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Nonlinear Resonances in a Forced Modified Rayleigh-Duffing Oscillator.

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Abstract
We investigate in this paper the parametric resonances of a forced modified Rayleigh-Duffing oscillator.We analyze the related equation by method of multiple scales and we obtain primary resonance, super-harmonic, sub-harmonic resonances order-two and order-three. We obtain also regions where steady-state subharmonic responses exist. We also use the amplitude-frequency curve to demonstrate the effect of various parameters on the response of the system. It is obtained the jump and hysteresis phenomenon in the system behaviors and bi-stability phenomenon in the evolution of the amplitude of the oscillations of the system. It is noted that the pure and unpure quadratic damping parameters and parametric excitation affect the nonlinear order-two sub-harmonic and order-two super-harmonic and cubic damping parameters, cubic restoring parameter and amplitude of external force influence all types of nonlinear resonances obtained.

***Keywords:*Modified Rayleigh-Duffing**, **parametric resonances**, **sub-harmonic**, **super-harmonic**, **resonanceshysteresis**.

1. Introduction

Many problems in physics, chemistry, biology, etc., are related to nonlinear self excited oscillators [1]. For example, the self-excited oscillations in bridges and airplane wings, the beating of a heart and the nonlinear model of a machine tool chatter [2]. A self-excited oscillator is a system which has some external source of energy upon which it can be drawn. Self-excited systems have a long history in the field of mechanics [3, 4]. One of key problems in the theory of nonlinear oscillations is a search of possibilities to estimate their amplitude and period analytically. In Refs. [5-7] the nonlinear ship rolling response can be rewritten as follow:

$$\ddot{x} + 2\mu|\dot{x}| + \beta_1|x|\dot{x} + \beta_2|\dot{x}|\dot{x} + \delta_1 x^2\,\dot{x} + \delta_2\dot{x}^3 + \cdots + \omega_0{}^2 x + a_3x^3 + a_5x^5 + \cdots += F\cos\omega\, t, \quad (1)$$

where ω_0 and $\boldsymbol{\omega}$are internal and external frequencies respectively, $\boldsymbol{\mu}$, β_i and δ_i arelinear, quadratic nonlinear and cubic nonlinear damping coefficients respectively and a_i are restoring coefficients. $F$is the external excitation amplitude. In this case, (Francescutto and Contento [5]) are used experimental results and parameter identification technique to study bifurcations in ship rolling, application of the extended Melnikov's method are used by W. Wu and L. McCue [7] to analyze ship motion without the constraint of small linear damping. Two roll motion models are analyzed here. One is a simple roll model with nonlinear damping and cubic restoring moment. The other is the model with biased restoring moment. In the other hand (Miguel and Sanjuan [8]) analyzed the effect of nonlinear damping on the universal escape oscillator.

$$\ddot{x} + \sum_{p=1}^{n} c_p\,\dot{x}|\dot{x}|^{p-1} + \sum_{j=1}^{m} a_j x^j = F\cos\Omega\, t, \quad (2)$$

where c_p is the nonlinear damping and a_j restoring coefficients. In these papers, the authors are obtained the resonances states in their system and they are shown the resonance curve is highly affected by nonlinear damping. Parametric perturbations are characterized by parameters periodically in time changing and they are described by homogeneous differential equations of motion. Many works on self-excited, parametrically and externally excited are well known and deeply investigated in the literature separately. Minorski [9] is one of the first authors considering the interaction between two different types of perturbations. Warminski [10] emphasizes the differences in modelling ideal and non-ideal systems for a chosen class of self-excited, parametric and externally excited vibrations (see also [11, 12]). Recently, many of those studies lead to the parametric excitation combined with self-excited system and subjected to an external force which quite often takes the form

$$\ddot{x} + \eta(x;\dot{x})\dot{x} + \left(1 - \mu\cos 2\omega t\right)(x + \alpha x^3) = F\cos\omega\, t \quad (3)$$

where $\eta(x;\dot{x})$is a nonlinear damping function. The effect of nonlinear damping on a nonlinear oscillator was investigated previously in [13], showing among other things how it affected the evolution of fractalization of phase space.

Autoparametric resonance plays an important part in nonlinear engineering while posing interesting mathematical challenges. The linear dynamics is already nontrivial whereas the nonlinear dynamics of such systems is extremely rich and largely unexplored [14-17]. Tina Marie Morrison in her thesis [18], have investigated the dynamics of a system consisting of a simple harmonic oscillator with small nonlinearity, damping and parametric forcing in the neighborhood of 2:1 resonance near a Hopf bifurcation:

$$\ddot{z} + \epsilon A\dot{z} + (1 + \epsilon k_1 + \epsilon B\cos 2\omega\, t)z + \epsilon(\beta_1 z^3 + \beta_2 z^2\dot{z} + \beta_3 z\dot{z}^2 + \beta_4\dot{z}^3 = 0. \quad (4)$$

In the present work we consider the modified Rayleigh-Duffing oscillator modeled by the following equation:

$$\ddot{x} + \epsilon\mu|(1 - \dot{x}| + \beta_1|x|\dot{x}^2)\dot{x} + \epsilon\beta\dot{x}^2 + \epsilon k_1 x\dot{x} + \epsilon k_2 x\,\dot{x}^2 + (\omega^2 + \epsilon\alpha\cos\Omega\, t) + \epsilon\lambda x^3 = F\cos\Omega t. \quad (5)$$

Our interest in understanding the behavior of this equation is motivated by many applications. The first is a model of the El Nino Southern Oscillation (ENSO) coupled tropical ocean-atmosphere weather phenomenon [19,20] in which the state variables are temperature and depth of a region of the ocean called the thermocline.

The annual seasonal cycle is the parametric excitation. The model exhibits a Hopf bifurcation in the absence of parametric excitation. The second application involves a MEMS device [21,22] consisting of a $30\boldsymbol{\mu m}$ diameter silicon disk which can be made to vibrate by heating it with a laser beam resulting in a Hopf bifurcation.

The parametric excitation is provided by making the laser beam intensity vary periodically in time. Other applications of our system are described in [23].

We focus our attention on the study of the differents resonances which can exist in the forced parametric modified Rayleigh-Duffing oscillator. We seek approximate solutions to Eq. (5) by using the method of multiple scales (MMS)and we find the peak amplitude of resonances phenomenon. We study the effects of certains parameters of this oscillator on the differents resonances. Finally, we study the chaotic motion of this oscillator by simulation in the sub-harmonic and super-harmonic regions.

2. Resonances of the forced modified Rayleigh-Duffing oscillator

We use the method of multiple scales (MMS) to seek approximate solutions to Eq.(5). The analysis reveals the existence of various super-harmonic and subharmonic resonances. The method of multiple scales supposed that the approximate steady solution of first order for Eq.(5) is of the form [24]

$$x(t,\epsilon) = x_0(T_0,T_1) + \epsilon x_1(T_0,T_1) \ldots. \quad (6)$$

where$T_n = \epsilon^n t$.. Then$\frac{d}{dt} = D_0 + \epsilon D_1(15)$, $\frac{d^2}{dt^2} = {D_0}^2 + 2\epsilon D_0 D_1\ldots$ with $D_n^m = \frac{\partial^m}{\partial {T_n}^m}$

Substituting Eq.(6) into Eq.(5) and equating the coefficients of the same power of small parameter ϵ, one obtains

In order ϵ^0 : $D_0{}^2 x_0 + \omega^2 x_0 = F \cos \Omega\, t$, (7)
In order ϵ :
$D_1{}^2 x_1 + \omega^2 x_1 = -2D_1 D_0 x_0 - \boldsymbol{\mu} D_0 x_0 + \boldsymbol{\mu}(D_0 x_0)^{\mathbf{3}} - \boldsymbol{\beta}(D_0 x_0)^{\mathbf{2}} - \boldsymbol{\alpha} x_0 \cos \Omega\, T_0 - k_1 x_0 D_0 x_0 - k_2 x_0 (D_0 x_0)^{\mathbf{2}} - \boldsymbol{\lambda} x_0{}^{\mathbf{3}}$ (8)

The solution for Eq.(7) is

$$x_0 = A(T_1)e^{iT_0} + \Lambda e^{i\Omega T_0} + CC \ (9),$$

where

$\Lambda = \frac{F}{2(\omega^2-\Omega^2)}$, $A = \frac{1}{2} a e^{i\theta}$,(10)
CC stands for complex conjugate of preceding terms. Substituting the solution x_0 from Eq.(9) into Eq.(8), and expanding the terms on the right hand side, we obtain

$D_1{}^2 x_1 + \omega^2 x_1 = [-2i\omega A' - i\mu\omega A + 3i\mu\omega^3 A^2 \bar{A} + 6i\mu\omega\Omega^2 A\Lambda^2 - 3\lambda A^2 \bar{A} - 6\lambda A\Lambda^2 - k_2\omega^2 A^2 \bar{A} - 2k_2\Omega^2 A\Lambda^2]e^{i\omega\, T_0}$+[-i$\boldsymbol{\mu}\Lambda\Omega$+6i$\boldsymbol{\mu}\omega^2\Omega A\bar{A}\Lambda + 3i\Omega^3\Lambda^3 - 6\lambda\Lambda A\bar{A} - 3\lambda\Lambda^3 - 2k_2\omega^2 \Lambda A\bar{A} + 3i\Omega^3\Lambda^3]e^{i\Omega\, T_0}$+[-i$\omega^3 A^3 - \lambda A^3 + k_2\omega^2 A^3]e^{3i\omega\, T_0} + [-i\mu\omega^3\Lambda^3 - \lambda\Lambda^3 + k_2\omega^2\Lambda^3]e^{3i\Omega\, T_0} + [-3i\boldsymbol{\mu\omega^2\bar{A}^2\Omega}\Lambda - \mathbf{3}\boldsymbol{\lambda\bar{A}^2}\Lambda + k_2\omega^2\boldsymbol{\bar{A}^2}\Lambda - \mathbf{2}k_2\boldsymbol{\omega\Omega\bar{A}^2}\boldsymbol{\Lambda}]e^{i(-2\omega+\Omega)\, T_0} + [-\frac{1}{2}\boldsymbol{\alpha}\bar{A} - ik_1\Omega\Lambda\bar{A} - ik_1\omega\Lambda\bar{A} - 2\beta\omega\Omega\Lambda\bar{A}]e^{i(-\omega+\Omega)\, T_0} + [-\frac{1}{2}\boldsymbol{\alpha}\Lambda - ik_1\Omega\Lambda^2 + \beta\Omega^2\Lambda^2]e^{2i\Omega\, T_0} + \boldsymbol{\beta A^2} e^{2i\omega\, T_0} - 2\boldsymbol{\beta}\omega^2 A\bar{A} - 2\boldsymbol{\beta}\Omega^2\Lambda^2 + CC + NST$ (11)

where NST is non resonance terms. We need to eliminate coefficients of $e^{i\omega\, T_0}$that constitute the secular terms and would make the solutions unbounded. Thesolvability condition is thus set by equating the coefficients of $e^{i\omega\, T_0}$terms to zero.

2.1 Superharmonic resonances

In this case, we consider first $2\Omega = \boldsymbol{\omega} + \epsilon\boldsymbol{\sigma}$ and after $3\Omega = \boldsymbol{\omega} + \epsilon\boldsymbol{\sigma}$ where $\boldsymbol{\sigma}$ is a detuning parameter.

2.1.1$2\Omega = \boldsymbol{\omega} + \epsilon\boldsymbol{\sigma}$
If $2\Omega = \boldsymbol{\omega} + \epsilon\boldsymbol{\sigma}$ the condition for elimination of secular terms in Eq.(11) is
$[-2i\omega A' - i\mu\omega A + 3i\mu\omega^3 A^2\bar{A} + 6i\mu\omega\Omega^2 A\Lambda^2 - 3\lambda A^2\bar{A} - 6\lambda A\Lambda^2 - k_2\omega^2 A^2\bar{A} - 2k_2\Omega^2 A\Lambda^2]$+$(-\frac{1}{2}\boldsymbol{\alpha}\Lambda - ik_1\Omega\Lambda^2 + \beta\Omega^2\Lambda^2)$=0, (12)
with $T_1 = \epsilon T_0$. To this order, A is considered to be a function to T_1 only. Then, substituting the polar form Eq.(10) into Eq.(12) and equating the real and imaginary parts, one gets

$a' = \left(-\frac{1}{2} + 3\Omega^2\Lambda^2\right)\boldsymbol{\mu a} + \frac{\mathbf{3}}{\mathbf{8}}\boldsymbol{\mu}\omega^2 a^3 - \frac{k_1\Omega\Lambda^2\cos\gamma}{\omega} + \frac{(-\frac{1}{2}\alpha\Lambda + \beta\Omega^2\Lambda^2)\sin\gamma}{\omega}$, (13)

$a\boldsymbol{\gamma}' = \boldsymbol{a\sigma} - (\mathbf{3}\boldsymbol{\lambda} + k_2\Omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}} a - \frac{1}{8}(\mathbf{3}\boldsymbol{\lambda} + k_2\omega^2)\frac{a^3}{\boldsymbol{\omega}} + \frac{k_1\Omega\Lambda^2\sin\gamma}{\omega} + \frac{(-\frac{1}{2}\alpha\Lambda + \beta\Omega^2\Lambda^2)\cos\gamma}{\omega}$, (14)

with$\boldsymbol{\gamma}$=$\boldsymbol{\sigma}T_1 - \boldsymbol{\theta}$ Writting $a' = \boldsymbol{\gamma}' = \mathbf{0}$ to find the stable period solution, we obtain

$\left(-\frac{1}{2} + 3\Omega^2\Lambda^2 + \frac{\mathbf{3}}{\mathbf{8}}\omega^2 a^2\right)\mu a = \frac{k_1\Omega\Lambda^2\cos\gamma}{\omega} - \frac{(-\frac{1}{2}\alpha\Lambda + \beta\Omega^2\Lambda^2)\sin\gamma}{\omega}$, (15)

$$[\sigma-(3\lambda+k_2\Omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}}-\frac{1}{8}(\boldsymbol{3\lambda}+k_2\omega^2)\frac{a^2}{\boldsymbol{\omega}}]a=-\frac{k_1\Omega\Lambda^2\sin\gamma}{\omega}-\frac{\left(-\frac{1}{2}\alpha\Lambda+\beta\Omega^2\Lambda^2\right)\cos\gamma}{\omega},(16)$$

Considering equations Eq.(15) and Eq.(16), the frequency-response curve for superharmonic resonance is

$$[[\sigma-(3\lambda+k_2\Omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}}-\frac{1}{8}(\boldsymbol{3\lambda}+k_2\omega^2)\frac{a^2}{\boldsymbol{\omega}}]^2+\left(-\frac{1}{2}+3\Omega^2\Lambda^2+\frac{3}{8}\omega^2a^2\right)^2\mu^2]a^2$$
$$=\frac{\left(-\frac{1}{2}\alpha\Lambda+\beta\Omega^2\Lambda^2\right)^2}{\omega^2}+\frac{\Omega^2{k_1}^2\Lambda^4}{\omega^2}(17).$$

At steady-state, the relationship between the response amplitude and the detuning parameter $\boldsymbol{\sigma}$ is

$$\sigma=(\boldsymbol{3\lambda}+k_2\Omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}}-\frac{1}{8}(\boldsymbol{3\lambda}+k_2\omega^2)\frac{a^2}{\boldsymbol{\omega}}\pm\frac{\Omega^2{k_1}^2\Lambda^4}{\omega^2\,a^2}+\frac{\left(-\frac{1}{2}\alpha\Lambda+\beta\Omega^2\Lambda^2\right)^2}{\omega^2\,a^2}$$
$$-[\left(-\frac{1}{2}+3\Omega^2\Lambda^2+\frac{3}{8}\omega^2a^2\right)^2\mu^2]^{\frac{1}{2}}.(18)$$

The peak amplitude would verify the following equation:

$$\frac{\Omega^2{k_1}^2\Lambda^4}{\omega^2\,a^2}+\frac{\left(-\frac{1}{2}\alpha\Lambda+\beta\Omega^2\Lambda^2\right)^2}{\omega^2\,a^2}=[\left(-\frac{1}{2}+3\Omega^2\Lambda^2+\frac{3}{8}\omega^2a^2\right)^2\mu^2]^{\frac{1}{2}},(19)$$

Thus, we obtain that the corresponding value of $\boldsymbol{\sigma}$ is

$$\boldsymbol{\sigma_p}=(\boldsymbol{3\lambda}+k_2\Omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}}-\frac{1}{8}(\boldsymbol{3\lambda}+k_2\omega^2)\frac{a_p^2}{\boldsymbol{\omega}}.(20)$$

We can conclude that:

- the peak value of $\boldsymbol{\sigma}$ is independent of k_2 and λ,
- k_2 and λ,affect the peak location and as they increase, $|\boldsymbol{\sigma_p}|$ increases.

