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ON $p-$VALENT FUNCTIONS IN DYNAMICAL SYSTEMS

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Abstract

In this work we introduce and investigate some subclasses of $p-$valent functions with positive coefficients in topological dynamical systems. We present some various characteristics for these subclasses such as the radii of meromorphically $p-$valent starlikeness, coefficients inequalities-inclusion properties, the growth-distortion theorems, subordination properties and convexity.

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1 Introduction

By (topological) dynamical system (X, T) we mean a compact metric space X with a metric d and a continuous self-surjection T. Let Σ_p denote the class of functions of the form

$$f(\omega) = \omega^{-p} + \sum_{k=p+1}^{\infty} a_k\ \omega^k \quad (a_k \geq 0; p \in N = \{1, 2, ...\}), \tag{1}$$

which are meromorphic and $p-$valent in the punctured unit disc $DS^* = \{\omega : \omega \in DS \text{ and } 0 < |\omega| < 1\}$, where DS denotes any dynamical system subset of a complex set C.

For all $f \in \Sigma_p^-$, we use a linear operator which is introduced by [5],

$$\begin{aligned} D^0 f(\omega) &= f(\omega), \\ D^1 f(\omega) &= \omega^{-p} + \sum_{k=p+1}^{\infty} (k+p+1)\ a_k\ \omega^k = \frac{(\omega^{p+1} f(\omega))'}{\omega^p}, \\ D^2 f(\omega) &= D^1[D^1(f(\omega))], \quad (p \in N) \end{aligned}$$

and (in general)

$$D^n f(\omega) = \omega^{-p} + \sum_{k=p+1}^{\infty} (k+p+1)^n\ a_k\ \omega^k \quad (n \in N_0). \tag{2}$$

Then we can observe easily that for $f \in \Sigma_p$,

$$\omega(D^n f(\omega))' = D^{n+1} f(\omega) - (p+1) D^n f(\omega) \quad (p \in N; n \in N_0). \tag{3}$$

This operator D^n has been studied for the class Σ_p of meromorphic $p-$valent functions with negative coefficients.in the case $p = 1$ by [4].

Recall [1] that a function $f \in \Sigma_p$ is said to be meromorphically starlike of order α if it is satisfying the following condition:

$$Re\left\{-\frac{\omega f'(\omega)}{f(\omega)}\right\} \quad > \quad \alpha \quad (\omega \in DS^*), \tag{4}$$

for some $\alpha(0 \leqq \alpha < 1)$. Similarly a function $f \in \Sigma_p$ is said to be is said to be meromorphically convex of order α if it is satisfying the following condition:

$$Re\left\{-1 - \frac{\omega f''(\omega)}{f'(\omega)}\right\} \quad > \quad \alpha \quad (\omega \in DS^*), \tag{5}$$

for some $\alpha(0 \leqq \alpha < 1)$.

Let $\Sigma_p(\alpha)$ be the subclass of Σ_p consisting the functions which satisfy the following inequality:

$$Re\left\{-\frac{\omega(D^n f(\omega))'}{D^n f(\omega)}\right\} \quad > \quad p\,\alpha \quad (\omega \in DS^*; \alpha \geq 0). \tag{6}$$

In the following definitions we will define subclass $\Delta_p(n, \alpha, \beta)$ by using the linear operator D^n.

Definition 1.1. For fixed parameters $\alpha \geq 0, 0 \leq \beta < 1$ the meromorphically $p-$valent function $f(\omega) \in \Sigma_p(\alpha)$ will be in the class $\Delta_p(n, \alpha, \beta)$ if it satisfies the following inequality:

$$Re\left\{-\frac{\omega(D^n f(\omega))'}{p(D^n f(\omega))}\right\} \quad \geq \quad \alpha\left|\frac{\omega(D^n f(\omega))'}{p(D^n f(\omega))} + 1\right| + \beta \quad (n \in N_0). \tag{7}$$

2 Coefficient inequalities

Here we determine a necessary and sufficient conditions for functions f in the class $\Delta_p(n, \alpha, \beta)$.

