bridge Engineering 2018; volume 6, issue 2:25-39.

[24] Valery V. Vasiliev and Evgeny V. Morozov 'Advanced Mechanics of Composite Materials, Second Edition' Book, Elsevier Ltd. All Rights Reserved, 2007.

[25] Valery V. Vasiliev and Evgeny V. Morozov. Mechanics and Analysis of Composite Materials. Book, Elsevier Science Ltd, 2001.

[26] U. Farooq, P. Myler. Damage and energy absorption behaviour of low-velocity impacted fibre-reinforced composite panels using analytical and data filtering techniques. International Journal of Recent Research and Applied Studies 2021; Volume 8, Issue 4(3): 13-27.

[27] U. Farooq, P. Myler. Stress base modelling and computer simulation of drop-weight impact response of composite panels with domain partitioning and adaptive meshing technique. International Journal of Recent Research and Applied Studies 2021; Volume 8, Issue 3(16): 38-58.

[28] Farooq U, Myler P. Explicit Dynamic Simulation of Flat Nose Low Velocity Impact of Relatively Thick Composite Panels Using Built-In Failure Criteria. International Journal of Scientific Engineering and Science 2021; Volume 5, Issue 6: 35-51.

[29] Farooq U, Myler P, Siddique M. Experimental and Micro-Macro Mechanics Methods in Prediction of Mechanical Properties of Carbon Fibre-Reinforced Composite Panels. International Journal of Scientific Engineering and Science 2021; Volume 5, Issue 7: 58-73.

[30] Farooq U, Myler P. Flat and Round Nose Low-Velocity Impact Response of Carbon Fibre-Reinforced Laminated Composite Panels Using Energy Approach Coupled With Data

Filtering Techniques. International Journal of Scientific Engineering and Science 2021; Volume 5, Issue 7, pp. 74-88.