We plot now the frequency-response curve from Eq.(17). In Fig. 1, the frequency-response curve are plotted for fixed values of linear and nonlinear parameters. The nonlinear resonance curve show characteristic jumps and pronounced hysteresis phenomena. The system hence has multi-stable solutions, and will be the site of oscillations increasingly important to achieve a state of equilibrium that depends on the dissipative components of the system, or to a breaking of a component of the system. Near the resonant frequency, the system can accumulate energy.

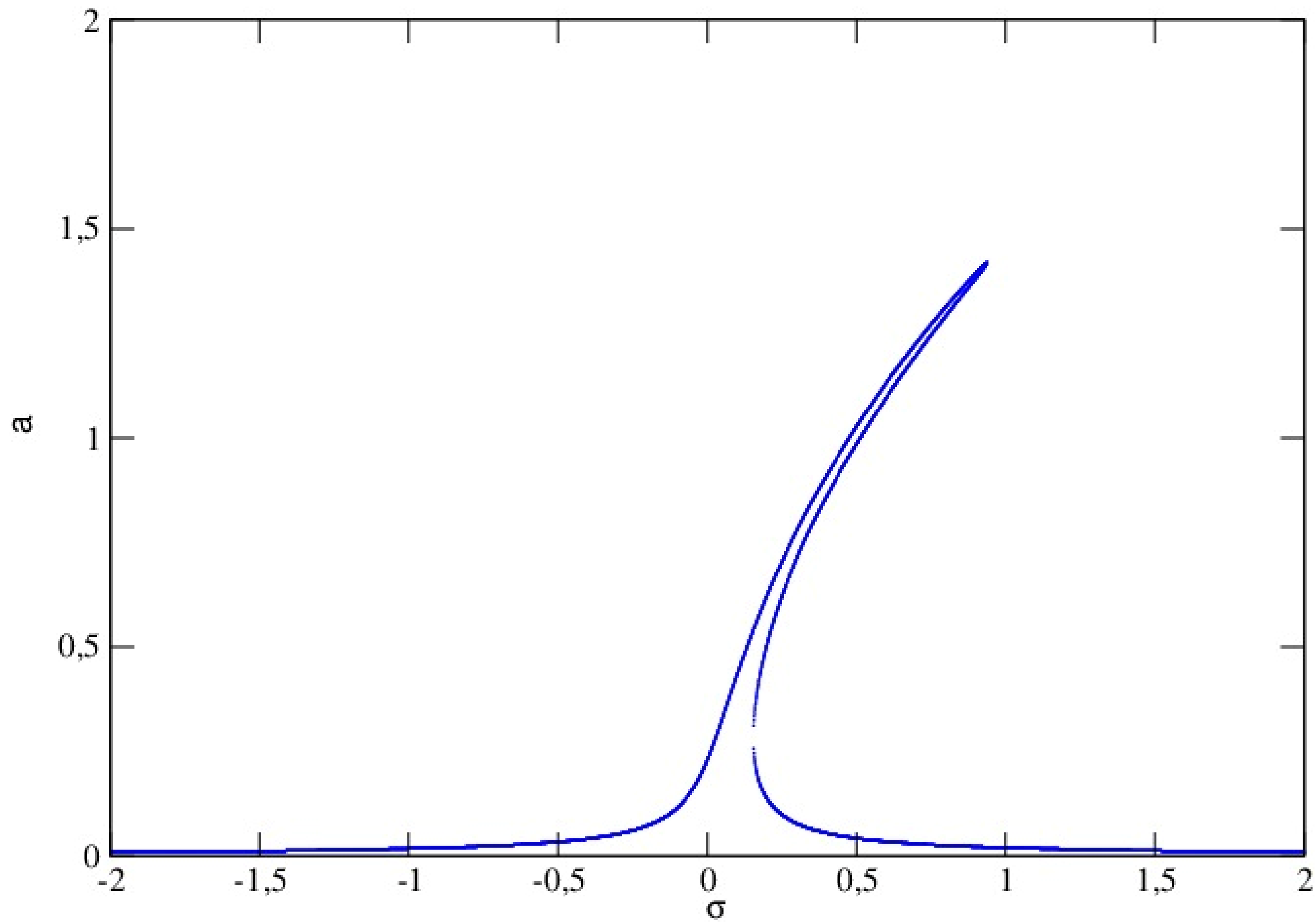

Fig. 1 Superharmonic resonance in the space (a,) for $\boldsymbol{\mu}$= 0.05; $\boldsymbol{\alpha}$= 0.3; λ = 1; $\boldsymbol{\beta}$ = 0.25; k_2 = 0.5; k_1 = 0.3; F = 0.2.

2.1.2 $3\Omega = \boldsymbol{\omega} + \epsilon\boldsymbol{\sigma}$

If the first term and the term which have $3\Omega T_0$ 0 as an exponential argument of the right member of Eq.(11) are the secular terms. The condition for the elimination of secular terms is

$$-2i\omega A' - i\mu\omega A + 3i\mu\omega^3 A^2\bar{A} + 6i\mu\omega\Omega^2 A\Lambda^2 - 3\lambda A^2\bar{A} - 6\lambda A\Lambda^2 - k_2\omega^2 A^2\bar{A} - 2k_2\Omega^2 A\Lambda^2 + (-i\mu\omega^3\Lambda^3 - \lambda\Lambda^3 + k_2\omega^2\Lambda^3)e^{i\sigma T_1}, \qquad (21)$$

Following the analysis done in the previous section for the 2 super-harmonic resonance, we rewrite A and separate the equation into real and imaginary parts.

Using $\boldsymbol{\gamma}$ =$\boldsymbol{\sigma}$ $\mathbf{T_1}$-$\boldsymbol{\Theta}$, we arrive at a homogeneous set of equations in $\boldsymbol{\gamma}$ and a

$$a' = \left(-\frac{1}{2} + 3\Omega^2\Lambda^2\right)\boldsymbol{\mu a} + \frac{\mathbf{3}}{\mathbf{8}}\boldsymbol{\mu}\omega^2 a^3 - \mu\omega^2\Lambda^3\cos\gamma - \frac{(\lambda - k_2\Omega^2)\Lambda^3\sin\gamma}{\omega}, \ (22)$$

$$a\boldsymbol{\gamma}' = \boldsymbol{a\sigma} - (\mathbf{3}\boldsymbol{\lambda} + k_2\Omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}}a - \frac{1}{8}(\mathbf{3}\boldsymbol{\lambda} + k_2\omega^2)\frac{a^3}{\boldsymbol{\omega}} + \mu\omega^2\Lambda^3\sin\gamma - \frac{(\lambda - k_2\Omega^2)\Lambda^3\cos\gamma}{\omega}. \ (23)$$

Steady-state solution $a' = \gamma' = 0$ lead to

$$[\boldsymbol{\sigma} - (\mathbf{3}\boldsymbol{\lambda} + k_2\Omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}} - \frac{1}{8}(\mathbf{3}\boldsymbol{\lambda} + k_2\omega^2)\frac{a^2}{\boldsymbol{\omega}}]^2 a^2 + \left(-\frac{1}{2} + 3\Omega^2\Lambda^2 + \frac{\mathbf{3}}{\mathbf{8}}\omega^2 a^2\right)^2 \mu^2 a^2 = [\mu^2\omega^6 + (\lambda - k_2\Omega^2)^2]\frac{\Lambda^6}{\omega^2}. \ (24)$$

We determine the detuning parameter $\boldsymbol{\sigma}$ from Eq.(24)

$$\sigma = (3\lambda + k_2\Omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}} + \frac{1}{8}(\boldsymbol{3\lambda} + k_2\omega^2)\frac{a^2}{\boldsymbol{\omega}} \pm [[\mu^2\omega^6 + (\lambda - k_2\Omega^2)^2]\frac{\Lambda^6}{\omega^2 a^2} - \left(-\frac{1}{2} + 3\Omega^2\Lambda^2 + \frac{\mathbf{3}}{\mathbf{8}}\omega^2 a^2\right)^2 \mu^2]^{\frac{1}{2}},$$
(26)

And corresponding value of $\boldsymbol{\sigma}$ is

$$\sigma_p = (3\lambda + k_2\Omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}} + \frac{1}{8}(\boldsymbol{3\lambda} + k_2\omega^2)\frac{a_p^2}{\boldsymbol{\omega}} \text{ (27).}$$

Therefore, we notice that the peak amplitude and frequency of this resonance are affected by third order non linearity parameters and by the forcing amplitude but the parametric excitation term and the coefficients of the quadratic nonlinear terms do not contribute to the resonance at first order. We plot in Fig. 2 the frequency-response curve giving by Eq.(24). This curve also proves that the amplitude of the resonance frequency increases with the external exciting force to which the order-three resonance super-harmonic. The nonlinear resonance curve show characteristic jumps and pronounced hysteresis phenomena.

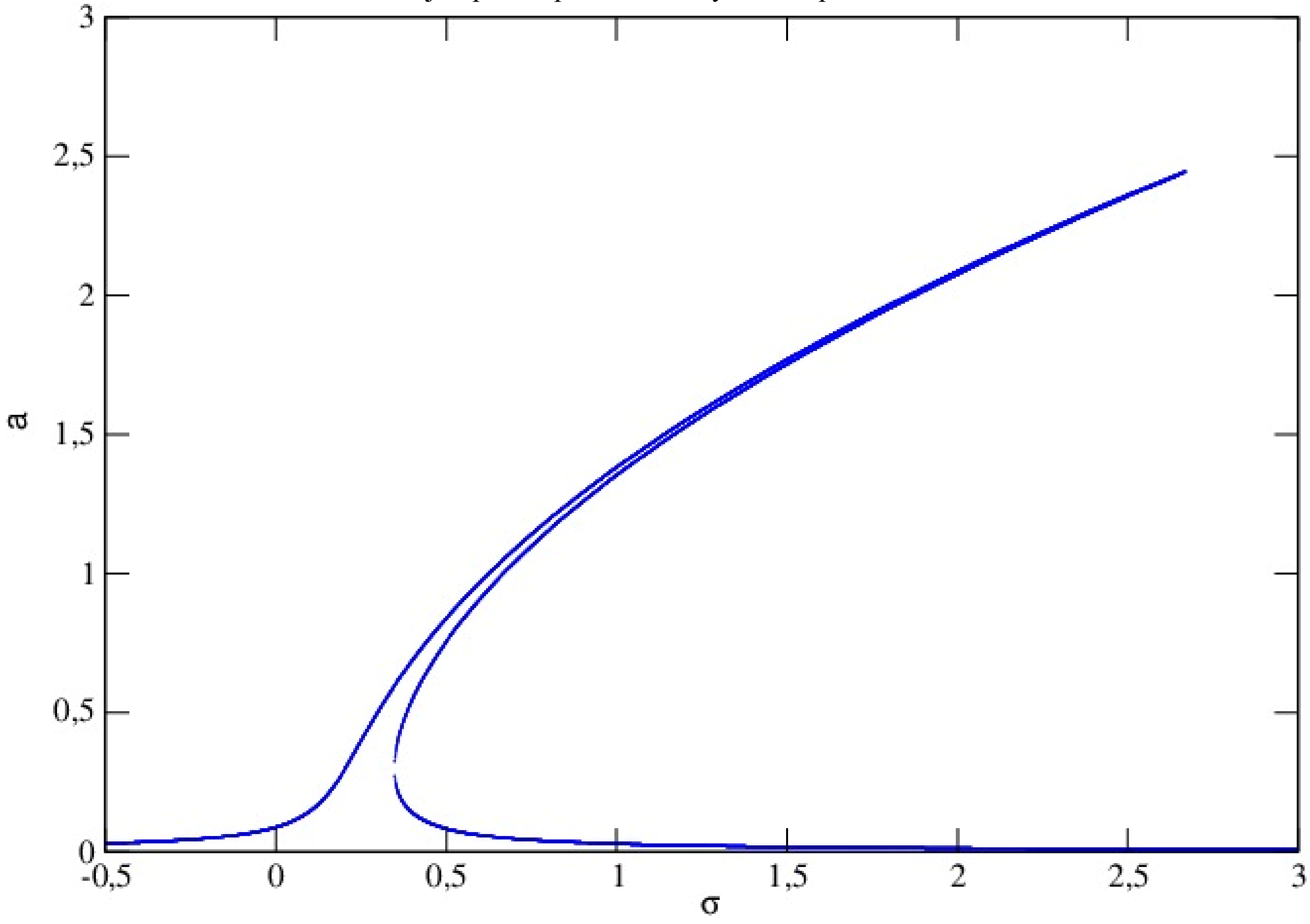

Fig. 2 Superharmonic resonance in the space(a, σ); for $\boldsymbol{\mu}$= 0.005; λ= 1; k_2 = 0.25; F = 0.5.