Lemma 2.1. (See [1]) Let

$$R_a = \begin{cases} a - \frac{\alpha+\beta}{1+\alpha}, & \text{for} \quad a \leq 1 + \frac{1-\beta}{\alpha(1+\alpha)} \\ \sqrt{(1-a)^2(1-\alpha^2) - 2(1-\beta)(1-a)}, & \text{for} \quad a \geq 1 + \frac{1-\beta}{\alpha(1+\alpha)}. \end{cases}$$

Then

$$\{w : |w - a| \leq R_a\} \subseteq \{w : Re(w) \geq \alpha|w - 1| + \beta\}.$$

Theorem 2.2. Let $f \in \Sigma_p$. Then f is in the class $\Delta_p(n, \alpha, \beta)$ if and only if

$$\sum_{k=p+1}^{\infty} [p(\alpha + \beta) + k(1 + \alpha)](k + p + 1)^n \; a_k \leq p(1 - \beta) \tag{8}$$

$(\alpha \geq 0; 0 \leq \beta < 1; p \in N; n \in N_0)$.

Proof. Suppose that $f \in \Delta_p(n,\alpha,\beta)$. Then by the inequalities (2) and (7), we get that

$$Re\left\{-\frac{\omega(D^n f(\omega))'}{p(D^n f(\omega))}\right\} \geq \alpha\left|\frac{\omega(D^n f(\omega))'}{p(D^n f(\omega))}+1\right|+\beta.$$

That is,

$$Re\left\{\frac{1-\sum\limits_{k=p+1}^{\infty}\frac{k}{p}(k+p+1)^n\ a_k\omega^{k+p}}{1+\sum\limits_{k=p+1}^{\infty}(k+p+1)^n\ a_k\omega^{k+p}}\right\}$$

$$\geq \alpha\left|\frac{\sum\limits_{k=p+1}^{\infty}(\frac{k}{p}+1)(k+p+1)^n\ a_k\omega^{k+p}}{1+\sum\limits_{k=p+1}^{\infty}(k+p+1)^n\ a_k\omega^{k+p}}\right|+\beta$$

$$\geq Re\left\{\alpha.\frac{\sum\limits_{k=p+1}^{\infty}(\frac{k}{p}+1)(k+p+1)^n\ a_k\omega^{k+p}}{1+\sum\limits_{k=p+1}^{\infty}(k+p+1)^n\ a_k\omega^{k+p}}+\beta\right\}$$

$$= Re\left\{\frac{\beta+\sum\limits_{k=p+1}^{\infty}[\alpha(\frac{k}{p}+1)+\beta](k+p+1)^n\ a_k\omega^{k+p}}{1+\sum\limits_{k=p+1}^{\infty}(k+p+1)^n\ a_k\omega^{k+p}}\right\},$$

that is,

$$Re\left\{\frac{p(1-\beta)-\sum\limits_{k=p+1}^{\infty}(k+k\alpha+p\alpha+p\beta)(k+p+1)^n\ a_k\omega^{k+p}}{1+\sum\limits_{k=p+1}^{\infty}(k+p+1)^n\ a_k\omega^{k+p}}\right\}\geq 0. \quad (9)$$

Taking ω to be real and putting $\omega \longrightarrow 1^-$ through real values, then the inequality (9) yields

$$\frac{p(1-\beta)-\sum\limits_{k=p+1}^{\infty}(k+k\alpha+p\alpha+p\beta)(k+p+1)^n\ a_k}{1+\sum\limits_{k=p+1}^{\infty}(k+p+1)^n\ a_k}\geq 0.$$

Hence

$$\sum_{k=p+1}^{\infty}[p(\alpha+\beta)+k(1+\alpha)](k+p+1)^n\ a_k \leq p(1-\beta)$$
$$(\alpha \geq 0; 0 \leq \beta < 1; p \in N; n \in N_0).$$

In order to prove the converse, we suppose that the inequality (8) holds true. In Lemma 2.1 above, since $1 \leq 1+\frac{1-\beta}{\alpha(1+\alpha)}$, we can put $a = 1$. Then for $p \in N$ and $n \in N_0$, let

