A Generalized Class of Product Type estimators for Product of Population Means using Auxiliary Variable and Attribute

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Abstract

In this paper a class of product type estimators has been proposed for studying the product of the population means of two study variables using the auxiliary information in form of both the attribute as well as the variable. We have also proposed an improved class of estimator using Jack-knife technique and it has been shown that the proposed Jack-knife estimator is unbiased. The proposed class has lesser mean square error under the optimum value of characterising parameters as compared to some commonly used estimators available in the literature. An empirical study is included as an illustration

1. Introduction

In survey sampling, the use of auxiliary information can increase the precision of an estimator when the study variable is highly correlated with the auxiliary variable, but in several practical situations, there also exists some auxiliary attribute which is also highly correlated with the study variable. For example: (i) amount of milk produced depends on the breed as well as the diet of the cow. (ii) Yield of wheat crop depends on the variety as well as the manure used. In such situations, taking the advantage of the available information on auxiliary variable and attribute, we can increase the efficiency or the estimator. . The use of auxiliary attribute is sometimes highly preferential as it is cheaper to obtain, it can be diagnosed by its presence or absence itself, moreover attributes are also free from measurement errors.

Consider the following notations

Y_1=First study variable

Y_2= second study variable

X= auxiliary variable

P= Auxiliary attribute

N= Size of the population

$\overline{Y}_1 = \frac{1}{N}\sum_{i=1}^{N} Y_{1i}$ = Population mean of first study variable

$\overline{Y}_2 = \frac{1}{N}\sum_{i=1}^{N} Y_{2i}$ =Population mean of second study variable

$\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$ =Population mean of auxiliary variable

$P = \frac{1}{N}\sum_{i=1}^{N}\phi_i$ = Population mean of auxiliary attribute

$S_{Y_1}^2 = \frac{1}{N}\sum_{i=1}^{N}(Y_{1i} - \overline{Y}_1)^2$ =Population variance of first study variable

$S_{Y_2}^2 = \frac{1}{N}\sum_{i=1}^{N}(Y_{2i} - \overline{Y}_2)^2$ = Population variance of second study variable

$S_X^2 = \frac{1}{N}\sum_{i=1}^{N}(X - \overline{X}_i)^2$ = Population variance of auxiliary variable

$S_P^2 = \frac{1}{N}\sum_{i=1}^{N}(\phi_i - P)^2$ = Population variance of auxiliary attribute

$P^* = \overline{Y}_1.\overline{Y}_2$ = Product of the population mean of two study variables (1.1)

Assuming that a simple random sample of size n is drawn from the population on the study variables and auxiliary characters X and φ , denote the following sample information as:

$\overline{y_1} = \frac{1}{n}\sum_{i=1}^{n} Y_{1i}$ = sample mean of first study variable

$\overline{y_2} = \frac{1}{n}\sum_{i=1}^{n} Y_{2i}$ = sample mean of second study variable

$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} X_i$ = sample mean of the auxiliary variable

$p = \frac{1}{n}\sum_{i=1}^{n} \phi_i$ = sample mean of the auxiliary attribute

$s_{Y_1}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_{1i} - \overline{y}_1)^2$ =sample variance of first study variable

$s_{Y_2}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_{2i} - \overline{y}_2)^2$ = sample variance of second study variable

$s_X^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{x})^2$ = sample variance of the auxiliary variable

$s_P^2 = \frac{1}{n}\sum_{i=1}^{n}(\phi_i - p)^2$ = sample variance of the auxiliary attribute (1.2)

Now consider some estimators of population mean developed in the past using the auxiliary information

(i). General estimator of mean in case of SRSWOR sample mean $\hat{Y}_1 = \bar{y}_1 . \bar{y}_2$

with $MSE\left(\hat{Y}_1\right) = P^{*2}.f_n\left[C_{Y_1}^2 + C_{Y_2}^2 + 2\rho_{Y_1Y_2}C_{Y_1}C_{Y_2}\right]$ (1.3)

(ii) Ratio estimator using auxiliary variable

$$\hat{Y}_2 = \bar{y}_1 . \bar{y}_2\left(\frac{\bar{X}}{\bar{x}}\right)$$

With $MSE\left(\hat{Y}_2\right) = P^{*2} f_n\left[C_{Y_1}^2 + C_{Y_2}^2 + C_X^2 + 2\rho_{Y_1Y_2}C_{Y_1}C_{Y_2} - 2\rho_{Y_1X}C_{Y_1}C_X + 2\rho_{Y_2X}C_{Y_2}C_X\right]$ (1.4)