2.2 Subharmonic resonances

The sub-harmonic resonance takes place if $\Omega = 2\boldsymbol{\omega} + \epsilon\boldsymbol{\sigma}$or $\Omega = 3\boldsymbol{\omega} + \epsilon\boldsymbol{\sigma}$

2.2.1 $= \Omega = 2\omega + \epsilon\sigma$

The first term and the term with $(\Omega - \omega)\, T_0$ in Eq.(11) contribute to secular terms. The condition of the elimination of secular terms is

$$-2i\omega A' - i\mu\omega A + 3i\mu\omega^3 A^2\bar{A} + 6i\mu\omega\Omega^2 A\Lambda^2 - 3\lambda A^2\bar{A} - 6\lambda A\Lambda^2 - k_2\omega^2 A^2\bar{A} - 2k_2\Omega^2 A\Lambda^2 + (-\frac{1}{2}\boldsymbol{\alpha}\bar{A} - ik_1\Omega\Lambda\bar{A} - ik_1\omega\Lambda\bar{A} - 2\beta\omega\Omega\Lambda\bar{A})e^{i\sigma T_1} = 0, (28)$$

. We substitute the polar notation for A Eq.(10) in Eq.(28), and equate the real and imaginary parts, then let with $\boldsymbol{\gamma}=\boldsymbol{\sigma}T_1 - 2\theta$

$$a' = \left(-1 + 6\Omega^2\Lambda^2 + \frac{\mathbf{3}}{\mathbf{4}}\omega^2 a^2\right)\mu a - \frac{(\omega+\Omega)}{\omega}k_1 a\Lambda\cos\gamma - (-\frac{1}{2}\boldsymbol{\alpha} + 2\beta\omega\Omega\Lambda)\frac{a}{\omega}\sin\gamma \quad , (29)$$

$$a\boldsymbol{\gamma}' = \boldsymbol{a}[\sigma - 2(3\lambda + k_2\Omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}} - \frac{1}{4}(\mathbf{3}\boldsymbol{\lambda} + k_2\omega^2)\frac{a^2}{\boldsymbol{\omega}}] + \frac{(\omega+\Omega)}{\omega}k_1 a\Lambda\sin\gamma - (-\frac{1}{2}\boldsymbol{\alpha} + 2\beta\omega\Omega\Lambda)\frac{a}{\omega}\cos\gamma\,. \quad (30)$$

Seeking steady-state, we let $a' = \gamma' = 0$and we eliminate $\boldsymbol{\gamma}$ dependence to get the frequency response equation as

$$[\sigma - 2(3\lambda + k_2\Omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}} - \frac{1}{4}(\mathbf{3}\lambda + k_2\omega^2)\frac{a^2}{\boldsymbol{\omega}}]^{\mathbf{2}}a^2 + \left(-1 + 6\Omega^2\Lambda^2 + \frac{\mathbf{3}}{\mathbf{4}}\omega^2 a^2\right)^2\mu^2 a^2 = \left[{k_1}^2\Lambda^2\frac{(\omega+\Omega)^2}{\omega^2} + \frac{(-\frac{1}{2}\boldsymbol{\alpha} + 2\beta\omega\Omega\Lambda)^2}{\omega^2}\right]a^2.\ (31)$$

For this equation we have the trivial solution $a = 0$ and another set of solutions which verify the following equation:

$$[\boldsymbol{\sigma} - \mathbf{2}(\mathbf{3}\boldsymbol{\lambda} + k_2\Omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}} - \frac{1}{4}(\mathbf{3}\boldsymbol{\lambda} + k_2\omega^2)\frac{a^2}{\boldsymbol{\omega}}]^{\mathbf{2}} + \left(-1 + 6\Omega^2\Lambda^2 + \frac{\mathbf{3}}{\mathbf{4}}\omega^2 a^2\right)^2\mu^2 = \left[{k_1}^2\Lambda^2\frac{(\omega+\Omega)^2}{\omega^2} + \frac{(-\frac{1}{2}\boldsymbol{\alpha}+2\beta\omega\Omega\Lambda)^2}{\omega^2}\right]. (32)$$

We obtain finaly the non trivial solutions of the form

$$a^2 = p_1 \pm (p_1^2 - q_1)^{\frac{1}{2}}, (33)$$

Where

$$p_1 = \frac{1}{9[\mu^2\omega^6 + 9\,(\mathbf{3}\boldsymbol{\lambda} + k_2\omega^2)^2]}\left[4\omega(\boldsymbol{\lambda} + k_2\omega^2)\left[\sigma - 2(\mathbf{3}\boldsymbol{\lambda} + k_2\omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}}\right] - 12\mu^2\omega^4(-1 + 24\omega^2\Lambda^2)\right], (34)$$

$$q_1 = \frac{1}{9[\mu^2\omega^6 + 9\,(\mathbf{3}\boldsymbol{\lambda} + k_2\omega^2)^2]}\left[16\,\omega^2\left[\sigma - 2(\mathbf{3}\boldsymbol{\lambda} + \mathbf{4}k_2\omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}}\right]^2 - 16\mu^2\omega^4(-1 + 24\omega^2\Lambda^2) - 16\,k_1^2\Lambda^2(\omega+\Omega)^2 - 8\alpha - 64\beta\omega^2\Lambda\right]. (35)$$

For non trivial solutions, it follows from Eq.(33) that both the radical and the first term must be positive, i.e. the non trivial solutions for a are real only when p_1>0and $p_1^2 \geq q_1$. In Fig. 3 the frequency-response (Eq. (33)) is plotted. This curve shows that the sub-harmonic resonance in order-two appear and the maximumamplitude corresponding to resonance increases as the resonance frequency augment remaining in the field imposed by the conditions of occurrence of this resonance with the order.

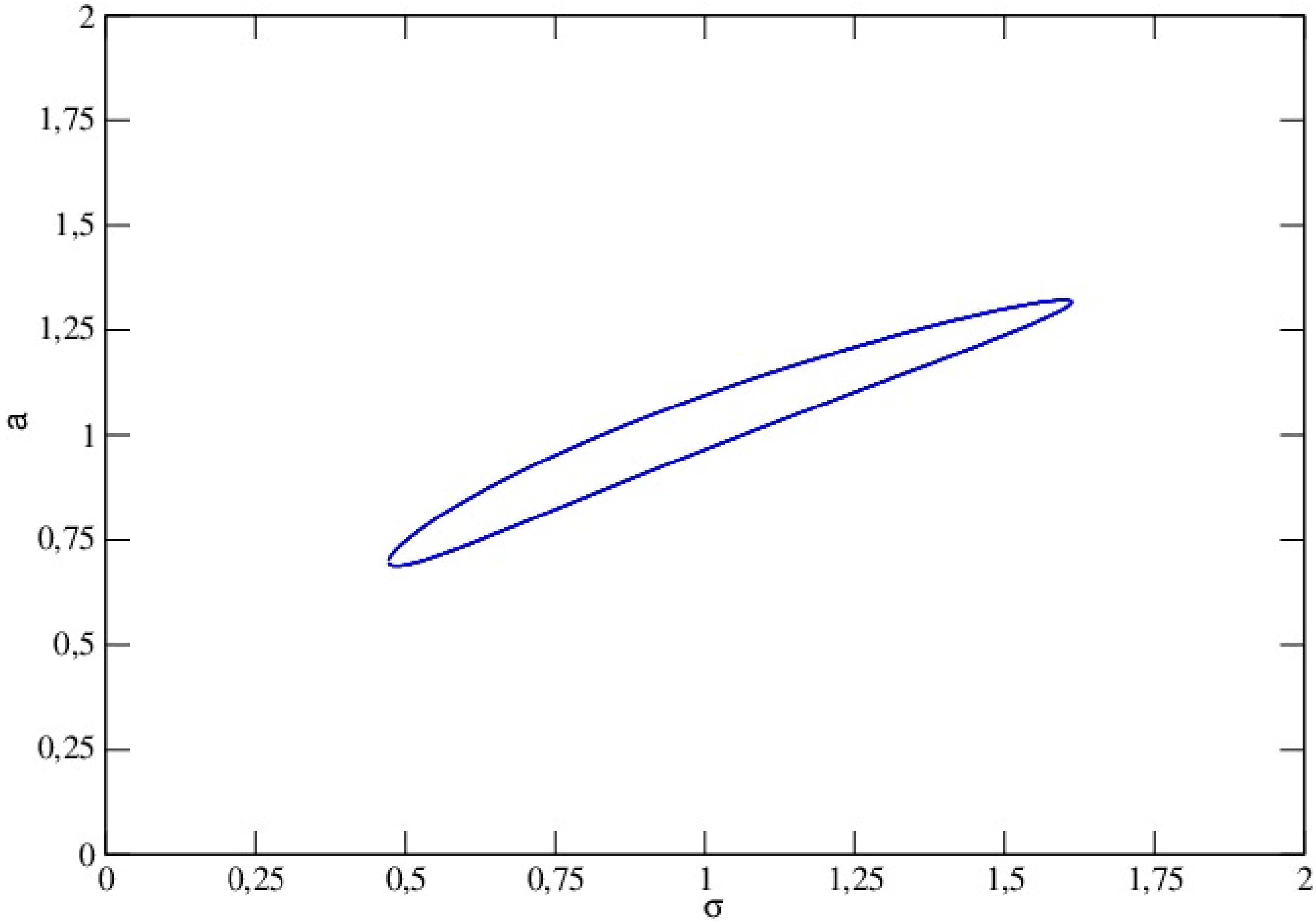

Fig. 3 Subharmonic resonance in the space (a, σ) for $\boldsymbol{\mu}$= 0.05; $\boldsymbol{\alpha}$= 0.3; λ = 1; $\boldsymbol{\beta}$ = 0.1; k_2 = 0.5; k_1 = 0.1; **F** = 0.5.

2.2.2 $\boldsymbol{\Omega = 3\omega + \epsilon\sigma}$

If we insert $\Omega = 3\omega + \epsilon\sigma$ in Eq.(11), the solvability condition takes the form

$$-2i\omega A' - i\mu\omega A + 3i\mu\omega^3 A^2\bar{A} + 6i\mu\omega\Omega^2 A\Lambda^2 - 3\lambda A^2\bar{A} - 6\lambda A\Lambda^2 - k_2\omega^2 A^2\bar{A} - 2k_2\Omega^2 A\Lambda^2 + \left(-3i\boldsymbol{\mu\omega^2\bar{A}^2\Omega}\Lambda - \boldsymbol{3\lambda\bar{A}^2}\Lambda + k_2\omega^2\boldsymbol{\bar{A}^2}\Lambda - \boldsymbol{2}k_2\boldsymbol{\omega\Omega\bar{A}^2}\Lambda\right)e^{i\sigma T_1}, (36)$$

Substituting the polar notation for A Eq.(10) into Eq.(36), and setting $\boldsymbol{\gamma}$=$\boldsymbol{\sigma}T_1 - 3\boldsymbol{\theta}$. Nest we put $a' = \gamma' = 0$ and eliminate, then the frequency-response equation is

$$[\boldsymbol{\sigma} - \boldsymbol{3}(\boldsymbol{3\lambda} + k_2\Omega^2)\frac{\Lambda^2}{\boldsymbol{\omega}} - \frac{3}{8}(\boldsymbol{3\lambda} + k_2\omega^2)\frac{a^2}{\boldsymbol{\omega}}]^{\boldsymbol{2}}a^2 + [\left(-\frac{3}{2} + 9\Omega^2\Lambda\right)\mu + \frac{\boldsymbol{9}}{\boldsymbol{8}}\omega^2 a^2]^2 a^2 = \frac{9}{16}\left[\boldsymbol{9}\mu^2\boldsymbol{\omega^4}\Omega^2 - (\boldsymbol{3\lambda} + k_2\omega^2 - \boldsymbol{2}k_2\boldsymbol{\omega\Omega})^{\boldsymbol{2}}\right]\frac{\Lambda^2\boldsymbol{a^4}}{\omega^2}. (37)$$

The solutions for Eq. (37) are either a = 0 or

$$a^2 = p_1 \pm (p_1^2 - q_1)^{\frac{1}{2}}, (38)$$

Where

$$p_1 = \frac{1}{9[\mu^2\omega^6 + 9\,(\mathbf{3\lambda} + k_2\omega^2)^2]}\left[-72\omega^4\left(-\frac{3}{2} + 9\Omega^2\Lambda\right)\mu + 24\omega(\mathbf{3\lambda} + k_2\omega^2)\left[\sigma - 3(\mathbf{3\lambda} + k_2\omega^2)\frac{\Lambda}{\boldsymbol{\omega}}\right]^2 + 18[9\mu^2\omega^4\Omega^2 - (\mathbf{3\lambda} + k_2\omega^2 + 2k_2\omega\Omega)^2]\Lambda^2\right], (39)$$

$$q_1 = \frac{1}{9[\mu^2\omega^6 + 9\,(\mathbf{3\lambda} + k_2\omega^2)^2]}\left[64\,\omega^2\left(-\frac{3}{2} + 9\Omega^2\Lambda\right)^2\mu^2 + 64\omega[\sigma - 3(\mathbf{3\lambda} + \mathbf{4}k_2\Omega^2\Lambda^2)]^2\right]. (40)$$

Since q_1 is always positive, we need $p_1 > 0$ and $p_1^2 \geq q_1$.
We numerically simulate Eq.(38) (see Fig. 4) and we note the same comments as in the case of the sub-harmonic resonance of order-two but the resonance amplitudes for resonance are significant in this case i.e. the order-three.