$w_{np} = -\frac{\omega(D^n f(\omega))'}{p(D^n f(\omega))}$. Now, if we let $\omega \in \partial DS^* = \{\omega \in DS : |\omega| = 1\}$, we get from the inequalities (2) and (8) that

$$\begin{aligned}
|w_{np} - 1| - R_1 &= \left| -\frac{\omega(D^n f(\omega))'}{p(D^n f(\omega))} - 1 \right| - 1 + \frac{\alpha+\beta}{1+\alpha} \\
&= \left| \frac{\sum\limits_{k=p+1}^{\infty} (k+p)(k+p+1)^n \ a_k \omega^{k+p}}{p + \sum\limits_{k=p+1}^{\infty} p(k+p+1)^n \ a_k \omega^{k+p}} \right| - 1 + \frac{\alpha+\beta}{1+\alpha} \\
&\leq \frac{\sum\limits_{k=p+1}^{\infty} (k+p)(k+p+1)^n \ a_k |\omega|^{k+p}}{p + \sum\limits_{k=p+1}^{\infty} p(k+p+1)^n \ a_k |\omega|^{k+p}} - 1 + \frac{\alpha+\beta}{1+\alpha} \\
&\leq \frac{\sum\limits_{k=p+1}^{\infty} (k+p)(k+p+1)^n \ a_k}{p + \sum\limits_{k=p+1}^{\infty} p(k+p+1)^n \ a_k} - 1 + \frac{\alpha+\beta}{1+\alpha} \\
&= \frac{-p(1-\beta) + \sum\limits_{k=p+1}^{\infty} [p(\alpha+\beta) + k(1+\alpha)](k+p+1)^n \ a_k}{p(1+\alpha) + \sum\limits_{k=p+1}^{\infty} p(1+\alpha)(k+p+1)^n \ a_k} \\
&\leq 0,
\end{aligned}$$

that is, $|w_{np} - 1| \leq R_1$. Thus by Lemma 2.1 above, we ge that

$$\begin{aligned}
Re\left\{ -\frac{\omega(D^n f(\omega))'}{p(D^n f(\omega))} - 1 \right\} &= Re\{w_{np}\} \\
&\geq \alpha |w_{np} - 1| + \beta \\
&= \alpha \left| -\frac{\omega(D^n f(\omega))'}{p(D^n f(\omega))} - 1 \right| + \beta \\
&= \alpha \left| \frac{\omega(D^n f(\omega))'}{p(D^n f(\omega))} + 1 \right| + \beta, \\
& \quad (\alpha \geq 0; 0 \leq \beta < 1; p \in N; n \in N_0).
\end{aligned}$$

Therefore by the maximum modulus theorem, we obtain $f \in \Delta_p(n, \alpha, \beta)$. □

Corollary 2.3. If $f \in \Delta_p(n, \alpha, \beta)$, then

$$\begin{aligned}
a_k &\leq \frac{p(1-\beta)}{[p(\alpha+\beta) + k(1+\alpha)](k+p+1)^n} \\
& \quad (\alpha \geq 0; 0 \leq \beta < 1; p \in N; n \in N_0).
\end{aligned} \tag{10}$$

Proof. Since $f \in \Delta_p(n, \alpha, \beta)$, then from Theorem 2.2 above, we get that

$$\sum_{k=p+1}^{\infty} [p(\alpha+\beta) + k(1+\alpha)](k+p+1)^n \ a_k \leq p(1-\beta).$$

Next, note that

$$\begin{aligned}&[p(\alpha+\beta)+k(1+\alpha)](k+p+1)^n\ a_k\\ \leq\ &\sum_{k=p+1}^{\infty}[p(\alpha+\beta)+k(1+\alpha)](k+p+1)^n\ a_k \leq p(1-\beta).\end{aligned}$$

Hence

$$a_k \leq \frac{p(1-\beta)}{[p(\alpha+\beta)+k(1+\alpha)](k+p+1)^n}$$
$$(\alpha \geq 0; 0 \leq \beta < 1; p \in N; n \in N_0).$$

□

That is, the equality (10) is attained for the function f given by

$$f(\omega) = \omega^{-p} + \sum_{k=p+1}^{\infty}\frac{p(1-\beta)}{[p(\alpha+\beta)+k(1+\alpha)](k+p+1)^n}\omega^k \quad (11)$$
$$(\alpha \geq 0; 0 \leq \beta < 1; p \in N; n \in N_0).$$