(iii) Naik and Gupta(1996) ratio estimator using auxiliary attribute

$$\hat{Y}_3 = \bar{y}_1 . \bar{y}_2\left(\frac{P}{p}\right)$$

With $MSE\left(\hat{Y}_3\right) = P^{*2} f_n\left[C_{Y_1}^2 + C_{Y_2}^2 + C_P^2 + 2\rho_{Y_1Y_2}C_{Y_1}C_{Y_2} - 2\rho_{Y_1P}C_{Y_1}C_P + 2\rho_{Y_2P}C_{Y_2}C_P\right]$ (1.5)

(iv) Product estimator using auxiliary variable

$$\hat{Y}_4 = \bar{y}_1 . \bar{y}_2\left(\frac{\bar{x}}{\bar{X}}\right)$$

With $MSE\left(\hat{Y}_4\right) = P^{*2}.f_n\left[C_{Y_1}^2 + C_{Y_2}^2 + C_X^2 + 2\rho_{Y_1Y_2}C_{Y_1}C_{Y_2} + 2\rho_{Y_1X}C_{Y_1}C_X + 2\rho_{Y_2X}C_{Y_2}C_X\right]$ (1.6)

(v) Naik and Gupta (1996)Product estimator using auxiliary attribute

$$\hat{Y}_5 = \bar{y}_1 . \bar{y}_2\left(\frac{p}{P}\right)$$

With $MSE\left(\hat{Y}_5\right) = P^{*2}.f_n\left[C_{Y_1}^2 + C_{Y_2}^2 + C_P^2 + 2\rho_{Y_1Y_2}C_{Y_1}C_{Y_2} + 2\rho_{Y_1P}C_{Y_1}C_P + 2\rho_{Y_2P}C_{Y_2}C_P\right]$ (1.7)

(vi)

$$\hat{Y}_6 = \bar{y}_1 . \bar{y}_2\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha}$$

With $MSE\left(\hat{Y}_6\right) = P^{*2}.f_n\left[C_{Y_1}^2\left(1-\rho_{Y_1X}^2\right) + C_{Y_2}^2\left(1-\rho_{Y_2X}^2\right) + 2C_{Y_1}C_{Y_2}\left(\rho_{Y_1Y_2} - \rho_{Y_1X}\rho_{Y_2X}\right)\right]$ (1.8)

(vii)

$$\hat{Y}_7 = \bar{y}_1 . \bar{y}_2\left(\frac{P}{p}\right)^{\alpha}$$

With $MSE\left(\bar{\bar{Y}}_7\right) = P^{*2}.f_n\left[C_{Y_1}^2\left(1-\rho_{Y_1P}^2\right) + C_{Y_2}^2\left(1-\rho_{Y_2P}^2\right) + 2C_{Y_1}C_{Y_2}\left(\rho_{Y_1Y_2} - \rho_{Y_1P}\rho_{Y_2P}\right)\right]$ (1.9)

We propose a class of estimator of P assuming that auxiliary population mean and auxiliary population proportion are known using the product of the two study variables

$$\bar{\bar{Y}}_P = \bar{y}_1.\bar{y}_2\left(\frac{\bar{x}}{\bar{X}}\right)^{\alpha}\left(\frac{p}{P}\right)^{\beta} \quad (1.10)$$

In order to obtain the bias and the mean square error (MSE), let us denote

$$\bar{y}_1 = \bar{Y}_1(1+e_0)$$
$$\bar{y}_2 = \bar{Y}_2(1+e_1)$$
$$\bar{x} = \bar{X}(1+e_2)$$
$$p = P(1+e_3)$$

With $E(e_0) = E(e_1) = E(e_2) = E(e_3) = 0$ (1.11)

And the results given in Sukhatme and Sukhatme (1991)

$$E(e_0^2) = f_nC_{Y_1}^2$$
$$E(e_1^2) = f_nC_{Y_2}^2$$
$$E(e_2^2) = f_nC_X^2$$
$$E(e_3^2) = f_nC_P^2$$
$$E(e_0e_1) = f_n\rho_{Y_1Y_2}C_{Y_1}C_{Y_2}$$
$$E(e_0e_2) = f_n\rho_{Y_1X}C_{Y_1}C_X$$
$$E(e_0e_3) = f_n\rho_{Y_1P}C_{Y_1}C_P$$
$$E(e_1e_2) = f_n\rho_{Y_2X}C_XC_{Y_2}$$
$$E(e_1e_3) = f_n\rho_{Y_2P}C_PC_{Y_2}$$
$$E(e_2e_3) = f_n\rho_{XP}C_XC_P$$

With $f_n = \left(\frac{1}{n} - \frac{1}{N}\right)$ (1.12)