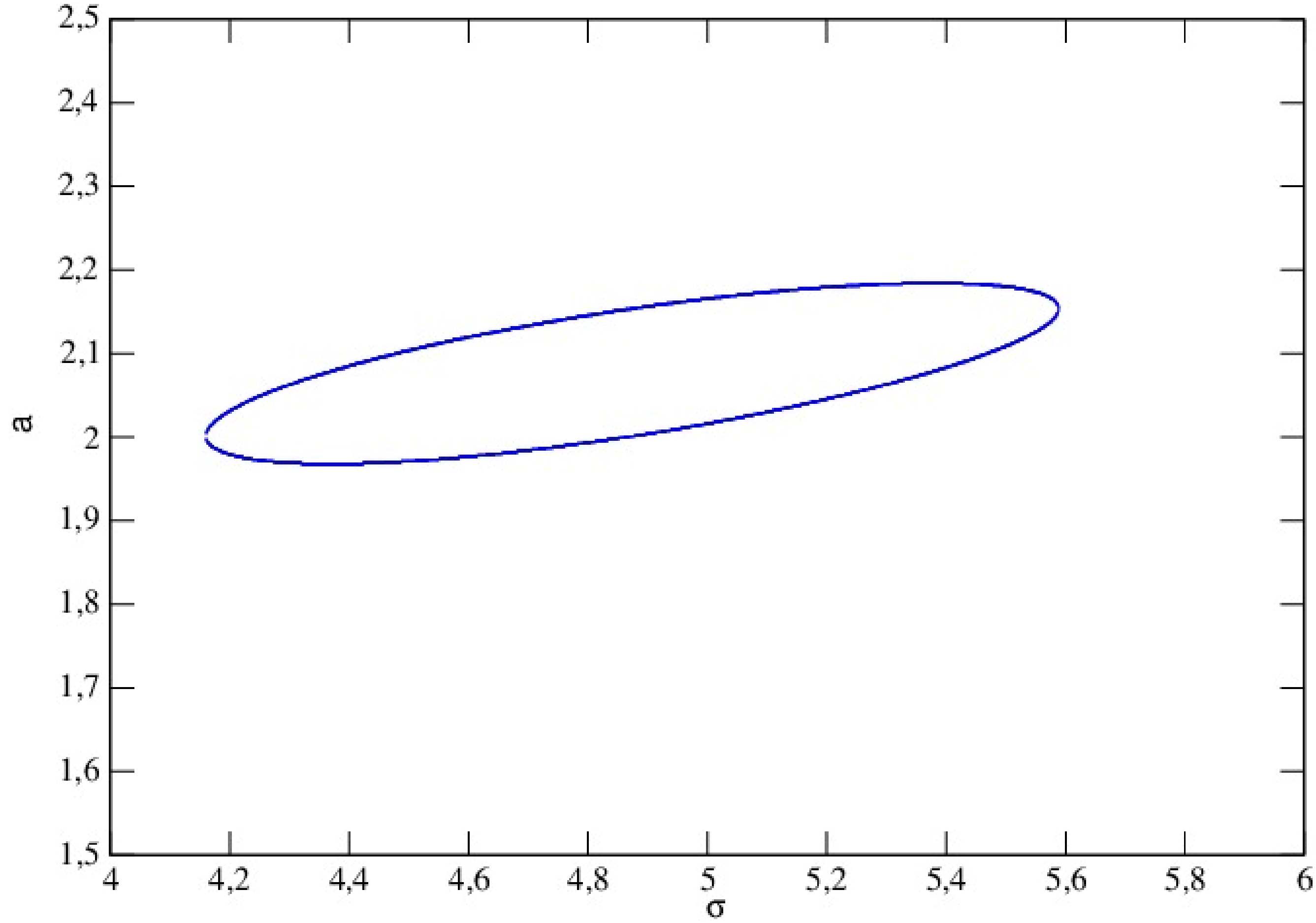

Fig. 4 Subharmonic resonance in the space(a, σ)) for $\boldsymbol{\mu}$= 1.2; λ = 1; k_2 = 0.001; ω = 1; F = 0.2.

2.3 **Primary resonance**

In this state, we put that$F = \epsilon F_0$. The closeness between both internal and
external frequencies is given by $\Omega = 1 + \epsilon\sigma$In these conditions after some algebaic manipulations, we obtain

$$D_1{}^2 x_1 + \omega^2 x_1 = [-2i\omega A' - i\mu\omega A + 3i\mu\omega^3 A^2\bar{A} - k_2\omega^2 A^2\bar{A} - 3\lambda A^2\bar{A} + \frac{1}{2}F_0]e^{i\omega\,T_0} + CC + NST. (41)$$

Equating resonant terms at 0 from Eq.(41), we obtain:

$$-2i\omega A' - i\mu\omega A + 3i\mu\omega^3 A^2\bar{A} - k_2\omega^2 A^2\bar{A} - 3\lambda A^2\bar{A} + \frac{1}{2}F_0]e^{i\sigma T_1} (42)$$

After the same algebraic manipulations in other resonant states, the amplitude of oscillations of primary resonant states is governed by the following nonlinear algebraic equation.

$$(-\frac{1}{2}\mu\omega a + \frac{3}{8}\mu\omega^3 a^3)^2 + \left(\omega\sigma a - \frac{1}{2}k_2\omega^2 a^3 - \frac{3}{8}\lambda a^3\right)^2 = \frac{F_0{}^2}{4}. (43)$$

The obtained results are reported in Fig.5 where a jump and hysteresis phenomena are found. These phenomena indicate that the solutions obtained are multiple and multistable. For example, when parameter$\boldsymbol{\sigma}$ is less than 0.529412, there is one solution but beyond the number of solution ranges from two to three. The primary resonance obtained is thus richer in number of solutions that sub-harmonic and super-harmonic resonances.

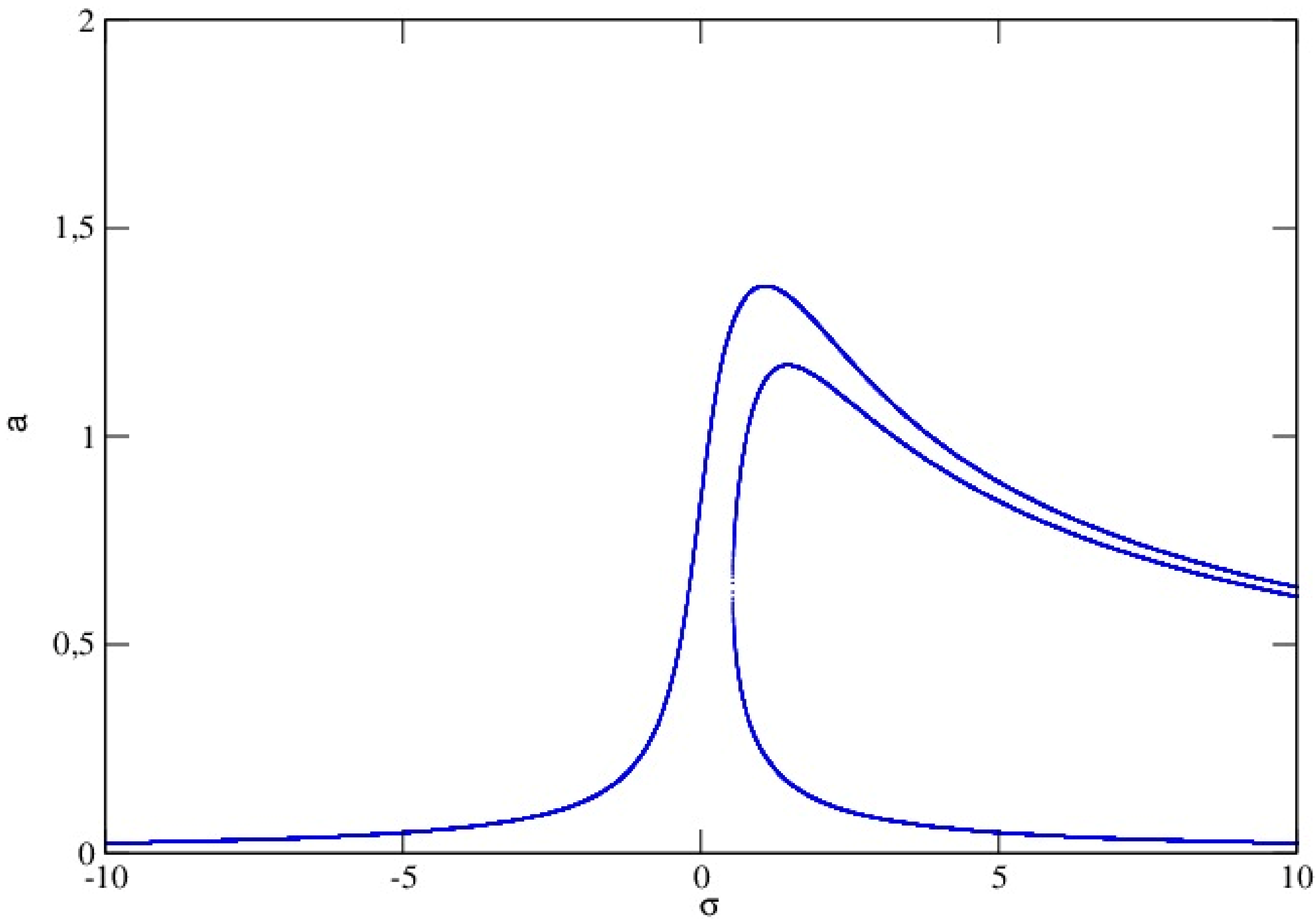

Fig. 5 Primary resonance in the space (a, σ)) for $\boldsymbol{\mu}$= 0.5; λ = 1; k_2 = 0.5; ω = 1; F = 0.5.

3.1 Effects of parameters on super-harmonic resonance

Figs. 6-9 show the effects of the parameters of the system on super-harmonic resonance at order two and Figs. 10 and 11 illustrate the effects of parameters system on super-harmonic resonance of order-three. Only cubic parameters affect the super-harmonic resonance of order three while all parameters of the systems have effects on super-harmonic resonance of order two. Figs. 6, 8, 9 show respectively that each of the system parameters F, $\alpha, k_1, \boldsymbol{\beta}$, λincreases the frequency and the resonant amplitude. The frequency and amplitude of resonance increases when one of the parameters is increasing. The hysteresis phenomenon becomes very pronounced when each of its parameters is increasing. Through Figs. 7 (a) and (b) the influences of the unpure cubic parameter k2 and pure cubic parameter $\boldsymbol{k_2}$ on such oscillationshas been checked. In these cases we note that the hybrid cubic parameterdoes not affect the resonance amplitude but increases the resonant frequency. The resonant amplitude decreases when the cubic parameter increases and therefore the hysteresis disappears. Figs.10(a); (b) and Figs. 11(a); (b) show the influences of λ, $\boldsymbol{\mu}$, F,k_2 respectively on super-harmonic resonance order-three. We noticed that these parameters are the same e_ects as the order-two super-harmonic resonance case.

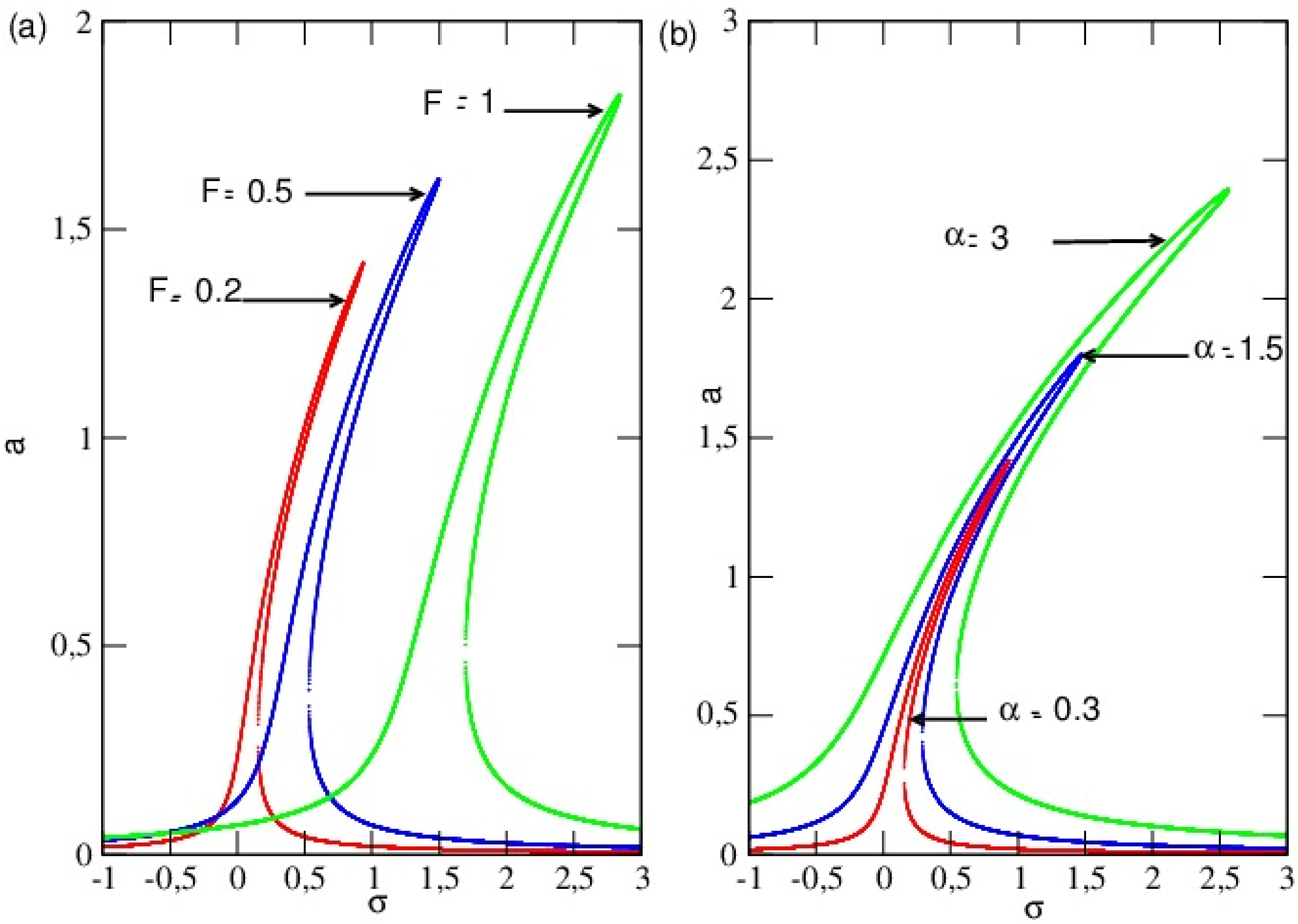

Fig. 6 Effects of (a) : F, (b) : $\boldsymbol{\alpha}$ on the frequency-response curves of the order-two super-harmonic resonance with the parameters of Fig.1

3.2 Effects of parameters on sub-harmonic resonance

In this part, we search for the effects of the parameters F, $\alpha, k_1, \boldsymbol{\beta}$, λ, k_2 *and* μ. Only cubic parameters affect the sub-harmonic resonance of order three (see Figs.16, 17) while all parameters of the systems have effects on super-harmonic resonance of order two (see Figs.12-15). From Figs. 12,14, 15, we conclude that the range of frequency where a response can be obtained is more important when each of parameters F, $\alpha, k_1, \boldsymbol{\beta}$, $\boldsymbol{\lambda}$ increase. When the amplitude of the parametric excitation force is zero ($\boldsymbol{\alpha}$ = 0), the order-two resonance sub-harmonic curve holds its shape and destroyed when this parameter is not zero and becomes large. When this parameter is set non-zero, the shape of the resonance curve is destroyed when the parameters other than the cubic damping parameters increase. Through Figs. 13 (a) and (b) the influences of the unpure cubic parameter k_2 and pure cubic parameter $\boldsymbol{\mu}$ on such oscillations has been checked. In these cases we note that the hybrid cubic parameter does not affect the resonance amplitude but the range of frequency where a response can be obtained is more important when parameter $\boldsymbol{\mu}$ discreases. It also noticed that the shape of the resonance curve is destroyed when $\boldsymbol{\mu}$ becomes small.

