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"Solutions of Polynomial Equation of 4th degree using Vedic Method"

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Introduction

This paper deals with the evaluation of all the four solutions of a Polynomial Equation of 4th degree by using Vedic Method as envisaged by Jagadguru Shankaracharya of Puri Mutt in applying both Straight Division method and Purana Apuranabhyam principle. The procedures that are adopted here are similar to those in solving the cubic polynomial equation, but a consideration of few variations in arriving at the second solution is necessitated to obtain a better accuracy. The details are well brought out.

Abstract

After finding out the first probable solution of the equation as x=a through "Vilokinam" by observing the differences between the RHS and LHS over a few values of x, the first dividend could be decided at a specific value of x. The method of working is digit by digit. The solution is assumed to be x=a.bcde where "a" is an integer and b,c,d,e.. are successive decimals.

1) Certain quantities by way of expressions in the expansion of $(a+b+c+d+e+f+g)^4$, where "a" is the integer part of the solution and $b \rightarrow g$ are the successive decimals and to be sorted out as deduction terms to be reduced from the Dividends depending on the positions of the decimals, which is to be carried out by way of the expansion terms. The expansion in general is $(a+b+c+d+e+f+g)^4$

2) Similarly one has to obtain the derivatives of the polynomial as the 1st derivative representative at x=a as the Common Divisor, CD, i.e. $4a^3$ at the selected value x=a, the second derivative representative $6a^2$ of the polynomial at the value x=a and the third derivative, 4a representative at the value x=a. Which derivative groups are to be deducted as the parts of the corresponding deduction terms. For this purpose, the expansion terms are to be sorted out to facilitate proper deductions to be carried out after which, the result is to be divided by the common divisor to obtain the corresponding decimal and remainder to be carried out to the next dividend, converting it as new dividend. This is to be continued till all the decimals could be reached. Thus the first solution could be obtained.

Then with the help of the remaining part of the polynomial – in this case, it is 3rd order, the third order polynomial is subjected to Straight Division, which could be converted it to the second solution, leading to a quadratic equation for the two remaining solutions.

By using Swamiji's Sutrams, Adyamadyena and Antyamantyena and also his expression for the quardratic solution, the remaining two solutions could be plugged out.

The same equation is used for the evaluation by Purana Apuranabhyam, the working details of which are clearly explained in the paper and all the 4 solutions obtained are well comparable with those obtained from the Straight Division method. It is noticed that the equation has two real and two complex solutions. The test given by Swamiji for the correctness of the solutions i.e. Gunitha Samuchaya, Samuchaya Gunaka is well agreeable. The full working details of both the methods are described in detail in this paper.

Let us consider the 4th order Polynomial Equation $x^4-4x^3-3x+23=0$

Method – A (Straight Division)

The given equation is x^4-$4x^3$-3x+23=0.

As a first step, the equation is treated as having RHS and LHS as RHS=-23 and LHS= x^4-$4x^3$-3x.

One can find out one most probable solution from RHS-LHS values at a definite value of "x", so that the remaining, three can be worked out. Applying Vilokanam, i.e. for a set of values of "x", the differences between the RHS-LHS can be worked out from which one can pick out one most probable value of "x=a" as one solution (Table – 1). This RHS-LHS can be considered as the starting Dividend for the application of Straight Division method. The method is confined to digit by digit. The following are the details of working. The RHS-LHS at x=2, with the 1st dividend as $\overline{10}$ is considered.

Table – 1

Value of x	L.HS.	R.H.S.	R.H.S. – L.H.S.
1	– 6	– 23	– 17
2	– 22		– 1
3	– 36		13
4	– 12		– 11
5	110		– 133
6	414		– 437
– 1	8		– 31
– 2	54		– 77
– 3	198		– 221
– 4	524		– 547

1) Considering that the solution is in the form of x=a.bcdef where 'a" is the integer part and b,c,d,e.. are the successive decimals, one proceeds with the work.

2) To facilitate the application of Straight Division method for the determination of the final solution, one has to arrive at a Common Divisor. This is achieved by the 1st derivative of the given polynomial x^4-$4x^3$-3x+23 at the value of x=2, the probable solution.

3) The Common Divisor (CD) is the first derivative of the polynomial x^4-$4x^3$-3x+23 at the value x=2 CD = $\overline{19}$

The difference between RHS and LHS at x=2 is -1. Starting with this as the 1st dividend $\overline{10}$, the Straight Division is carried out digit by digit. The details are shown in the Table – 2. For the 6 decimals (b $\rightarrow$ g), the terms from the general expansion $(a+b+c+d+e+f+g)^4$ (Refer Table – 13) are identified as derivatives of the polynomial and are as $4a^3$, $6a^2$ and 4a representatives together with the decimal values which are to be deducted from the respective dividends and the results are divided by the CD to give rise to the decimals and remainders, which are to be added to the following dividend to form the new dividends.