Lemma 2.4. [1] Let $\mu > \delta$ and

$$R_a = \begin{cases} a-\delta, & \text{for} \quad a \leq 2\mu+\delta \\ 2\sqrt{\mu(a-\mu-\delta)}, & \text{for} \quad a \geq 2\mu+\delta. \end{cases}$$

Then

$$\{w : |w-a| \leq R_a\} \subseteq \{w : |w-(\mu+\delta)| \leq Re\{w+\mu-\delta)\}\}.$$

Lemma 2.5. Let $\alpha \geq 0$ and $0 \leq \beta < 1$

$$R_a = \begin{cases} a-\alpha\beta, & \text{for} \quad a \leq 2\alpha+\alpha\beta \\ 2\sqrt{\alpha(a-\alpha-\alpha\beta)}, & \text{for} \quad a \geq 2\alpha+\alpha\beta. \end{cases}$$

Then

$$\{w : |w-a| \leq R_a\} \subseteq \{w : |w-(\alpha+\alpha\beta)| \leq Re\{w+\alpha-\alpha\beta)\}\}.$$

Proof. Since $\alpha \geq 0$ and $0 \leq \beta < 1$, then $\alpha > \alpha\beta$. Then in Lemma 2.4, put $\mu = \alpha$ and $\delta = \alpha\beta$. □

Theorem 2.6. The class $\Delta_p(n, \alpha, \beta)$ is closed under convex linear combinations.

Proof. Suppose every of the function

$$f(\omega) = \omega^{-p} + \sum_{k=p+1}^{\infty} a_k\ \omega^{k,j} \quad (a_{k,j} \geq 0; j = 1, 2; p \in N), \quad (12)$$

be in the class $\Delta_p(n, \alpha, \beta)$. It is sufficient to show that the function $h(\omega)$ defined by

$$h(\omega) = (1-\delta)f_1(\omega) + \delta f_2(\omega) \quad (0 \leq \delta \leq 1), \quad (13)$$

is also in the class $\Delta_p(n,\alpha,\beta)$. Since

$$h(\omega) = \omega^{-p} + \sum_{k=p+1}^{\infty} [(1-\delta)a_{k,1} + \delta a_{k,2}]\ \omega^{k,j} \quad (0 \le \delta \le 1), \tag{14}$$

and by Theorem 2.2, we get that

$$\begin{aligned} & \sum_{k=p+1}^{\infty} [p(\alpha+\beta) + k(1+\alpha)](k+p+1)^n\ [(1-\delta)a_{k,1} + \delta a_{k,2}] \\ = & \sum_{k=p+1}^{\infty} (1-\delta)[p(\alpha+\beta) + k(1+\alpha)](k+p+1)^n\ a_{k,1} \\ + & \sum_{k=p+1}^{\infty} \delta[p(\alpha+\beta) + k(1+\alpha)](k+p+1)^n\ a_{k,2} \\ \le & (1-\delta)p(1-\beta) + \delta p(1-\beta) \\ = & p(1-\beta) \\ & (\alpha \ge 0; 0 \le \beta < 1; p \in N; n \in N_0). \end{aligned} \tag{15}$$

Hence $f \in \Delta_p(n,\alpha,\beta)$. □

3 Growth and Distortion theorems

We prove the following growth and distortion theorems for the class $\Delta_p(n,\alpha,\beta)$.

Theorem 3.1. If $f \in \Delta_p(n,\alpha,\beta)$, then

$$\begin{aligned} & \left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(1-\beta)}{(2\alpha+\beta+1)(p+1)^{n-1}} . \frac{p!2^{-n}}{(p-m)!} r^{2p}\right\} r^{-(p+m)} \\ & \le |f^{(m)}(\omega)| \le \\ & \left\{\frac{(p+m-1)!}{(p-1)!} + \frac{(1-\beta)}{(2\alpha+\beta+1)(p+1)^{n-1}} . \frac{p!2^{-n}}{(p-m)!} r^{2p}\right\} r^{-(p+m)} \\ & (0 < |\omega| = r < 1; \alpha \ge 0; 0 \le \beta < 1; p \in N; n, m \in N_0; p > m) . \end{aligned} \tag{16}$$