From these pictures, we notice that the unstables amplitude scan been obtained when the shape of the resonance curve is destroyed. In the case of three-order sub-harmonic resonance, the cubic damping parameters, the cubic restoring parameter and amplitude of external force influenced the frequency response curve and shown the multistable solutions but the shape of this curve is not destroyed (see Figs.16, 17).

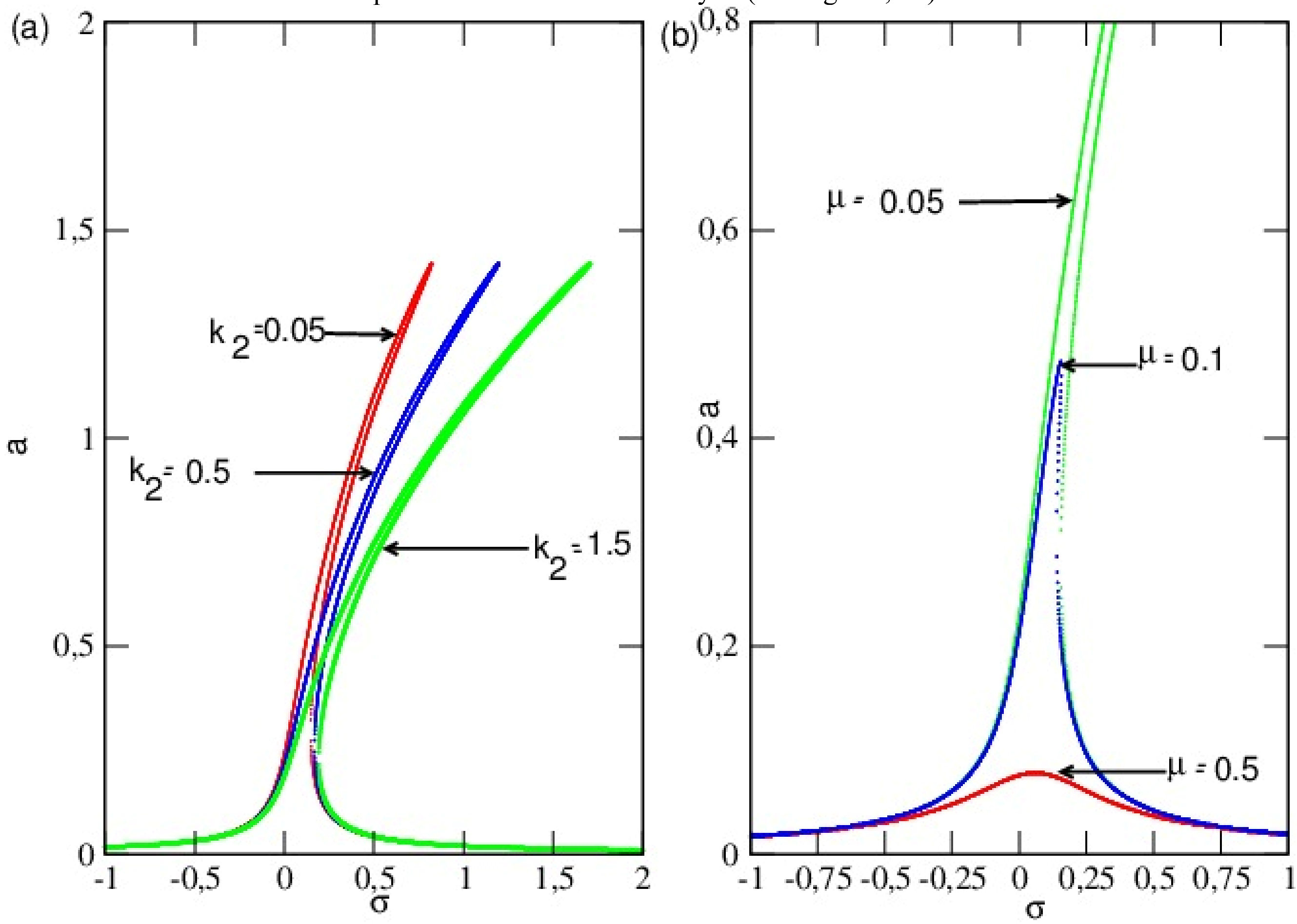

Fig. 7 Effect of (a) :$\boldsymbol{k_2}$, (b) : $\boldsymbol{\mu}$ on the frequency-response curves of the order-two super-harmonic resonance with the parameters of Fig.1.

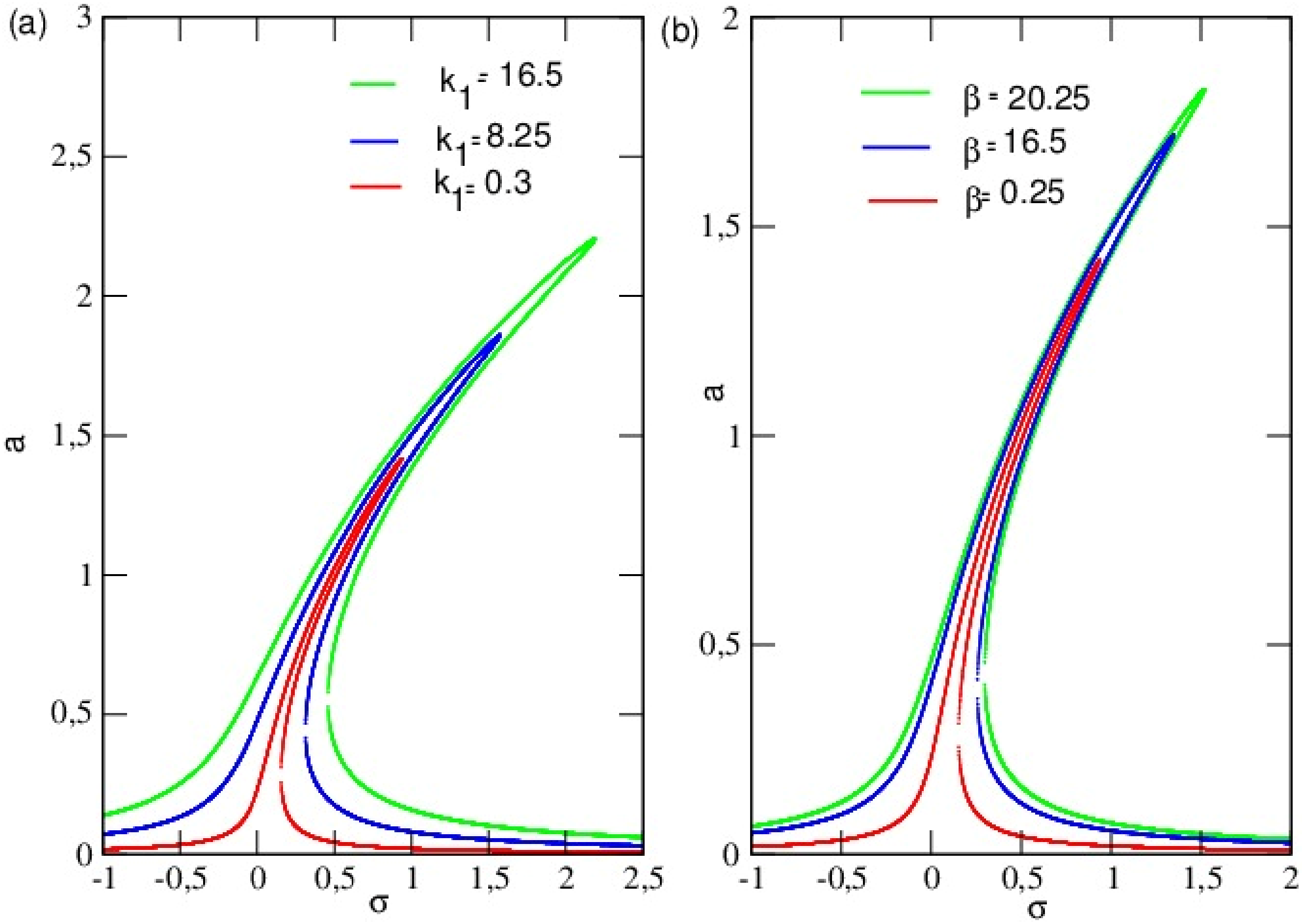

Fig. 8 Effects of (a) : $\boldsymbol{k_1}$, (b) : $\boldsymbol{\beta}$ on the frequency-response curves of the order-two super-harmonic resonance with the parameters of Fig.1.

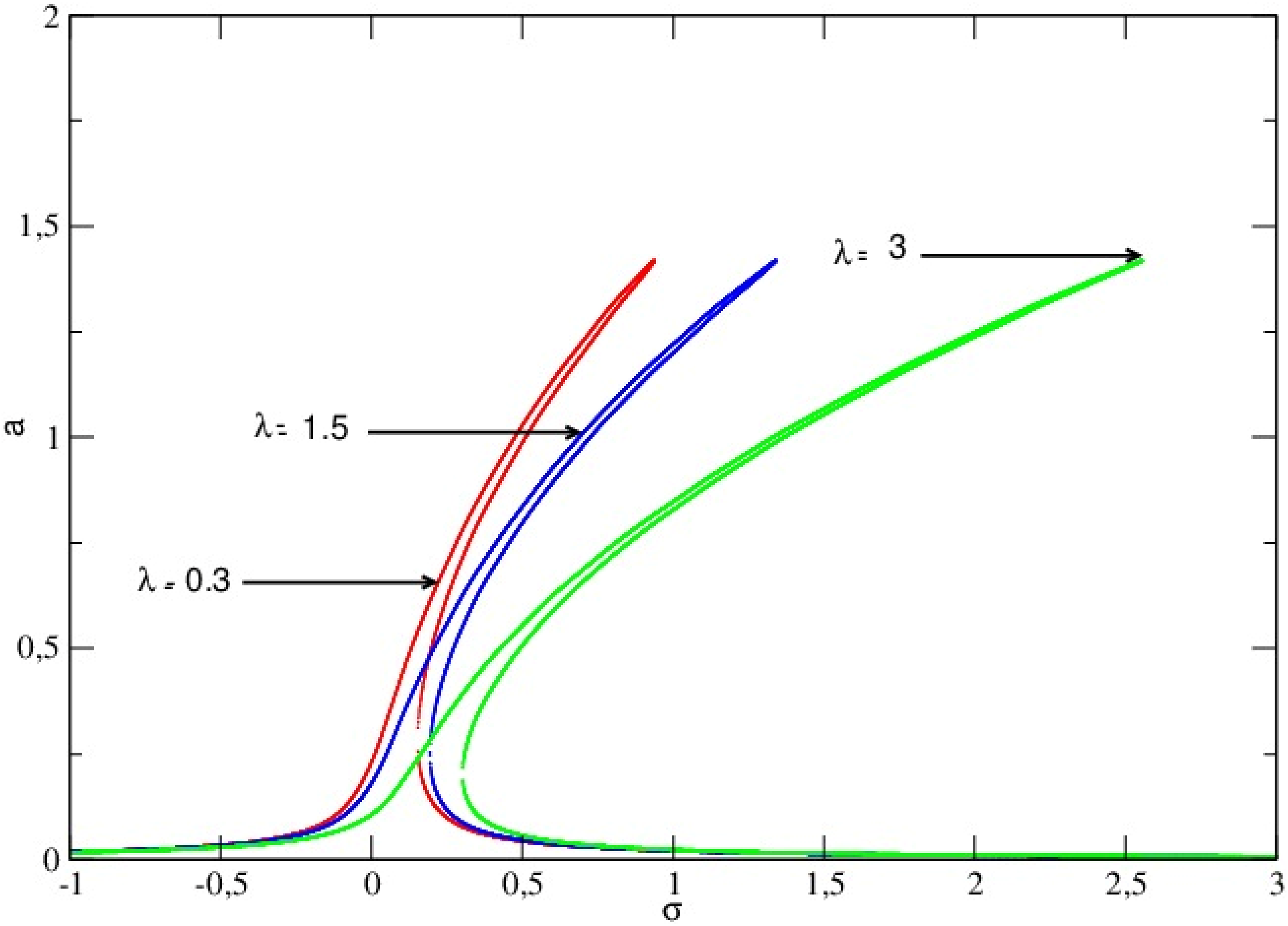

Fig. 9 Effects of λ on the frequency-response curves of the order-two super-harmonic reso-nance with the parameters of Fig.1.

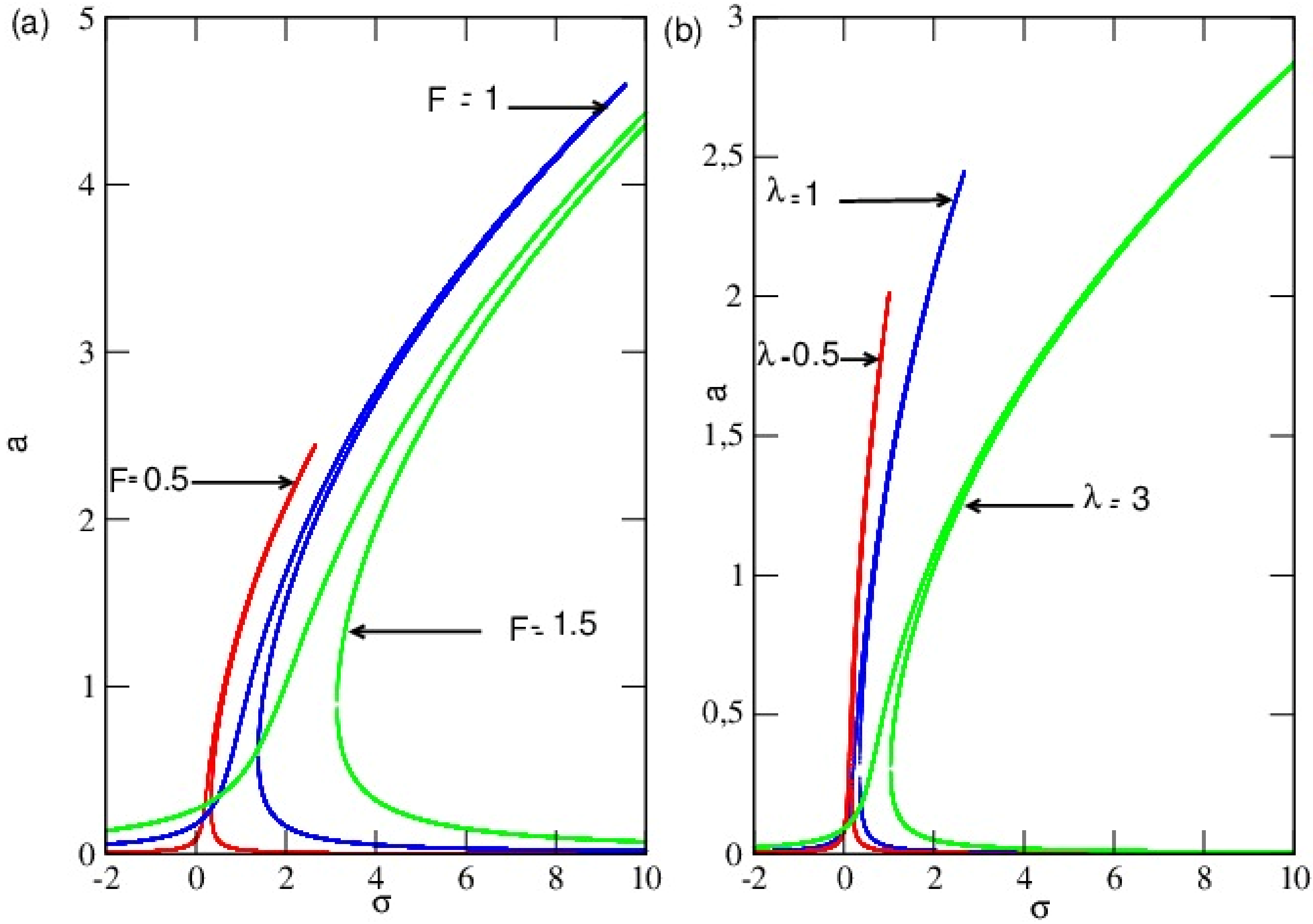

Fig. 10 Effects of (a) : F; (b) : λ on the frequency-response curves of the order-three super-harmonic resonance with the parameters of Fig. 2.