One has to consider the following : F(x)= x^4-$4x^3$-3x+23=0

The 1st derivative of the equation F(x) is F'(x) (at x=2)=$4x^3$-$12x^2$-3=$\overline{19}$

F'(x) : $4a^3$ representation at x=2 is -19

F"(x) : $6a^2$ representation = $\frac{1}{2}(12x^2-24x)$ =0 at x=2

F'" (x) : 4a representation = $\frac{1}{6}(24x-24)$=4 at x=2

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$F^{iv}(x) : \frac{1}{24}(24)=1$

The expansion terms deducted inorder to obtain the new dividends and decimals are shown in Table – 2.

Table – 2

CD= $\overline{19}$	0 $\overline{1}$	0 $\overline{10}$	0 $\overline{5}$	0 $\overline{12}$	0 $\overline{6}$	0 $\overline{3}$	0 $\overline{17}$
		0 $6a^2b^2$	0 $4a.b^3+6a^2.2bc$	0 $b^4+6a^2.c^2+$ $6a^2 2bd+$ $4a.3b^2c$	0 $4b^3c+6a^2.2be$ $+6a^2.2cd+$ $4a.3b^2d+$ $4a.3bc^2$	$\overline{500}$ $4ac^3+4b^3d+$ $6b^2c^2+6a^2.2bf$ $+6a^2.2ce+6a^2d^2$ $+4a6bcd+4a.3b^2e$	
2. a=2 (x=2) Integer part	0 b	5 c	2 d	6 e	3 f	27 g	

x=2.052657

The first decimal "b" is obtained by dividing the 1st dividend -10 directly by the CD -19 $\therefore$b=0 with the next dividend is -100. The deductions of the terms are the values from $4a^3$, $6a^2$ and 4a representations of the equation at x=2 and are evaluated along with the related decimals to form the deduction terms and subtracted from the respective dividends and the results are divided by the CD to get the decimals as the quotients and remainders to be carried over to the next dividends for the formation of new dividends. Table – 2 gives the details of deductions from the expansion terms of $(a+b+c+d+e+f+g)^4$ Thus the entire calculations from b to g decimals are evaluated and finally the 1st solution is 2.052657.

This value is subjected to a check, given as E=0.000108709 ~0, quite acceptable.

Then one can choose another value for "x" as the second solution from the table-1, as such, we have a=4. Proceeding in the same way, with x=a=4.
All the necessary data required for the evaluation of the solution are given below :

i) From Vilokanam, a = 4 . Considering at x = 4, RHS – LHS = -11

$4a^3$ representation : $F^{/}(x)$ = 61 (CD)

$6a^2$ representation : $\frac{1}{2}F^{//}(x) = \frac{1}{2}(96) = 48$

4a representation : $\frac{1}{6}F^{///}(x) = \frac{1}{6}(72) = 12$

1 representation : $\frac{1}{24}F^{iv}(x) = \frac{1}{24}(24) = 1$

x = a**.** b c d e f g h ………..

Table – 3

	0	0	0	0	0	0	0	0
CD=F'(a)=61	$\overline{11}$	$\overline{49}$	$\overline{50}$	$\overline{36}$	$\overline{2}$	$\overline{43}$	$\overline{55}$	$\overline{16}$
$6a^2$ representation= 48		$\overline{48}$	$\overline{768}$	$\overline{4992}$	$\overline{23,328}$	$\overline{114912}$	$\overline{564768}$	$\overline{2874480}$
4a representation =12			12	288	3024	20652	120276	673884
Representation of 1				$\overline{1}$	$\overline{32}$	$\overline{464}$	$\overline{4300}$	$\overline{31156}$
4	$\overline{1}$	$\overline{8}$	$\overline{20}$	$\overline{83}$	$\overline{333}$	$\overline{1559}$	$\overline{7366}$	$\overline{36588}$
a=4	b	c	d	e	f	g	h	i
(x=4)								

∴ x = 3.78570852 E = 0.016443352

When we consider x=a=4, the value of x=3.78570852, the check gives E=0.016443352 which needs refinement.