Substituting the values from (1.11) in (1.10) we get

$$\bar{\bar{Y}}_P = \bar{Y}_1.\bar{Y}_2\begin{pmatrix}1+e_0+e_1+\alpha e_2+\beta e_3+\frac{\alpha(\alpha-1)}{2}e_2^2+\frac{\beta(\beta-1)}{2}e_3^2\\ +\alpha e_2e_0+\alpha e_1e_2+\beta e_0e_3+\beta e_1e_3+\alpha\beta e_2e_3+e_0e_1\end{pmatrix} \quad (1.13)$$

Taking expectation on both sides we get

$$E\left(\hat{\bar{Y}}_P\right)=\bar{Y}_1.\bar{Y}_2\left\{\begin{array}{l}1+E(e_0)+E(e_1)+\alpha E(e_2)+\beta E(e_3)+\dfrac{\alpha(\alpha-1)}{2}E(e_2^2)+\dfrac{\beta(\beta-1)}{2}E(e_3^2)\\+\alpha E(e_2e_0)+\alpha E(e_1e_2)+\beta E(e_0e_3)+\beta E(e_1e_3)+\alpha\beta E(e_2e_3)+E(e_0e_1)\end{array}\right\}$$

(1.14)

Substituting the values from (1.11) and (1.12), we get

$$Bias\left(\hat{\bar{Y}}_P\right)=E\left(\hat{\bar{Y}}_P\right)-P^*$$

$$=f_n.P^*.\left\{\begin{array}{l}\alpha\rho_{Y_2X}C_{Y_2}C_X+\alpha\rho_{Y_1X}C_{Y_1}C_X+\beta\rho_{Y_2P}C_{Y_2}C_P+\beta\rho_{Y_1P}C_{Y_1}C_P\\+\dfrac{\alpha(\alpha-1)}{2}C_X^2+\dfrac{\beta(\beta-1)}{2}C_P^2+\alpha\beta\rho_{XP}C_XC_P+\rho_{Y_1Y_2}C_{Y_1}C_{Y_2}\end{array}\right\}$$

$$=f_n.P^*.A \quad \text{(say)} \tag{1.15}$$

Where

$$A=\left\{\begin{array}{l}\alpha\rho_{Y_2X}C_{Y_2}C_X+\alpha\rho_{Y_1X}C_{Y_1}C_X+\beta\rho_{Y_2P}C_{Y_2}C_P+\beta\rho_{Y_1P}C_{Y_1}C_P\\+\dfrac{\alpha(\alpha-1)}{2}C_X^2+\dfrac{\beta(\beta-1)}{2}C_P^2+\alpha\beta\rho_{XP}C_XC_P+\rho_{Y_1Y_2}C_{Y_1}C_{Y_2}\end{array}\right\}$$

Squaring (1.13) to the first order of approximation we have

$$\left(\hat{\bar{Y}}_P-P^*\right)^2=P^{*2}.(e_0+e_1+\alpha e_2+\beta e_3)^2 \tag{1.16}$$

Taking expectation we get

$$MSE(\hat{\bar{Y}}_P)=P^{*2}\left\{\begin{array}{l}E(e_0^2)+E(e_1^2)+\alpha^2E(e_2^2)+\beta^2E(e_3^2)+2\alpha E(e_0e_2)+2\alpha E(e_1e_2)\\+2\beta E(e_0e_3)+2\beta E(e_1e_3)+2\alpha\beta E(e_2e_3)+2E(e_0e_1)\end{array}\right\}$$

Substituting values from (1.12) we have

$$MSE(\hat{\bar{Y}}_P)=f_nP^{*2}\left\{\begin{array}{l}C_{Y_1}^2+C_{Y_2}^2+\alpha^2C_X^2+\beta^2C_P^2+2\rho_{Y_1Y_2}C_{Y_1}C_{Y_2}+2\alpha\rho_{Y_1X}C_{Y_1}C_X\\+2\alpha\rho_{Y_2X}C_{Y_2}C_X+2\beta\rho_{Y_1P}C_{Y_1}C_P+2\beta\rho_{Y_2P}C_{Y_2}C_P+2\alpha\beta\rho_{XP}C_XC_P\end{array}\right\}$$

$$=f_n.P^{*2}.B \quad \text{(say)} \tag{1.17}$$

Where

$$B=\begin{Bmatrix} C_{Y_1}^2+C_{Y_2}^2+\alpha^2C_X^2+\beta^2C_P^2+2\rho_{Y_1Y_2}C_{Y_1}C_{Y_2}+2\alpha\rho_{Y_1X}C_{Y_1}C_X \\ +2\alpha\rho_{Y_2X}C_{Y_2}C_X+2\beta\rho_{Y_1P}C_{Y_1}C_P+2\beta\rho_{Y_2P}C_{Y_2}C_P+2\alpha\beta\rho_{XP}C_XC_P \end{Bmatrix}$$