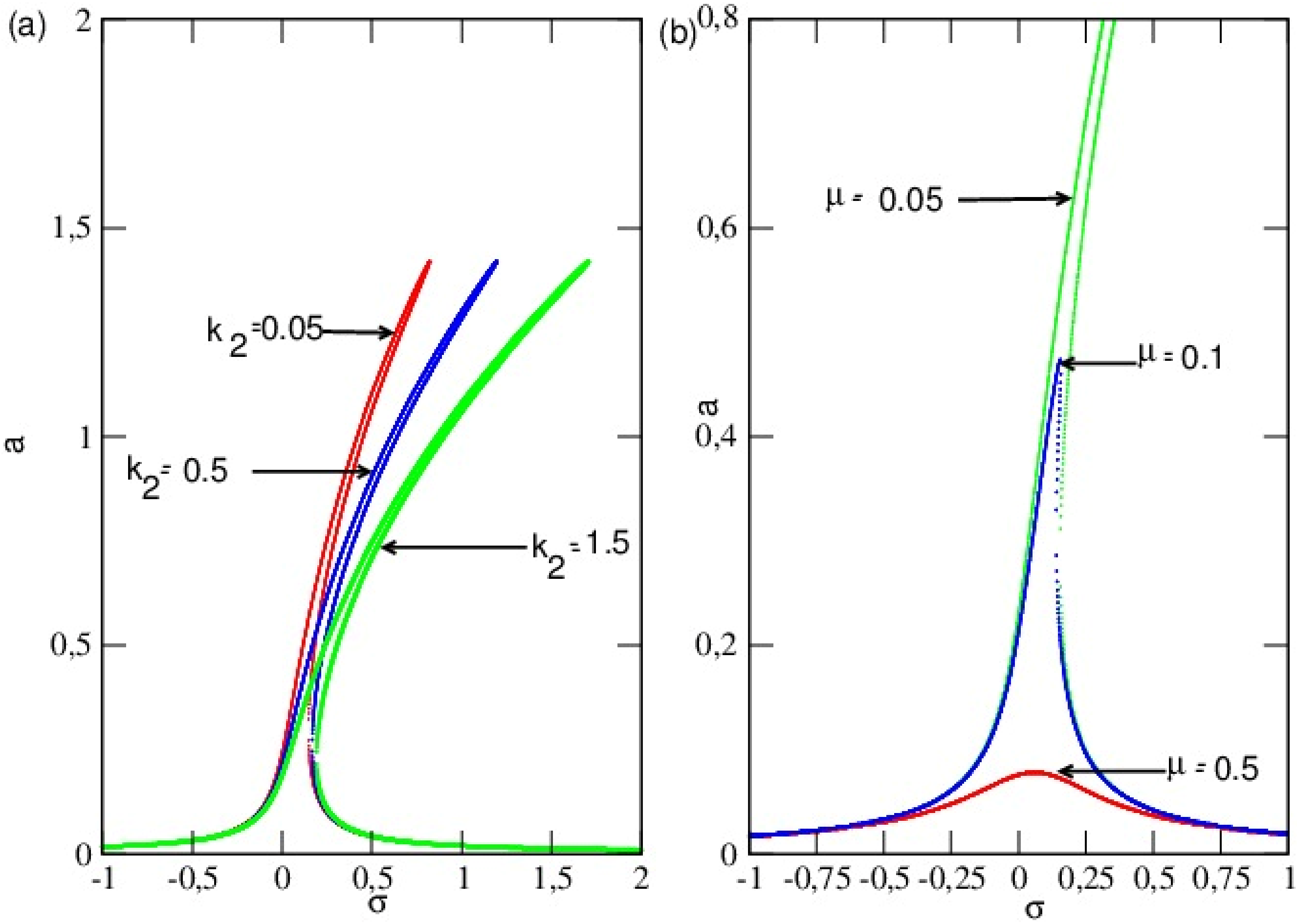

Fig. 11 Effects of (a) : $\boldsymbol{k_2}$; (b) : $\boldsymbol{\mu}$ on the frequency-response curves of the order-three super-harmonic resonance with the parameters of Fig. 2.

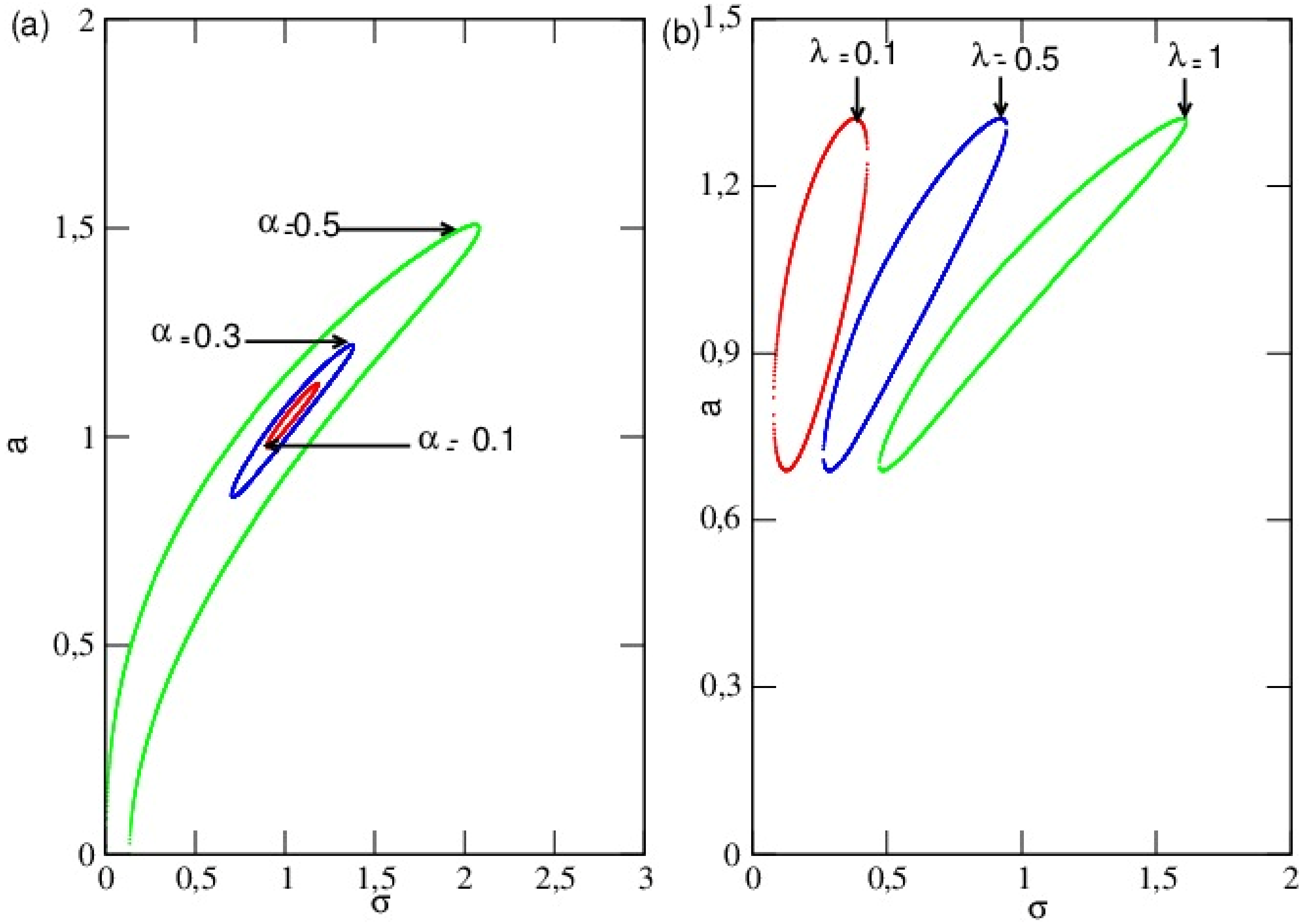

Fig. 12 Effects of (a) : α; (b) : λ on the frequency-response curves of the order-two sub-harmonic resonance with the parameters of Fig. 3.

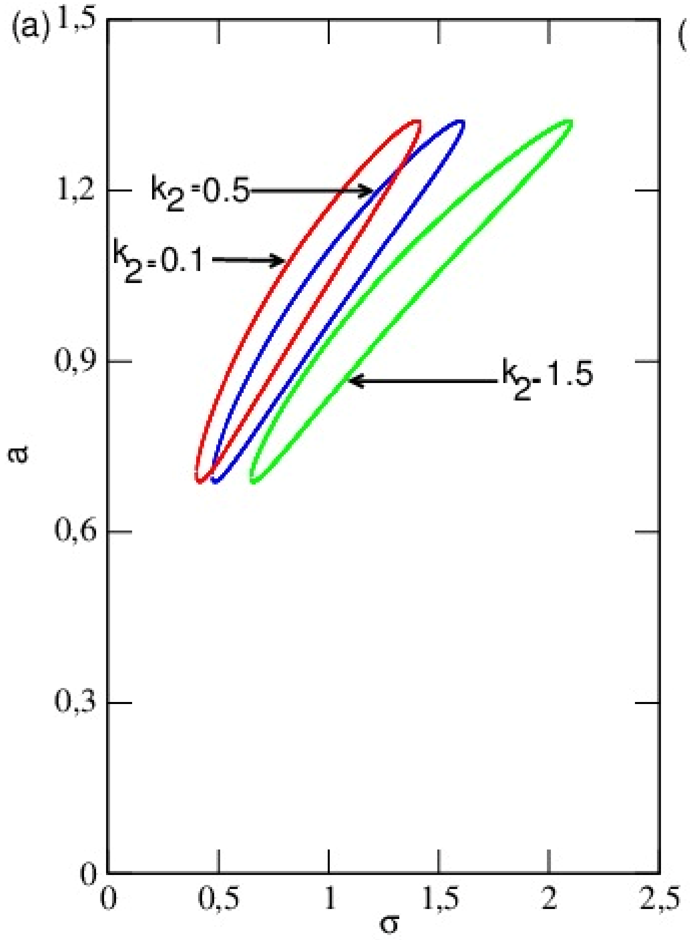

Fig. 13 Effects of (a) :$\boldsymbol{k_2}$; (b) : $\boldsymbol{\mu}$ on the frequency-response curves of the order-two sub-harmonic resonance with the parameters of Fig. 3.

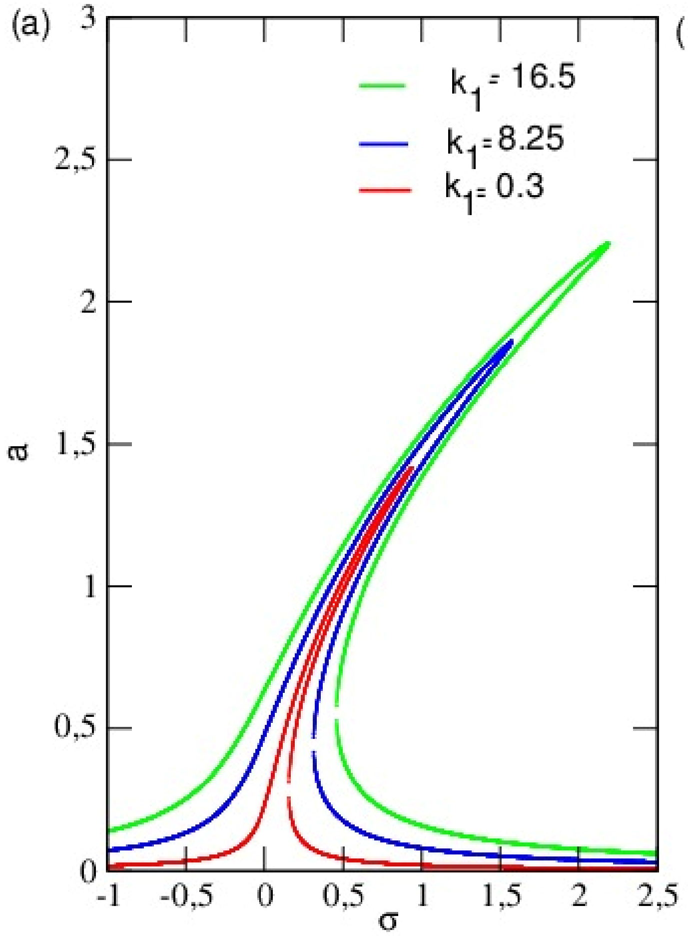

Fig. 14 Effects of (a) :$\boldsymbol{k_1}$; (b) : $\boldsymbol{\beta}$ on the frequency-response curves of the order-two sub-harmonic resonance with the parameters of Fig. 3.