Hence, This can be further refined with $x = \frac{z}{2}$ substitution

The given equation E is

$$\left(\frac{z}{2}\right)^4 - 4\left(\frac{z}{2}\right)^3 - 3\left(\frac{z}{2}\right) + 23 = 0 \qquad \Rightarrow z^4 + - 8z^3 - 24z + 368 = 0$$

$F(z) = z^4 - 8z^3 - 24z = -368$

if z = 8 F(z) = – 192 ⇒ RHS – LHS = – 176

ii) From Vilokanam, z =a= 8

With a=8,

$4a^3$ representation : $F'(z) = 4z^3 - 24z^2 - 24 = 488$ (CD – Common Divisor)

$6a^2$ representation : $\frac{1}{2}F''(z) = \frac{1}{2}(12z^2 - 48z) = 192$

4a representation : $\frac{1}{6}F'''(z) = \frac{1}{6}(24z - 48) = 24$

1 representation : $\frac{1}{24}F^{iv}(z) = \frac{1}{24}(24) = 1$

Application of these values along with the decimals, the respective deduction terms are evaluated to get the final new dividends and decimal values.

Table – 4

	0	0	0	0	0	0	0	0
CD=F' (8)= 488	$\overline{176}$	$\overline{296}$	$\overline{296}$	$\overline{480}$	$\overline{457}$	$\overline{46}$	$\overline{294}$	$\overline{356}$
6a2 representation = 192		$\overline{1728}$	$\overline{10368}$	$\overline{44352}$	$\overline{187776}$	$\overline{800832}$	$\overline{3477120}$	

4a representation = 24			648	5832	33696	171720	834840
Representation of 1				$\overline{81}$	$\overline{972}$	$\overline{7074}$	$\overline{42552}$
8.	$\overline{3}$	$\overline{9}$	$\overline{25}$	$\overline{88}$	$\overline{327}$	$\overline{1304}$	$\overline{5507}$
a.	b	c	d	e	f	g	h

z = 7.5710753 $\quad$ $x = \frac{z}{2} = 3.78553765$ $\quad$ E = 0.009260

The value for the x when subjected to the test, it is also suggestive of variation for a better accuracy and hence, $x=\frac{z}{100}$ is tried

The integer value corresponding to the variable value $x=\frac{z}{100}$ is considered from the Table – 5.

$$E = x^4 - 4x^3 - 3x = -23$$

$$\frac{z^4}{10^8} - \frac{4z^3}{10^6} - \frac{3z}{10^2} = -23$$

$$z^4 - 400z^3 - 3000000z = -23 \times 10^8$$

Table - 5

	LHS F(z)	RHS -23×10^8	RHS – LHS
z= 377	– 2363400559		63400559
z = 378	– 2322223344		22223344
z = 379	– 2280238719		– 19761281
z = 380	– 2237440000		– 62560000

iii) From Vilokanam, z=a = 378

The details of the data pertaind to the CD and other derivatives of the polynomial in "z" at z=378 are given below :

$$F(z) = z^4 - 400z^3 - 3000000z = -23 \times 10^8$$

$4a^3$ representation : $F^{/}(z) = 4z^3 - 1200z^2 - 3000000 = 41579808$ (CD)

$6a^2$ representation : $\frac{1}{2}F^{//}(z) =: \frac{1}{2}(12z^2 - 2400z) = 403704$

4a representation : $\frac{1}{6}F^{///}(z) = \frac{1}{6}(24z - 2400) = 1112$

1 representation : $\frac{1}{24}F^{iv}(z) = \frac{1}{24}(24) = 1$

The evaluation of the solution using the required deductions and the decimal values together with the new dividends using Straight Division method are given in Table - 6.

Table – 6

	0	0	0	0
CD=F/(a)=41579808	22223344	14334400	8511976	31289832
$6a^2$ representation =403704		$\overline{10092600}$	$\overline{12111120}$	$\overline{7670376}$
4a representation =1112			$\overline{139000}$	$\overline{250200}$
Representation of 1				$\overline{625}$
378	5	3	1	7
a	b	c	d	e

Upto e: z = 378.5317 x = 3.785317 E = – 0.000010639

This value is considered to be more accurate than the previous values. Hence this is finally adopted as the second solution. The results of the trials for the second solution are shown in Table – 14. First and Second solutions are used to get the information on the remaining two solutions. As a result, a quadratic equation could be solved using Swamiji's sutrams Adyamadyena and Antyamantyena.