The minimum value of MSE is obtained if optimum values of α and β are

$$opt(\alpha)=\frac{C_{Y_1}\left(\rho_{Y_1P}\rho_{XP}-\rho_{Y_1X}\right)-C_{Y_2}\left(\rho_{XP}\rho_{Y_2P}-\rho_{Y_2X}\right)}{C_X\left(1-\rho_{XP}^2\right)}$$

$$opt(\beta)=\frac{C_{Y_1}\left(\rho_{Y_1X}\rho_{XP}-\rho_{Y_1P}\right)-C_{Y_2}\left(\rho_{XP}\rho_{Y_2X}-\rho_{Y_2P}\right)}{C_P\left(1-\rho_{XP}^2\right)}$$

And the minimum mean square error under the optimising values of the characterising scalars is given by

$$\min MSE\left(\overline{\overline{Y}}_P\right)=\left(\frac{1}{n}-\frac{1}{N}\right)P^{*2}\begin{Bmatrix}(1-R_{Y_2.XP}^2)C_{Y_2}^2+(1-R_{Y_1.XP}^2)C_{Y_1}^2+2\rho_{Y_1Y_2}C_{Y_1}C_{Y_2} \\ -\dfrac{2C_{Y_1}C_{Y_2}(\rho_{Y_1X}\rho_{Y_2X.P}P_1+\rho_{Y_1P}\rho_{Y_2P.X}P_2)}{(1-\rho_{XP}^2)^{1/2}}\end{Bmatrix}$$

where

$$P_1=(1-\rho_{Y_2P}^2)^{1/2}$$
$$P_2=(1-\rho_{Y_2X}^2)^{1/2} \tag{1.18}$$

Provided that

$$\left\{(1-R_{Y_2.XP}^2)C_{Y_2}^2+(1-R_{Y_1.XP}^2)C_{Y_1}^2>\frac{2C_{Y_1}C_{Y_2}(\rho_{Y_1X}\rho_{Y_2X.P}P_1+\rho_{Y_1P}\rho_{Y_2P.X}P_2)}{(1-\rho_{XP}^2)^{1/2}}-2\rho_{Y_1Y_2}C_{Y_1}C_{Y_2}\right\}$$

(1.19)

2. The Proposed Jack-Knife Estimator

Consider a random sample of size *n=2m* drawn from the finite population of size N by SRSWOR and split it into two random sub-sample of size m each .

Define the following estimators

$$\overline{Y}_1^{(1)}=\overline{y}_1^{(1)}.\overline{y}_1^{(1)}\left(\frac{\overline{x}^{(1)}}{\overline{\overline{X}}}\right)^{\alpha}\left(\frac{p^{(1)}}{P}\right)^{\beta}$$

$$\overline{Y}_1^{(2)} = \overline{y}_1^{(2)} . \overline{y}_1^{(2)} \left(\frac{\overline{x}^{(2)}}{\overline{X}} \right)^{\alpha} \left(\frac{p^{(2)}}{P} \right)^{\beta}$$

$$\overline{Y}_1^{(3)} = \overline{y}_1 . \overline{y}_1 \left(\frac{\overline{x}}{\overline{X}} \right)^{\alpha} \left(\frac{p}{P} \right)^{\beta} \tag{2.1}$$

Where $\overline{y}_1^{(1)}, \overline{y}_2^{(1)}, \overline{y}_1^{(2)}, \overline{y}_1^{(2)}, \overline{x}^{(1)}, \overline{x}^{(2)}, p^{(1)}, p^{(2)}$ are the respective means of the study variables, auxiliary variables and the auxiliary attribute based in the sample 1 and 2 each of size and $\overline{y}, \overline{x}, p$ are the respective means of the study variable, auxiliary variable and auxiliary attribute based on the entire sample.