The result is sharp for the function f given by

$$\begin{aligned} f(\omega) &= \omega^{-p} + \sum_{k=p+1}^{\infty} \frac{(1-\beta)}{(2\alpha+\beta+1)(2p+2)^n} \omega^p \\ & (n \in N_0; p \in N). \end{aligned} \tag{17}$$

Proof. From Theorem 2.2, we get that

$$\begin{aligned} \frac{p(2\alpha+\beta+1)(2p+2)^n}{(p+1)!} \sum_{k=p+1}^{\infty} k!a_k &\le \sum_{k=p+1}^{\infty} [p(\alpha+\beta) + k(1+\alpha)](k+p+1)^n\ a_k \\ &\le p(1-\beta), \end{aligned}$$

that is,

$$\sum_{k=p+1}^{\infty} k!a_k \leq \frac{p(1-\beta)(p+1)!}{p(2\alpha+\beta+1)(2p+2)^n} = \frac{(1-\beta)p!2^{-n}}{(2\alpha+\beta+1)(p+1)^{n-1}}. \tag{18}$$

By the differentiating the function f in the form (1) m times with respect to ω, we get that

$$f^m(\omega) = (-1)^m \frac{(p+m-1)!}{(p-1)!}\omega^{-(p+m)} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!} a_k\ \omega^{k-m} \quad (m \in N_0; p \in N). \tag{19}$$

From (18) and (19), we get that

$$\begin{aligned} |f^{(m)}(\omega)| &\leq \frac{(p+m-1)!}{(p-1)!} r^{-(p+m)} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!} a_k\ r^{k-m} \\ &\leq \left\{ \frac{(p+m-1)!}{(p-1)!} + \sum_{k=p+1}^{\infty} \frac{k!}{(p-m)!} a_k\ r^{2p} \right\} r^{-(p+m)} \\ &\leq \left\{ \frac{(p+m-1)!}{(p-1)!} + \frac{(1-\beta)}{(2\alpha+\beta+1)(p+1)^{n-1}} . \frac{p!2^{-n}}{(p-m)!} r^{2p} \right\} \\ &. \quad r^{-(p+m)}, \end{aligned} \tag{20}$$

and

$$\begin{aligned} |f^{(m)}(\omega)| &\geq \frac{(p+m-1)!}{(p-1)!} r^{-(p+m)} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!} a_k\ r^{k-m} \\ &\geq \left\{ \frac{(p+m-1)!}{(p-1)!} - \sum_{k=p+1}^{\infty} \frac{k!}{(p-m)!} a_k\ r^{2p} \right\} r^{-(p+m)} \\ &\geq \left\{ \frac{(p+m-1)!}{(p-1)!} - \frac{(1-\beta)}{(2\alpha+\beta+1)(p+1)^{n-1}} . \frac{p!2^{-n}}{(p-m)!} r^{2p} \right\} \\ &. \quad r^{-(p+m)}. \end{aligned} \tag{21}$$

We can easily prove that the bounds of (16) are attained for the function $f(\omega)$ given by the form (17). □

4 Radii of meromorphically starlikeness and convexity

We determine the radii of $p-$valent starlikeness of order $\mu(0 \leq \mu < p)$ and $p-$valent convexity of order $\mu(0 \leq \mu < p)$ for the class $\Delta_p(n, \alpha, \beta)$.

Theorem 4.1. If $f \in \Delta_p(n, \alpha, \beta)$, then f is meromorphically $p-$valent starlike of order $\mu (0 \leq \mu < 1)$ in the disk $|\omega| < r_1$, that is,

$$Re\left\{-\frac{\omega f'(\omega)}{f(\omega)}\right\} \quad > \quad \mu \quad (0 \leq \mu < p; |\omega| < r_1; p \in N), \tag{22}$$

where

$$r_1 \quad = \quad \inf_{k \geq p+1} \left\{\frac{(p-\mu)[p(\alpha+\beta)+k(1+\alpha)](k+p+1)^n}{p(k+\mu)(1-\beta)}\right\}^{\frac{1}{k+p}}. \tag{23}$$