$\therefore$ E = (x – 2.052657) (x – 3.785317) A, A should have x^2, x and constant terms. A is quadratic

Applying Argumentation, Adyamadyena and comparing the coefficients of like terms

$E = (x^2 - 5.837974x + 7.769957)(x^2 + \alpha x + 2.960119)$
Comparing the coefficients of like terms,

$-17.28109776 + 7.769957\alpha = -3, \alpha = 1.837989$

$E = (x - 2.052657)(x - 3.785317)(x^2 + 1.837989x + 2.960119)$

The Quadratic Equation can be factorized by using differential relation.

$2x + 1.837989 = \pm\sqrt{3.378204 - 11.840476} = \pm 2.908998$

$x = -0.918995 \pm 1.454499i$

$E = (x^2 - 2.052657)(x - 3.785317)(x + 0.918995 + 1.454499i)(x + 0.918995 - 1.454499i)$

Finally the four solutions are two real, while the other two are complex.

Applying Gunita Samuccaya Sutram for final verification,

$S_c = 17 = (1-2.052657)(1-3.785317)(1.918995+1.454499i)(1.918995 - 1.454499i) = 16.99996 \sim 17$

Method – B (Purana Apuranabhyam)

This polynomial equation is also subjected to the second method. The working details are as follows. The first two terms of the given equation in the descending order could be identified as equivalent to the 1st two terms of $(x-1)^4$, a standard. These are substituted for the given polynomial equation and the $(x-1)^4$, the standard equation is completely represented. This is Purana Apuranabhyam. The equation is written in the form of another variable which is connected with the standard form (x-1) as (x-1)=y, x=y+1. Now the equation is $Y^4-6y^2-11y+17=0$
This 4th order equation needs to be solved using Straight Division Method.

E = x^4-4x^3-3x+23=0 $\Rightarrow x^4-4x^3=3x-23$ ---(1)

$(x-1)^4=x^4-4x^3+6x^2-4x+1$ ---(2)

Substituting the value of (1) in (2)

$(x-1)^4=3x-23+6x^2-4x+1 = 6x^2-x-22$

Let (x-1) = y; x=y+1

$Y^4=6(y+1)^2-y-1-22=6y^2+11y-17 \Rightarrow Y^4-6y^2-11y+17=0$

$y^4-6y^2-11y=-17$

solving y^4 equation.

Table – 7

	LHS	RHS	RHS-LHS
y=1	-16	-17	-1
y=2	-30	-17	13
y=3	-6	-17	-11

The required data including the differences between RHS – LHS and for the derivatives and the corresponding deduction terms as per the decimal positions are evaluated to determine the new dividends and other decimal values for the supposed values for y=1 & 3 are given in Tables – 7, 8 & 9 respectively.

with y=1;

$4a^3$ representation CD: F'(y) : $4y^3-12y-11 = -19$

$6a^2$ representation : $\frac{1}{2}$ F"(y) : $\frac{1}{2}(12y^2-12) = 0$

4a representation : $\frac{1}{6}$ F'"(y) : $\frac{1}{6}(24y) = 4$

1 representation : $\frac{1}{24}$ $F^{iv}(y)=1$

Table – 8

$CD = \overline{19}$	$\frac{0}{1}$	$\frac{0}{10}$	$\frac{0}{5}$	$\frac{0}{12}$	$\frac{0}{6}$	$\frac{0}{3}$	$\frac{0}{17}$
Y=1 a	0 b	5 c	2 d	6 e	3 f	27 g	

* Deduction expressions are the same as in the Table – 2, but their values at y=1

To check the value of y=1.052657, E=0.000109 ∴x=2.052657
with y=3;

$4a^3$ represetnation : CD=F'(y) : $4y^3-12y-11 = 61$

$6a^2$ representation : $\frac{1}{2}$ F"(y) : = $\frac{1}{2}(12y^2-12) = 48$

4a representation : $\frac{1}{6}$ F'"(y) : $\frac{1}{6}(24y) = 12$

1 representation : $\frac{1}{24}$ $F^{iv}(y)=1$

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Table – 9

CD=61	$\frac{0}{11}$	$\frac{0}{49}$	$\frac{0}{50}$	$\frac{0}{36}$	$\frac{0}{2}$	$\frac{0}{43}$	$\frac{0}{55}$	$\frac{0}{16}$
Y=3 a	$\overline{1}$ b	$\overline{8}$ c	$\overline{20}$ d	$\overline{83}$ e	$\overline{333}$ f	$\overline{1559}$ g	$\overline{7366}$ h	$\overline{36588}$ i

Y=2.78570852

To check the value of y=2. 78570852, x=3. 78570852, E=0.016443352, which needs refinement.

Which, finally, $y=\frac{z}{100}$ is made use of after evaluating all the necessary details for the deduction, new dividends and decimal values, the evaluation that shows that y=278.5317, x=3.785317.