From (1.13) we have

$$\begin{aligned} Bias(\overline{Y}_1^{(1)}) &= P^{*}.f_m.A \\ Bias(\overline{Y}_1^{(2)}) &= P^{*}.f_m.A \\ Bias(\overline{Y}_1^{(3)}) &= P^{*}.f_n.A = K_1 \end{aligned} \tag{2.2}$$

Let us define

$$\overline{Y}_1^{1} = \frac{\overline{Y}_1^{(1)} + \overline{Y}_1^{(2)}}{2} \tag{2.3}$$

This is an alternative estimator of population mean ratio. So,

$$Bias(\overline{Y}_1^{1}) = P^{*}.f_m.A = K_2 \quad \text{(say)} \tag{2.4}$$

Define the ratio

$$K = \frac{K_1}{K_2} = \frac{f_n}{f_m} = \frac{N-2m}{2(N-m)} \tag{2.5}$$

Using K, we propose the Jack-knife estimator

$$\overline{Y}_P^{*} = \frac{\overline{Y}_1^{(3)} - K\overline{Y}_1^{1}}{1-K} \tag{2.6}$$

Taking expectation on both sides, we get

$$E(\overline{Y}_P^{*}) = \frac{E(\overline{Y}_1^{(3)}) - KE(\overline{Y}_1^{1})}{1-K}$$

Substituting values from (2.2) and (2.4)

$$E(\overline{Y}_P^*) = \frac{\left(1-\frac{(N-2m)}{2(N-m)}\right)}{\left(1-\frac{(N-2m)}{2(N-m)}\right)}.P^* = P^* \tag{2.7}$$

Showing that the proposed Jack-knife estimator is an unbiased estimator of population mean product. Mean square error of the proposed estimator is defined as

$$MSE\left(\overline{Y}_P^*\right) = E\left(\overline{Y}_P^* - P^*\right)^2$$

$$= E\left(\frac{\overline{Y}_1^{(3)} - K\overline{Y}_1^1}{1-K} - P^*\right)^2$$

$$= \frac{1}{(1-K)^2}\left[E\left(\overline{Y}_1^{(3)} - P^*\right)^2 + K^2E\left(\overline{Y}_1^1 - P^*\right)^2 - 2KE\left(\overline{Y}_1^{(3)} - P^*\right)\left(\overline{Y}_1^1 - P^*\right)\right] \tag{2.8}$$

From (1.15) we have

$$E\left(\overline{Y}_1^{(3)} - P^*\right)^2 = MSE\left(\overline{Y}_1^{(3)}\right) = \left(\frac{1}{2m} - \frac{1}{N}\right).P^{*2}.B \tag{2.9}$$

Consider

$$E\left(\overline{Y}_1^1 - P^*\right)^2 = E\left(\frac{\overline{Y}_1^{(1)} + \overline{Y}_1^{(2)}}{2} - P^*\right)^2$$

(2.10)

From (1.15)

$$E\left(\overline{Y}_1^{(i)} - P^*\right)^2 = MSE\left(\overline{Y}_1^{(i)}\right) = \left(\frac{1}{m} - \frac{1}{N}\right).P^{*2}.B \qquad i=1,2 \tag{2.11}$$

Taking

$$\overline{y}_1^{(i)} = \overline{Y}_1(1+e_0^{(i)})$$
$$\overline{y}_2^{(i)} = \overline{Y}_2(1+e_1^{(i)})$$
$$\overline{x}^{(i)} = \overline{X}(1+e_2^{(i)})$$
$$p^{(i)} = P(1+e_3^{(i)})$$

With $E(e_0^{(i)}) = E(e_1^{(i)}) = E(e_2^{(i)}) = E(e_3^{(i)}) = 0$ i=1,2 (2.12)

Using (2.12) we can write

$$\left(\overline{Y}_1^{(i)}-P^*\right)=P^*\begin{pmatrix} e_0^{(i)}+e_1^{(i)}+\alpha e_2^{(i)}+\beta e_3^{(i)}+\frac{\alpha(\alpha-1)}{2}e_2^{(i)2}+\frac{\beta(\beta-1)}{2}e_3^{(i)2} \\ +\alpha e_0^{(i)}e_2^{(i)}+\alpha e_1^{(i)}e_2^{(i)}+\beta e_0^{(i)}e_3^{(i)}+\beta e_1^{(i)}e_3^{(i)}+\alpha\beta e_2^{(i)}e_3^{(i)}+e_0^{(i)}e_1^{(i)} \end{pmatrix} \quad (2.13)$$