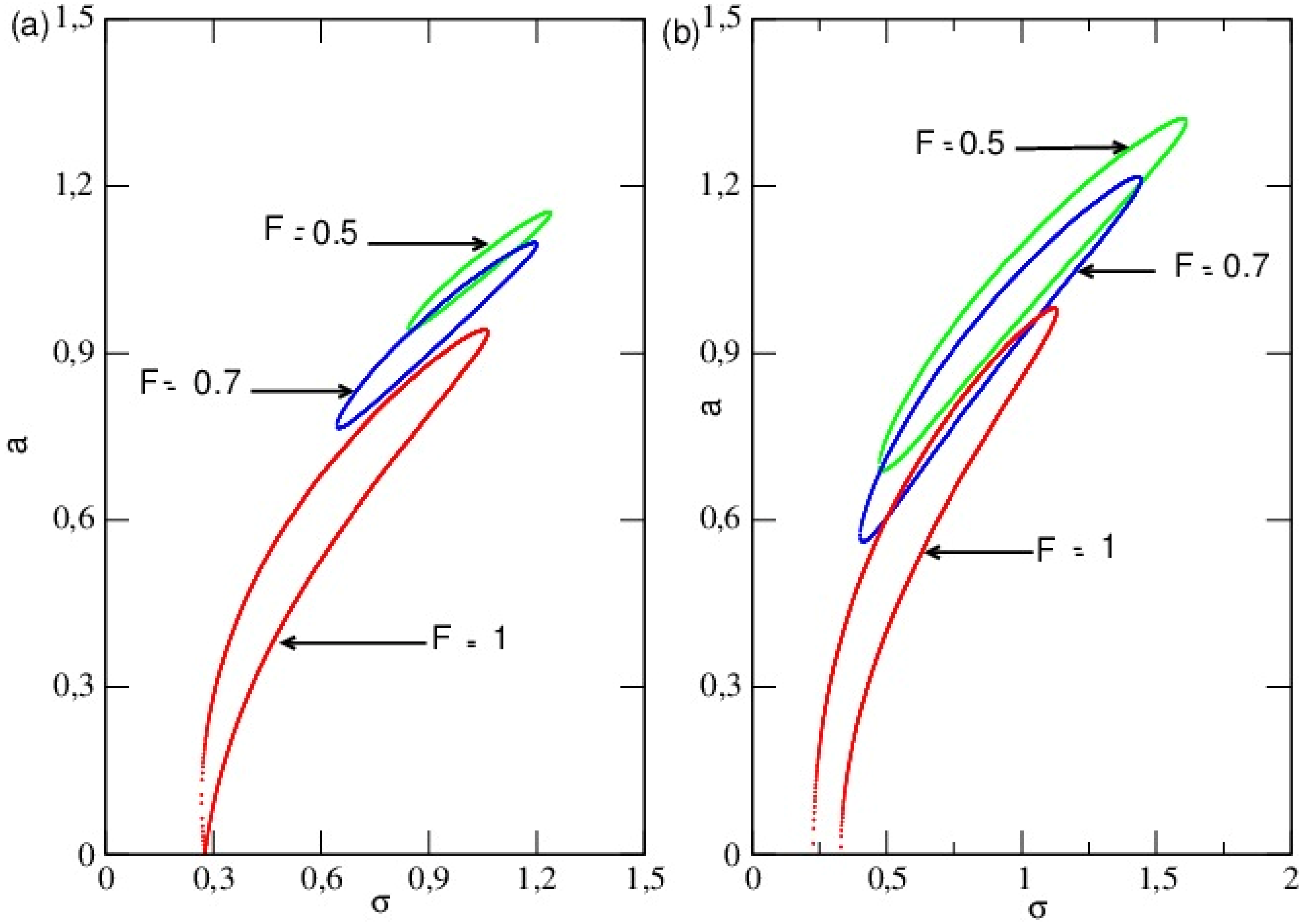

Fig. 15 Effects of F on the frequency-response curves of the order-two sub-harmonic resonance with the parameters of Fig. 3; (a) : $\boldsymbol{\alpha} = 0$; (b) :$\boldsymbol{\alpha} = 0.3$.

3.3 Effects of parameters on primary resonant state

In this subsection, we found the effects of parameters F;k_2; λ and $\boldsymbol{\mu}$ on primary resonance. Figs.18-21, show respectively the influence of F;k_2; λ and $\boldsymbol{\mu}$ on the frequency-response curves of the primary resonance in ($\boldsymbol{\sigma}$; a) space. From these figures we noticed that the jump and hysteresis phenomenon are appeared when the amplitude of external excitation is increased. When the cubic damping parameters and cubic restoring parameter increase the peak value of a discreases. When$k_2 = 0$ or $\boldsymbol{\mu}$ becomes large, the number of multiple solutions obtained discreases and the hysteresis phenomenon disappears (see Figs.19, 21). We conclude that the cubic damping parameters are seriously influenced the number of solutions obtained and affect the primary resonance.

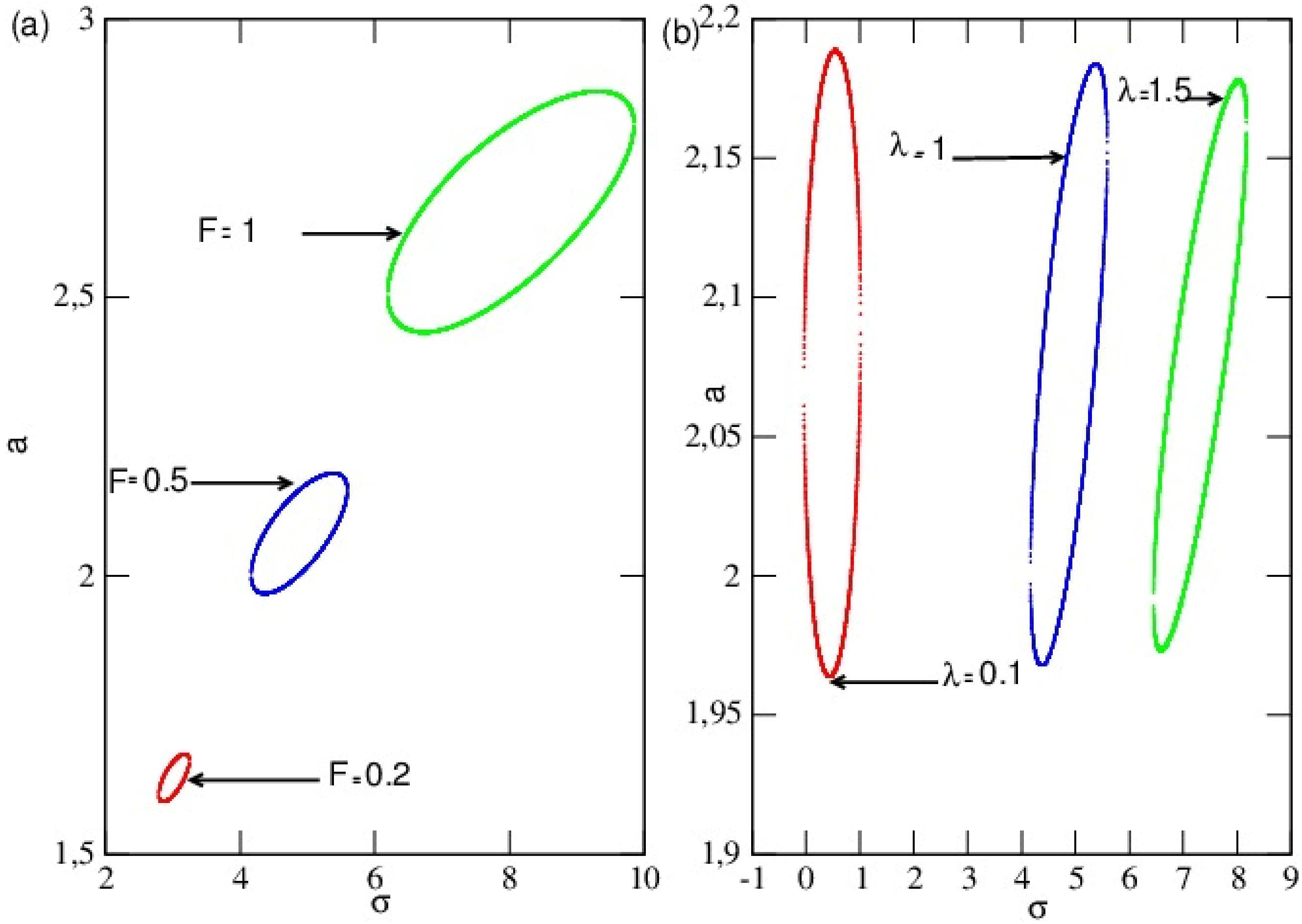

Fig. 16 Effects of (a) : F; (b) : λ on the frequency-response curves of the order-three sub-harmonic resonancewith the parameters of Fig. 4.

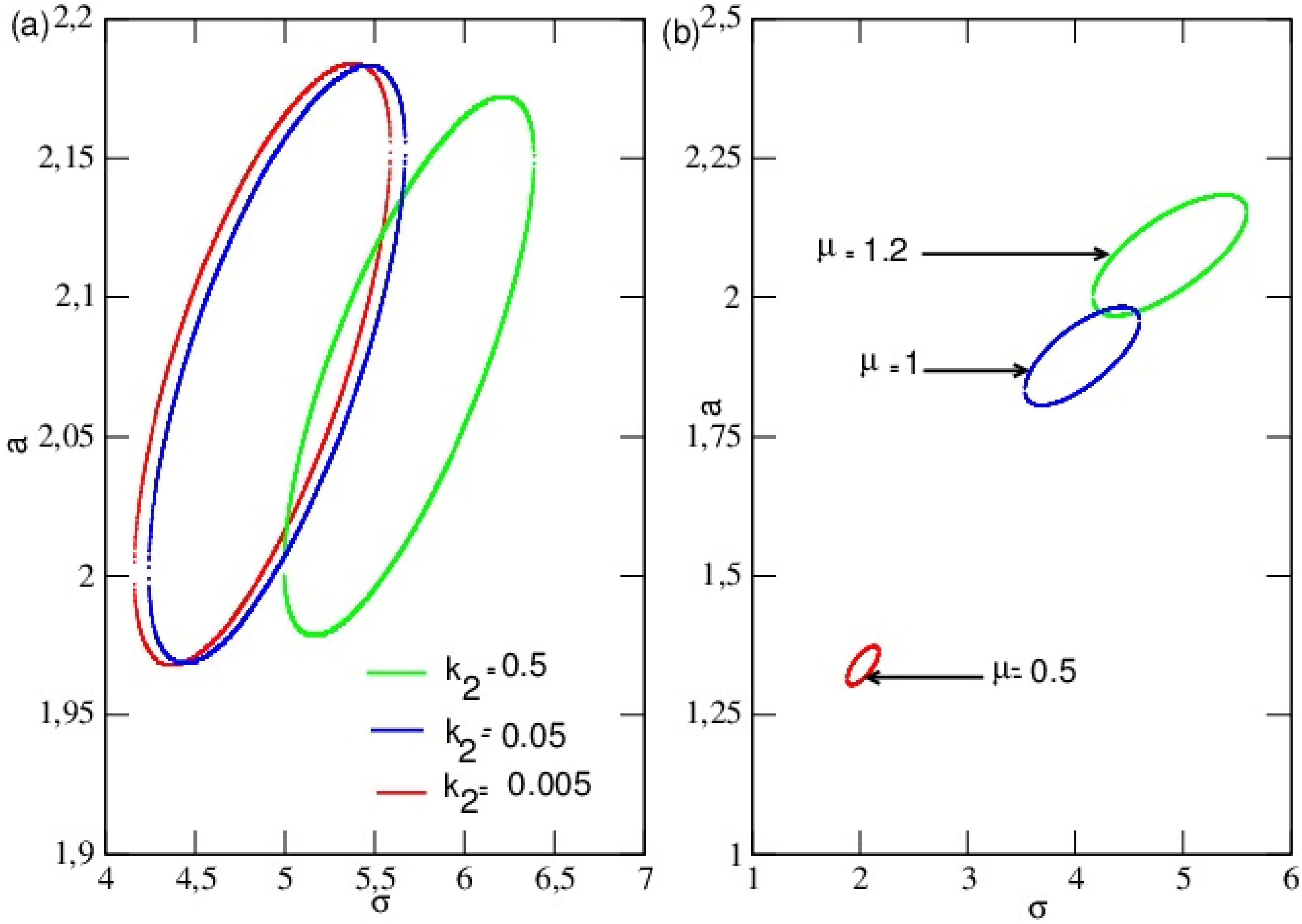

Fig. 17 Effects of (a) : k_2; (b) : $\boldsymbol{\mu}$ on the frequency-response curves of the order-three subharmonic resonance with the parameters of Fig. 4.

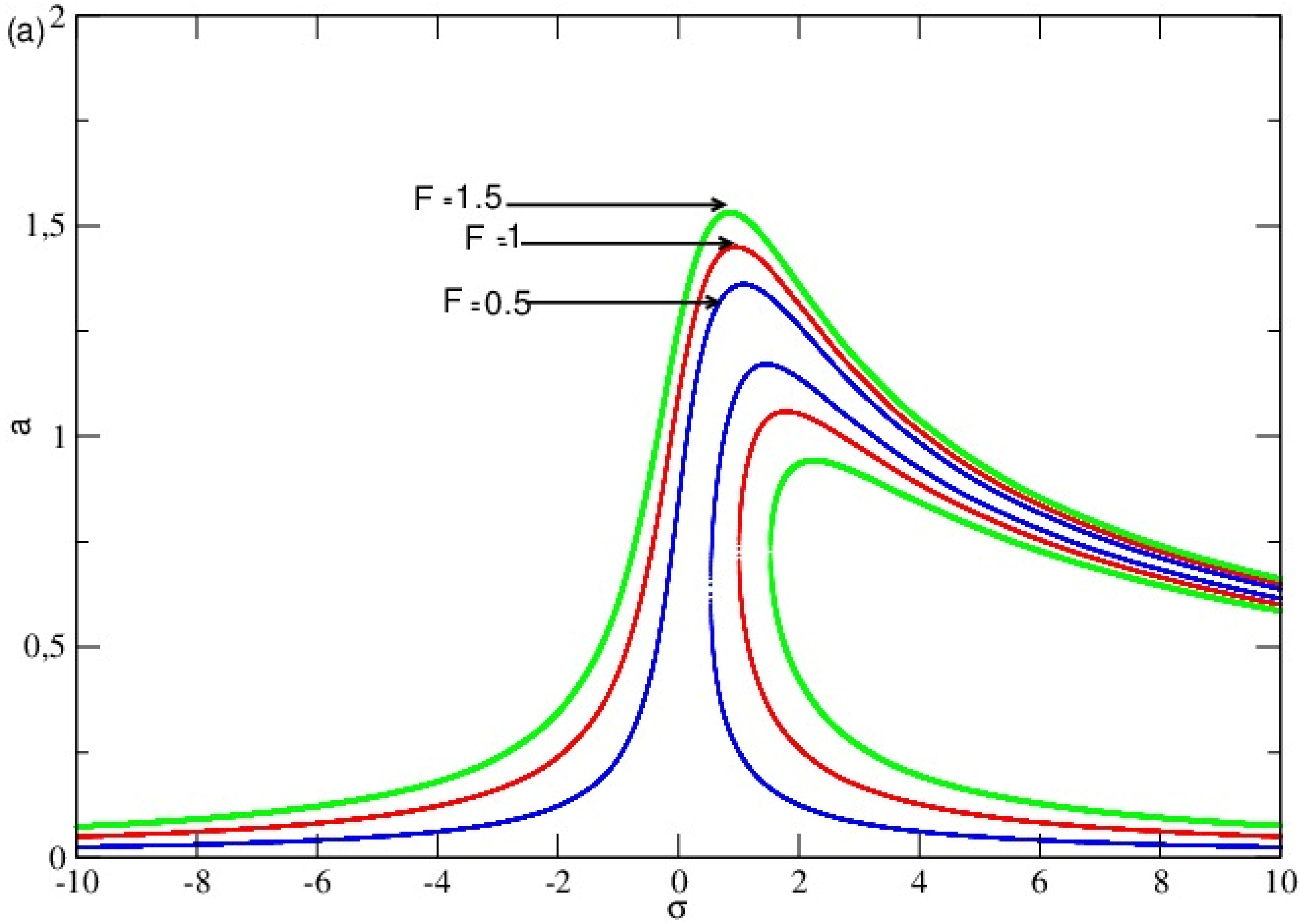

Fig. 18 Effects of F on the frequency-response curves of primary resonance with the parameters of Fig. 5.