Proof. By the form (1), we get that

$$\begin{aligned}
\left|\frac{\frac{\omega f'(\omega)}{f(\omega)}+p}{\frac{\omega f'(\omega)}{f(\omega)}-p+2\mu}\right| &= \left|\frac{\sum\limits_{k=p+1}^{\infty}(k+p)a_k\omega^k}{2(p-\mu)\omega^{-p}-\sum\limits_{k=p+1}^{\infty}(k-p+2\mu)a_k\omega^k}\right| \\
&\leq \frac{\sum\limits_{k=p+1}^{\infty}(k+p)|\omega|^k}{2(p-\mu)a_k|\omega|^{-p}-\sum\limits_{k=p+1}^{\infty}(k-p+2\mu)a_k|\omega|^k} \\
&= \frac{\sum\limits_{k=p+1}^{\infty}(k+p)a_k|\omega|^{k+p}}{2(p-\mu)-\sum\limits_{k=p+1}^{\infty}(k-p+2\mu)a_k|\omega|^{k+p}}.
\end{aligned} \tag{24}$$

Then the following incurability

$$\left|\frac{\frac{\omega f'(\omega)}{f(\omega)}+p}{\frac{\omega f'(\omega)}{f(\omega)}-p+2\mu}\right| \quad \leq \quad 1 \quad (0 \leq \mu < p; p \in N) \tag{25}$$

will be hold if

$$\sum_{k=p+1}^{\infty}\frac{(k+\mu)}{(p-\mu)}a_k|\omega|^{k+p} \quad \leq \quad 1 \quad (0 \leq \mu < p; p \in N). \tag{26}$$

Then by Corollary 2.3 the inequality (26) will be true if

$$\frac{(k+\mu)}{(p-\mu)}|\omega|^{k+p} \quad \leq \quad \frac{[p(\alpha+\beta)+k(1+\alpha)](k+p+1)^n}{p(1-\beta)} \quad (0 \leq \mu < p; p \in N), \tag{27}$$

that is,

$$|\omega|^{k+p} \quad \leq \quad \frac{(p-\mu)[p(\alpha+\beta)+k(1+\alpha)](k+p+1)^n}{p(k+\mu)(1-\beta)} \quad (0 \leq \mu < p; p \in N). \tag{28}$$

Therefore the inequality (28) leads us to the disc $|\omega| < r_1$, where r_1 is given by the form (23). □

Theorem 4.2. If $f \in \Delta_p(n,\alpha,\beta)$, then f is meromorphically $p-$valent convex of order $\mu(0 \leq \mu < 1)$ in the disk $|\omega| < r_2$, that is,

$$Re\left\{-1-\frac{\omega f''(\omega)}{f'(\omega)}\right\} \;>\; \mu \quad (0 \leq \mu < p; |\omega| < r_2; p \in N), \tag{29}$$

where

$$r_2 \;=\; \inf_{k\geq p+1}\left\{\frac{(p-\mu)[(\alpha+\beta)+k(1+\alpha)](k+p+1)^n}{k(k+\mu)(1-\beta)}\right\}^{\frac{1}{k+p}}. \tag{30}$$

Proof. By the form (1), we get that

$$\begin{aligned}\left|\frac{1+\frac{\omega f''(\omega)}{f'(\omega)}+p}{1+\frac{\omega f''(\omega)}{f'(\omega)}-p+2\mu}\right| &= \left|\frac{\sum\limits_{k=p+1}^{\infty} k(k+p)a_k\omega^k}{2p(p-\mu)\omega^{-p}-\sum\limits_{k=p+1}^{\infty}k(k-p+2\mu)a_k\omega^k}\right| \\ &\leq \frac{\sum\limits_{k=p+1}^{\infty} k(k+p)|\omega|^k}{2p(p-\mu)a_k|\omega|^{-p}-\sum\limits_{k=p+1}^{\infty}k(k-p+2\mu)a_k|\omega|^k} \\ &= \frac{\sum\limits_{k=p+1}^{\infty} k(k+p)a_k|\omega|^{k+p}}{2p(p-\mu)-\sum\limits_{k=p+1}^{\infty}k(k-p+2\mu)a_k|\omega|^{k+p}}.\end{aligned} \tag{31}$$