To the first order of approximation

$$E\left(\overline{Y}_1^{(1)}-P^*\right)\left(\overline{Y}_1^{(2)}-P^*\right)=P^{*2}E\left[\left(\beta e_3^{(1)}+\alpha e_2^{(1)}+e_1^{(1)}+e_0^{(1)}\right)\left(\beta e_3^{(2)}+\alpha e_2^{(2)}+e_1^{(2)}+e_0^{(2)}\right)\right]$$

$$=P^{*2}\begin{pmatrix} \beta^2E\left(e_3^{(1)}e_3^{(2)}\right)+\alpha\beta E\left(e_3^{(1)}e_2^{(2)}\right)+\beta E\left(e_3^{(1)}e_0^{(2)}\right)+\beta E\left(e_3^{(1)}e_1^{(2)}\right)+\alpha\beta E\left(e_2^{(1)}e_3^{(2)}\right) \\ +\alpha^2E\left(e_2^{(1)}e_2^{(2)}\right)+\alpha E\left(e_2^{(1)}e_0^{(2)}\right)+\alpha E\left(e_2^{(1)}e_1^{(2)}\right)+\beta E\left(e_0^{(1)}e_3^{(2)}\right)+\alpha E\left(e_0^{(1)}e_2^{(2)}\right) \\ +E\left(e_0^{(1)}e_0^{(2)}\right)+E\left(e_0^{(1)}e_1^{(2)}\right)+\beta E\left(e_1^{(1)}e_3^{(2)}\right)+\alpha E\left(e_1^{(1)}e_2^{(2)}\right)+E\left(e_1^{(1)}e_0^{(2)}\right)+E\left(e_1^{(1)}e_1^{(2)}\right) \end{pmatrix}$$

Using the results given in Sukhatme and Sukhatme (1997)

$$E\left(e_0^{(1)}e_0^{(2)}\right)=-\frac{1}{N}C_{Y_1}^2$$

$$E\left(e_0^{(1)}e_1^{(2)}\right)=-\frac{1}{N}\rho_{Y_1Y_2}C_{Y_1}C_{Y_2}$$

$$E\left(e_0^{(1)}e_2^{(2)}\right)=-\frac{1}{N}\rho_{Y_1X}C_{Y_1}C_X$$

$$E\left(e_0^{(1)}e_3^{(2)}\right)=-\frac{1}{N}\rho_{Y_1P}C_{Y_1}C_P$$

$$E\left(e_0^{(2)}e_1^{(1)}\right)=-\frac{1}{N}\rho_{Y_1Y_2}C_{Y_1}C_{Y_2}$$

$$E\left(e_0^{(2)}e_2^{(1)}\right)=-\frac{1}{N}\rho_{Y_1X}C_{Y_1}C_X$$

$$E\left(e_0^{(2)}e_3^{(1)}\right)=-\frac{1}{N}\rho_{Y_1P}C_{Y_1}C_P$$

$$E\left(e_1^{(1)}e_1^{(2)}\right)=-\frac{1}{N}C_{Y_2}^2$$

$$E\left(e_1^{(1)}e_2^{(2)}\right)=-\frac{1}{N}\rho_{Y_2X}C_{Y_2}C_X$$

$$E\left(e_1^{(1)}e_3^{(2)}\right)=-\frac{1}{N}\rho_{Y_2P}C_{Y_2}C_P$$

$$E\left(e_1^{(2)}e_2^{(1)}\right)=-\frac{1}{N}\rho_{Y_2X}C_{Y_2}C_X$$

$$E\left(e_1^{(2)}e_3^{(1)}\right)=-\frac{1}{N}\rho_{Y_2P}C_{Y_2}C_P$$

$$E\left(e_2^{(1)}e_2^{(2)}\right)=-\frac{1}{N}C_X^2$$

$$E\left(e_2^{(1)}e_3^{(2)}\right)=-\frac{1}{N}\rho_{XP}C_XC_P$$

$$E\left(e_2^{(2)}e_3^{(1)}\right)=-\frac{1}{N}\rho_{XP}C_XC_P$$

$$E\left(e_3^{(1)}e_3^{(2)}\right)=-\frac{1}{N}C_P^2$$

We get

$$E\left(\overline{Y}_1^{(1)}-P^*\right)\left(\overline{Y}_1^{(2)}-P^*\right)=-\frac{P^{*2}}{N}.B \quad (2.14)$$

Putting the values from (2.11)and (2.14) in (2.10) we get

$$E\left(\overline{Y}_1^1-P^*\right)^2=MSE\left(\overline{Y}_1^1\right)=\frac{1}{4}\left(2\left(\frac{1}{m}-\frac{1}{N}\right)P^{*2}B-\frac{2}{N}P^{*2}B\right)$$

$$=\left(\frac{1}{2m}-\frac{1}{N}\right)P^{*2}B \quad (2.15)$$

Now consider

$$E\left(\overline{Y}_1^{(3)}-P^*\right)\left(\overline{Y}_1^1-P^*\right)=E\left(\overline{Y}_1^{(3)}-P^*\right)\left(\frac{\overline{Y}_1^{(1)}+\overline{Y}_1^{(2)}}{2}-P^*\right)$$

$$=\frac{1}{2}\left[E\left(\overline{Y}_1^{(3)}-P^*\right)\left(\overline{Y}_1^{(1)}-P^*\right)+E\left(\overline{Y}_1^{(3)}-P^*\right)\left(\overline{Y}_1^{(2)}-P^*\right)\right] \quad (2.16)$$