Let $y=\frac{z}{100}$

$\frac{z^4}{10^8} - 6\frac{z^2}{10^4} - 11\frac{z}{10^2} + 17 = 0 \Rightarrow z^4-60000z^2-11000000z+1700000000=0$

$\Rightarrow z^4-60000z^2-11000000z=-1700000000$

Table – 10

	LHS	RHS	RHS-LHS
Z=276	-1803787024	-1700000000	103787024
z=277	-1763400559	-1700000000	63400559
z=278	-1722223344	-1700000000	22223344
z=279	-1680238719	-1700000000	-1976128

with z=278,

$4a^3$ represetnation : CD=F'(z) : $4z^3$-120000zy-11000000 = 41579808 (CD – Common Divisor)

$6a^2$ representation : $\frac{1}{2}$ F''(z) : = $\frac{1}{2}(12z^2-120000)$ = 403704

4a representation : $\frac{1}{6}$ F'''(z) : $\frac{1}{6}(24z)$ = 1112

1 representation : $\frac{1}{24}$ $F^{iv}(z)=1$

Table – 11

CD=41579808	$\frac{0}{22223344}$	$\frac{0}{14334400}$	$\frac{0}{8511976}$	$\frac{0}{31289832}$	$\frac{0}{13918463}$
278 a=278 (z=278)	5 b	3 c	1 d	7 e	

Z=278.5317, E=0.0000106376

$y=\frac{z}{100}$ = 2.785317

x=y+1 = 3.785317

The two acceptable solutions derived (i.e. 2.052657 & 3.785317) for this 4^{th} degree polynomial equation can be written as

E = (x – 2.052657) (x – 3.785317) A (Where A is quadratic and should have x^2, x and constant terms)

Applying Argumentation, Adyamadyena and comparing the coefficients of like terms

$E= (x^2 - 5.837974x + 7.769957)(x^2 + \alpha x + 2.960119)$

Comparing the coefficients of like terms,

$-17.28109776 + 7.769957\alpha = -3,\ \alpha = 1.837989$

$E = (x - 2.052657)(x - 3.785317)(x^2 + 1.837989x + 2.960119)$

The Quadratic Equation can be factorized by using differential relation.

$2x + 1.837989 = \pm\sqrt{3.378204 - 11.840476} = \pm 2.908998$

$x = -0.918995 \pm 1.454499i$

$E = (x^2 - 2.052657)(x - 3.785317)(x + 0.918995 + 1.454499i)(x + 0.918995 - 1.454499i)$

Applying Gunita Samuccaya Sutram for final verification

$S_c = 17 = (1-2.052657)(1-3.785317)(1.918995+1.454499i)(1.918995 - 1.454499i) = 16.99996 \sim 17$

Both the methods gives the same values.

Conclusion : The four solutions of the given polynomial of 4^{th} order equations are given in Table – 12

Table - 12

Solution	Straight Division Mehtod	Purana Apuranabhyam Method
1	2.052657	2.052657
2	3.785317	3.785317
3	-0.918995 - 1.454499i	-0.918995 - 1.454499i
4	-0.918995 + 1.454499i	-0.918995 + 1.454499i

Table – 13

Expansion Table

$(a+b+c+d+e+f+g)^4$

Coefficients→	1	4	6	12	24
10^0	a^4				
10^1		a^3b			
10^2		a^3c	a^2b^2		
10^3		a^3d, ab^3		a^2bc	

10^4	b^4	a^3e,	a^2c^2	a^2bd, b^2ac	
10^5		a^3f, b^3c		a^2be, a^2cd, b^2ad, c^2ab	
10^6		a^3g, ac^3, b^3d	a^2d^2, b^2c^2	a^2bf, a^2ce, b^2ae	abcd
10^7		b^3e, bc^3		a^2bg, a^2cf, a^2de, b^2af, b^2cd, c^2ad, d^2ab	abce
10^8	c^4	b^3f	a^2e^2, b^2d^2	a^2cg, a^2df, b^2ag, b^2ce, c^2ae, c^2bd, c^2ae, d^2ac	abcf, abde

Table – 14 (Trials – Straight Division)

S.No.	"a" value	Solution (x)	E value
i	x=a=4	3.78570852	0.016443352
ii	$x=\frac{z}{2}$, z=8	3.78553765	0.009260
iii	$x=\frac{z}{100}$, z=378	3.785317	-0.000010639

* Out of the above, the 3rd one is the most acceptable as the 2nd solution of the given polynomial

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