Then the following incurability

$$\left|\frac{1+\frac{\omega f''(\omega)}{f'(\omega)}+p}{1+\frac{\omega f''(\omega)}{f'(\omega)}-p+2\mu}\right| \;\leq\; 1 \quad (0 \leq \mu < p; p \in N) \tag{32}$$

will be hold if

$$\sum_{k=p+1}^{\infty}\frac{k(k+\mu)}{p(p-\mu)}a_k|\omega|^{k+p} \;\leq\; 1 \quad (0 \leq \mu < p; p \in N). \tag{33}$$

Then by Corollary 2.3 the inequality (33) will be true if

$$\frac{k(k+\mu)}{p(p-\mu)}|\omega|^{k+p} \;\leq\; \frac{[p(\alpha+\beta)+k(1+\alpha)](k+p+1)^n}{p(1-\beta)} \quad (0 \leq \mu < p; p \in N), \tag{34}$$

that is,

$$|\omega|^{k+p} \;\leq\; \frac{(p-\mu)[(\alpha+\beta)+k(1+\alpha)](k+p+1)^n}{k(k+\mu)(1-\beta)} \quad (0 \leq \mu < p; p \in N). \tag{35}$$

Therefore the inequality (35) leads us to the disc $|\omega| < r_2$, where r_2 is given by the form (30). □

5 Subordination properties

If f and g are analytic functions in DS, we say that f is *subordinate* to g, written symbolically as follows:

$$f \prec g \quad \text{in } DS \qquad \text{or} \qquad f(\omega) \prec g(\omega) \quad (\omega \in DS),$$

if there exists a function w which is analytic in DS with

$$w(0) = 0 \qquad \text{and} \qquad |w(\omega)| < 1 \quad (\omega \in DS),$$

such that

$$f(\omega) = g(w(\omega)) \quad (\omega \in DS).$$

Indeed it is known that

$$f(\omega) \prec g(\omega) \quad (\omega \in DS) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(DS) \subset g(DS).$$

In particular, if the function g is univalent in DS we have the following equivalence.

$$f(\omega) \prec g(\omega) \quad (\omega \in DS) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(DS) \subset g(DS).$$

Let $\phi : C^2 \longrightarrow C$ be a function and let h be univalent in DS. If J is analytic function in DS and satisfied the differential subordination $\phi(J(\omega), J'(\omega)) \prec h(\omega)$ then J is called a *solution of the differential subordination* $\phi(J(\omega), J'(\omega)) \prec h(\omega)$. The univalent function q is called a *dominant* of the solution of the differential subordination, $J \prec q$.

Lemma 5.1. [6] Let $q(\omega) \neq 0$ be univalent in DS. Let θ and ϕ be analytic in a domain D containing $q(DS)$ with $\phi(w) \neq 0$ when $w \in q(DS)$. Set

$$Q(\omega) = \omega q'(\omega)\phi(q(\omega)) \quad \text{and} \quad h(\omega) = \theta(q(\omega)) + Q(\omega). \tag{36}$$

Suppose that

1. $Q(\omega)$ is starlike univalent in DS, and
2. $Re\{\frac{\omega h'(\omega)}{Q(\omega)}\} > 0$ for $\omega \in DS$.

If J is analytic function in DS and

$$\theta(J(\omega)) + \omega J'(\omega)\phi(J(\omega)) \prec \theta(q(\omega)) + \omega q'(\omega)\phi(q(\omega)), \tag{37}$$

then $J(\omega) \prec q(\omega)$ and q is the best dominant.

Theorem 5.2. Let $q(\omega) \neq 0$ be univalent in DS such that $\frac{\omega q'(\omega)}{q(\omega)}$ is starlike univalent in DS and

$$Re\{1 + \frac{\epsilon}{\gamma}q(\omega) + \frac{\omega q''(\omega)}{q'(\omega)} - \frac{\omega q'(\omega)}{q(\omega)}\} > 0, \tag{38}$$
$$(\epsilon, \gamma \in C, \gamma \neq 0).$$

If $f \in \Sigma_p$ satisfies the subordination:

$$\epsilon\frac{\omega[D^n f(\omega)]'}{[D^n f(\omega)]} + \gamma[1 + \frac{\omega[D^n f(\omega)]''}{[D^n f(\omega)]'} - \frac{\omega[D^n f(\omega)]'}{[D^n f(\omega)]}] \prec \epsilon q(\omega) + \frac{\gamma\omega q'(\omega)}{q(\omega)}, \tag{39}$$

then $\frac{\omega[D^n f(\omega)]'}{[D^n f(\omega)]} \prec q(\omega)$ and q is the best dominant.