To the first order of terms we have

$$E\left(\overline{Y}_1^{(3)}-P^*\right)\left(\overline{Y}_1^{(i)}-P^*\right)=P^{*2}E\left[\left(\beta e_3+\alpha e_2+e_1+e_0\right)\left(\beta e_3^{(i)}+\alpha e_2^{(i)}+e_1^{(i)}+e_0^{(i)}\right)\right]$$

$$=P^{*2}\begin{pmatrix}\beta^2E\left(e_3e_3^{(i)}\right)+\alpha\beta E\left(e_3e_2^{(i)}\right)+\beta E\left(e_3e_0^{(i)}\right)+\beta E\left(e_3e_1^{(i)}\right)+\alpha\beta E\left(e_2e_3^{(i)}\right)\\ +\alpha^2E\left(e_2e_2^{(i)}\right)+\alpha E\left(e_2e_0^{(i)}\right)+\alpha E\left(e_2e_1^{(i)}\right)+\beta E\left(e_0e_3^{(i)}\right)+\alpha E\left(e_0e_2^{(i)}\right)\\ +E\left(e_0e_0^{(i)}\right)+E\left(e_0e_1^{(i)}\right)+\beta E\left(e_1e_3^{(i)}\right)+\alpha E\left(e_1e_2^{(i)}\right)+E\left(e_1e_0^{(i)}\right)+E\left(e_1e_1^{(i)}\right)\end{pmatrix} \quad \text{i=1,2}$$

Substituting the following results given by Sukhatme and Sukhatme (1997)

$$E\left(e_0e_0^{(i)}\right)=f_nC_{Y_1}^2 \qquad E\left(e_1e_2^{(i)}\right)=f_n\rho_{Y_2X}C_{Y_2}C_X$$

$$E\left(e_0e_1^{(i)}\right)=f_n\rho_{Y_1Y_2}C_{Y_1}C_{Y_2} \qquad E\left(e_1e_3^{(i)}\right)=f_n\rho_{Y_2P}C_{Y_2}C_P$$

$$E\left(e_0e_2^{(i)}\right)=f_n\rho_{Y_1X}C_{Y_1}C_X \qquad E\left(e_1e_2^{(i)}\right)=f_n\rho_{Y_2X}C_{Y_2}C_X$$

$$E\left(e_0e_3^{(i)}\right)=f_n\rho_{Y_1P}C_{Y_1}C_P \qquad E\left(e_1e_3^{(i)}\right)=f_n\rho_{Y_2P}C_{Y_2}C_P$$

$$E\left(e_0e_1^{(i)}\right)=f_n\rho_{Y_1Y_2}C_{Y_1}C_{Y_2} \qquad E\left(e_2e_2^{(i)}\right)=f_nC_X^2$$

$$E\left(e_0e_2^{(i)}\right)=f_n\rho_{Y_1X}C_{Y_1}C_X \qquad E\left(e_2e_3^{(i)}\right)=f_n\rho_{XP}C_XC_P$$

$$E\left(e_0e_3^{(i)}\right)=f_n\rho_{Y_1P}C_{Y_1}C_P \qquad E\left(e_2e_3^{(i)}\right)=f_n\rho_{XP}C_XC_P$$

$$E\left(e_1e_1^{(i)}\right)=f_nC_{Y_2}^2 \qquad E\left(e_3e_3^{(i)}\right)=f_nC_P^2$$

We get

$$E\left(\overline{Y}_1^{(3)}-P^*\right)\left(\overline{Y}_1^{(i)}-P^*\right)=\left(\frac{1}{2m}-\frac{1}{N}\right)P^{*2}B \quad (2.17)$$

Putting the values from (2.17) in (2.16) we get

$$E\left(\overline{Y}_1^{(3)}-P^*\right)\left(\overline{Y}_1^1-P^*\right)=\frac{1}{2}\left[\left(\frac{1}{2m}-\frac{1}{N}\right)P^{*2}.B+\left(\frac{1}{2m}-\frac{1}{N}\right)P^{*2}.B\right]$$

$$=\left(\frac{1}{2m}-\frac{1}{N}\right)P^{*2}.B \quad (2.18)$$

Substituting the values from (2.9), (2.15)and (2.18)in (2.8) we get

$$MSE\left(\overline{Y}_P^*\right)=\frac{1}{(1-K)^2}\left(\frac{1}{2m}-\frac{1}{N}\right)P^{*2}.\left(1+K^2-2K\right)B$$

$$= f_n P^{*2}.B \qquad (2.19)$$

The optimum values of mean square error is obtained for

$$opt(\alpha)=\frac{C_{Y_1}\left(\rho_{Y_1P}\rho_{XP}-\rho_{Y_1X}\right)-C_{Y_2}\left(\rho_{XP}\rho_{Y_2P}-\rho_{Y_2X}\right)}{C_X\left(1-\rho_{XP}^2\right)}$$

$$opt(\beta)=\frac{C_{Y_1}\left(\rho_{Y_1X}\rho_{XP}-\rho_{Y_1P}\right)-C_{Y_2}\left(\rho_{XP}\rho_{Y_2X}-\rho_{Y_2P}\right)}{C_P\left(1-\rho_{XP}^2\right)}$$

And the minimum value of mean square error under the optimum values of the characterising scalars is given by

$$\min MSE\left(\overline{Y}_P\right)=\left(\frac{1}{n}-\frac{1}{N}\right)P^{*2}\left\{\begin{array}{l}(1-R^2_{Y_2.XP})C^2_{Y_2}+(1-R^2_{Y_1.XP})C^2_{Y_1}+2\rho_{Y_1Y_2}C_{Y_1}C_{Y_2}\\ -\dfrac{2C_{Y_1}C_{Y_2}(\rho_{Y_1X}\rho_{Y_2X.P}P_1+\rho_{Y_1P}\rho_{Y_2P.X}P_2)}{(1-\rho^2_{XP})^{1/2}}\end{array}\right\}$$

where

$$P_1=(1-\rho^2_{Y_2P})^{1/2}$$
$$P_2=(1-\rho^2_{Y_2X})^{1/2}$$

Provided

$$\left\{(1-R^2_{Y_2.XP})C^2_{Y_2}+(1-R^2_{Y_1.XP})C^2_{Y_1}>\frac{2C_{Y_1}C_{Y_2}(\rho_{Y_1X}\rho_{Y_2X.P}P_1+\rho_{Y_1P}\rho_{Y_2P.X}P_2)}{(1-\rho^2_{XP})^{1/2}}-2\rho_{Y_1Y_2}C_{Y_1}C_{Y_2}\right\}$$

This is same as that of mimimum MSE of the family of estimator

Gain in efficiency can be calculated by the formula:

$$\frac{MSE\left(\hat{P}^{*}\right)-MSE\left(\hat{\bar{Y}}_{P}\right)}{MSE\left(\hat{P}^{*}\right)}*100$$

where

$$MSE\left(\hat{P}^{*}\right)=f_{n}P^{*2}\left(C_{Y_1}^{2}+C_{Y_2}^{2}+2\rho_{Y_1Y_2}C_{Y_1}C_{Y_2}\right)$$

Empirical study

Source : Advance Data from Vital and Health Statistics Number 347, October 7, 2004 (CDC)

Y_1 = Height of the people

Y_2 = Weight of the people

X =Age of the people

P = Sex of the people

N =35, n =15

$\bar{Y}_1$ =54.92 $\bar{Y}_2$ =39.06

$\bar{X}$ =10.26

$C_{Y_1}^{2}$ =0.037363 $\rho_{Y_1Y_2}^{2}$ =0.945906

$C_{Y_2}^{2}$ =0.238729 $\rho_{Y_1X}^{2}$ =0.96162

C_{X}^{2} =0.0500114 $\rho_{Y_1P}^{2}$ =0.004977

C_{P}^{2} =0.972222 $\rho_{Y_2X}^{2}$ =0.964356

$\rho_{Y_2P}^{2}$ =0.005785 ρ_{XP}^{2} =0.0012588

Table: PRE of various estimators with respect to sample mean ratio

Estimators	PRE
$\hat{\bar{Y}}_1$	100
$\hat{\bar{Y}}_2$	218.42
$\hat{\bar{Y}}_3$	34.53
$\hat{\bar{Y}}_4$	56.83

$\hat{\bar{Y}}_5$	30.01
$\hat{\bar{Y}}_6$	3911.33
$\hat{\bar{Y}}_7$	100.56
$\bar{Y}_P^*$	**4167.03**

Gain in efficiency of $\bar{Y}_P^*$ with respect to $\hat{P}^*$ = 97.60 %

Conclusion

The comparative study of the proposed Jack-Knife estimator establishes its superiority in the sense of unbiasedness and minimum mean square error over various estimators available in the literature using information regarding auxiliary variable and attribute under the optimum conditions.

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