Proof. Our aim is to apply Lemma 5.1. Setting

$$J(\omega) = \frac{\omega[D^n f(\omega)]'}{[D^n f(\omega)]} = \frac{-p + \sum_{k=p+1}^{\infty} k(k+p+1)^n \ a_k \ \omega^{k+p}}{1 + \sum_{k=p+1}^{\infty} (k+p+1)^n \ a_k \ \omega^{k+p}}$$
$$(n \in N_0; p \in N),$$

$\theta(w) = w$ and $\phi(w) = \frac{\gamma}{w}$, $\gamma \neq 0$. It can be easily observed that J is analytic in DS, θ is analytic in C, ϕ is analytic in $C/\{0\}$ and $\phi(w) \neq 0$. By computation shows that:

$$\frac{\omega J'(\omega)}{J(\omega)} = 1 + \frac{\omega[D^n f(\omega)]''}{[D^n f(\omega)]'} - \frac{\omega[D^n f(\omega)]'}{[D^n f(\omega)]}$$

which yields, by (39), the following subordination:

$$\epsilon J(\omega) + \gamma \frac{\omega J'(\omega)}{J(\omega)} \prec \epsilon q(\omega) + \frac{\gamma \omega q'(\omega)}{q(\omega)},$$

that is,

$$\theta(J(\omega)) + \omega J'(\omega)\phi(J(\omega)) \prec \theta(q(\omega)) + \omega q'(\omega)\phi(q(\omega)).$$

Now by letting

$$Q(\omega) = \omega q'(\omega)\phi(q(\omega)) = \frac{\gamma \omega q'(\omega)}{q(\omega)}$$

and

$$h(\omega) = \theta(q(\omega)) + Q(\omega) = \epsilon q(\omega) + \frac{\gamma \omega q'(\omega)}{q(\omega)}.$$

We find Q i starlike univalent in DS and that:

$$Re\{\frac{\omega h'(\omega)}{Q(\omega)}\} = Re\{1 + \frac{\epsilon}{\gamma} q(\omega) + \frac{\omega q''(\omega)}{q'(\omega)} - \frac{\omega q'(\omega)}{q(\omega)}\} > 0.$$

Hence by Lemma 5.1, $\frac{\omega[D^n f(\omega)]'}{[D^n f(\omega)]} \prec q(\omega)$ and q is the best dominant. □

Corollary 5.3. If $f \in \Sigma_p$ and assume that (38) holds, then:

$$1 + \frac{\omega[D^n f(\omega)]''}{[D^n f(\omega)]'} \prec \frac{1 + A\omega}{1 + B\omega} + \frac{(A - B)\omega}{(1 + A\omega)(1 + B\omega)}$$

implies then $\frac{\omega[D^n f(\omega)]'}{[D^n f(\omega)]} \prec \frac{1+A\omega}{1+B\omega}$, $-1 \leq B < A \leq 1$ and $\frac{1+A\omega}{1+B\omega}$ is the best dominant.

Proof. By setting $\epsilon = \gamma = 1$ and $q(\omega) = \frac{1+A\omega}{1+B\omega}$ in Theorem 5.2, where $-1 \leq B < A \leq 1$. □

Corollary 5.4. If $f \in \Sigma_p$ and assume that (38) holds, then:

$$1 + \frac{\omega[D^n f(\omega)]''}{[D^n f(\omega)]'} \prec e^{\alpha\omega} + \alpha\omega$$

implies then $\frac{\omega[D^n f(\omega)]'}{[D^n f(\omega)]} \prec e^{\alpha\omega}$, $|\alpha| < \pi$ and $e^{\alpha\omega}$ is the best dominant.

Proof. By setting $\epsilon = \gamma = 1$ and $q(\omega) = e^{\alpha\omega}$ in Theorem 5.2, where $|\alpha| < \pi$. □

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