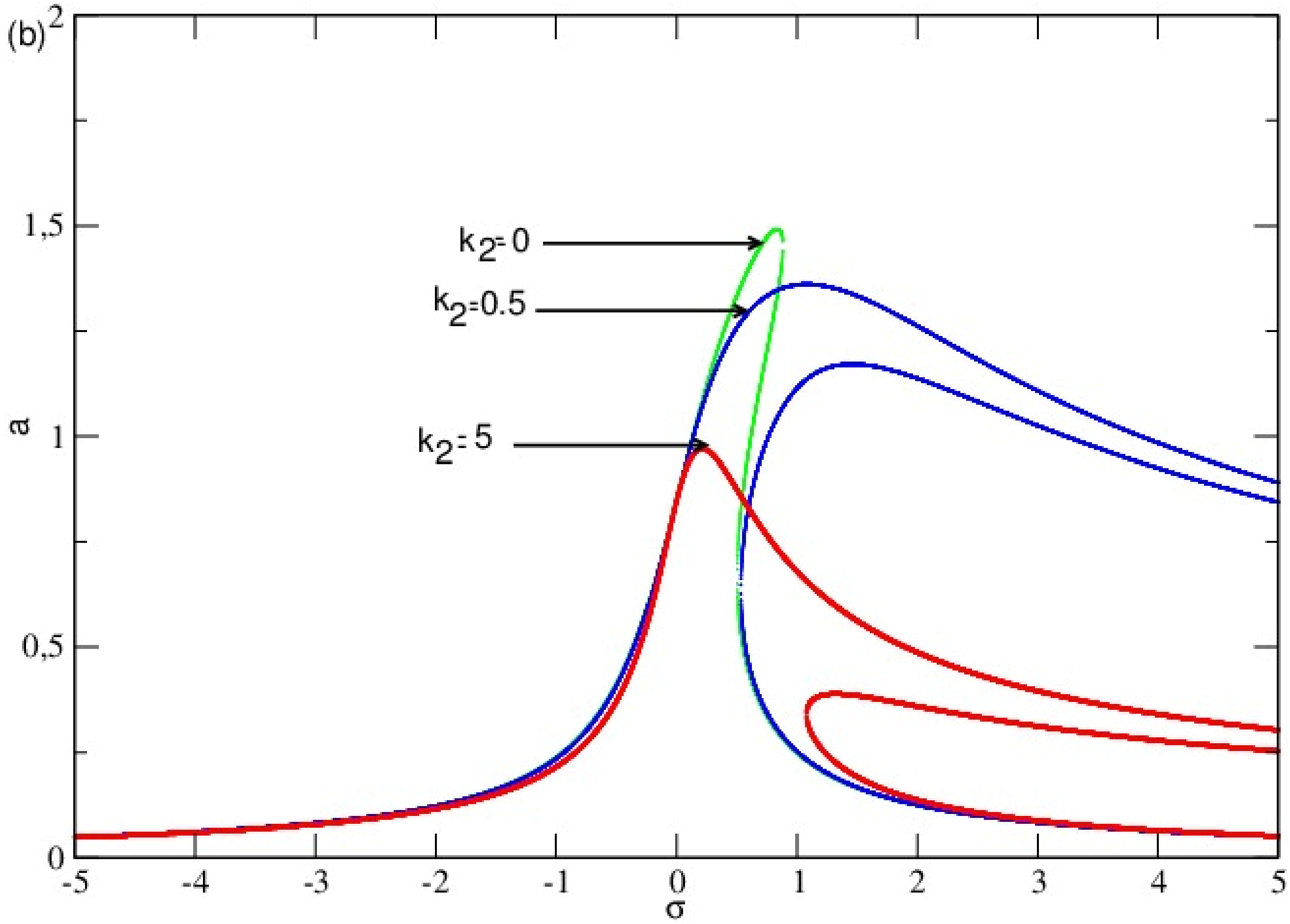

Fig. 19 Effects of k_2 on the frequency-response curves of primary resonance with the parameters of Fig. 5.

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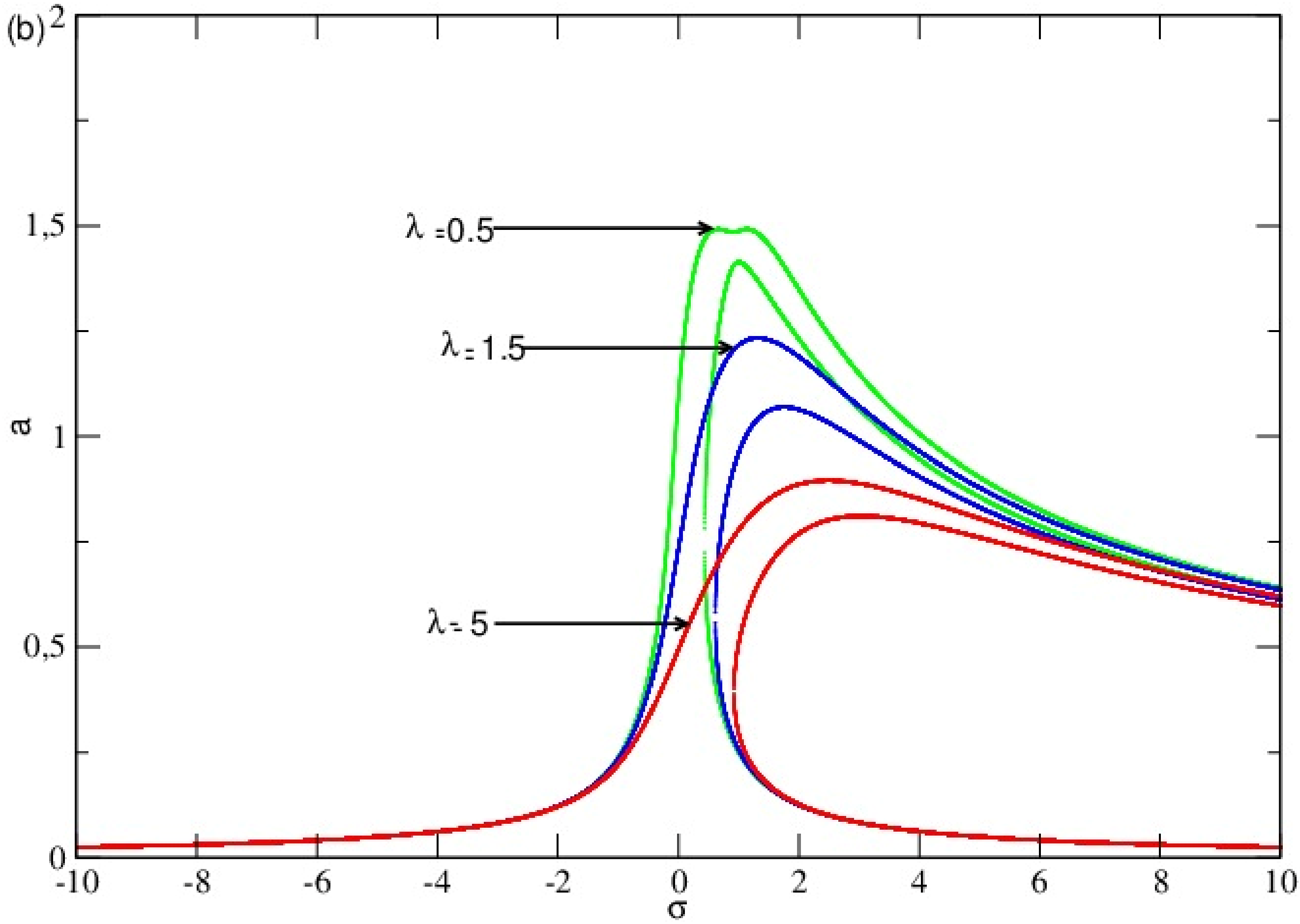

Fig. 20 Effects of λ on the frequency-response curves of primary resonance with the parameters of Fig. 5.

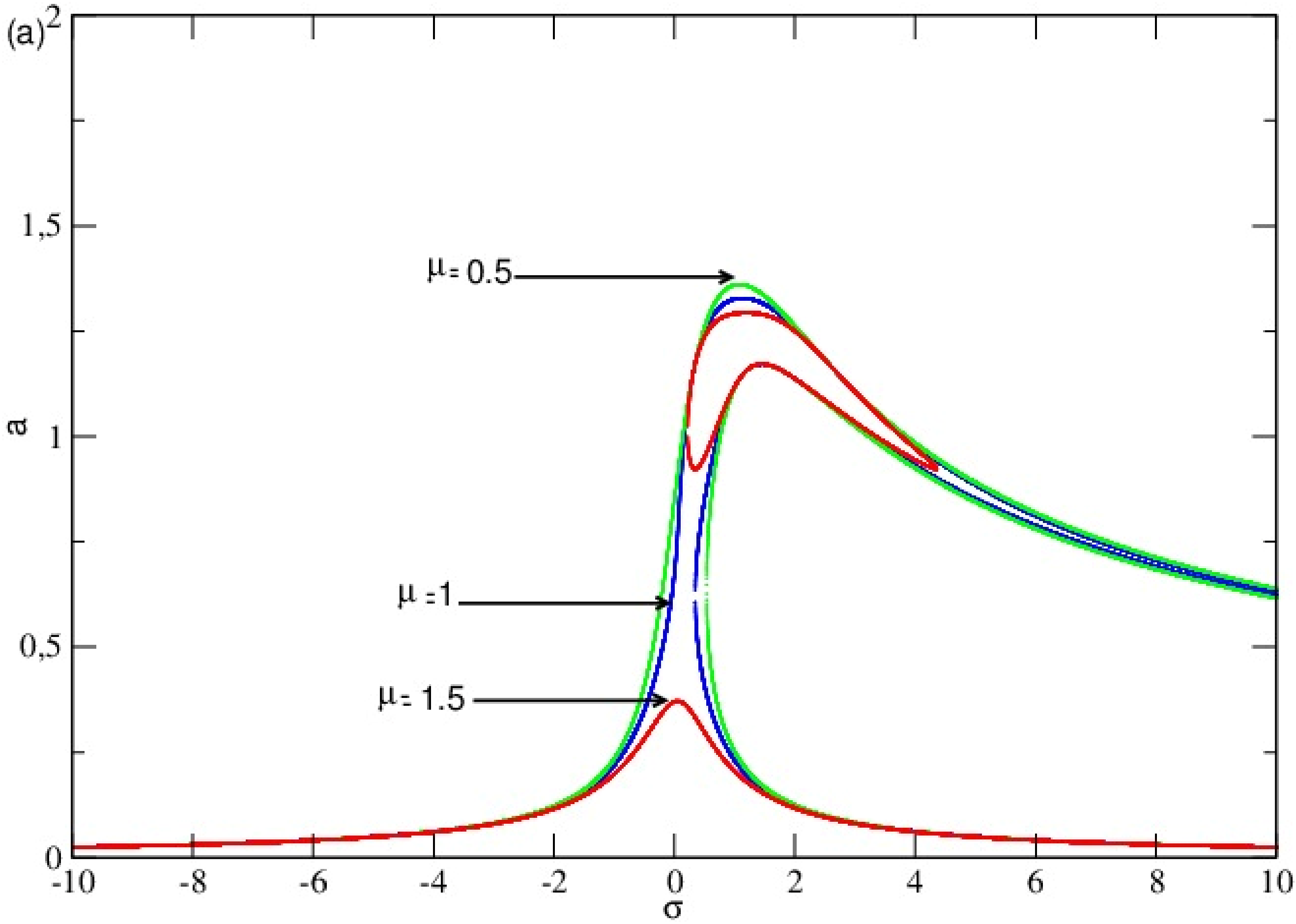

Fig. 21 Effects of $\boldsymbol{\mu}$ on the frequency-response curves of primary resonance with the parameters of Fig. 5.

4 Conclusion

In this paper, super-harmonic, sub-harmonic and primary resonant sates have been studied. Using the method of multiple scales, we obtained the primary resonance and the order-two and order-three for each type of other resonance. We found also in each case the maximum value of the amplitude of the oscillations for the system. We noted that in the case of two-order super-harmonic or sub-harmonic resonance, this maximum value depends on all the parameters of the system but in the case of order three, only the coefficients of the cubic terms parameters affect the maximum amplitude of the resonance. By fixing all the parameters of the system and varying only the amplitude of the parametric excitation above the critical value, the increasing amplitude of the parametric excitation provokes a rapid change in the amplitude of the response to the resonances. We obtained the jump and hysteresis phenomenon in the system behaviors and bi-stability phenomenon in the evolution of the amplitude of the oscillations of the system. The detection of each phenomena occurred due to the nonlinearity is very capital in the physical system motion. Because engineer must be able to recognize these phenomena when they occur and should understand their consequences and recommend appropriate measures to control or minimize large amplitudes motions. In general we note that effects due to cubic nonlinearities on the response curves have a significant from physical point of view. Our results confirm the different effect of damping parameters on the resonances states obtained by A. Francescutto et al. [5], K.W. Holappa et al.[6], Miguel A. F. Sanjuan [8], C. H. Miwadinou et al. [26], A. Zborowski and M. Taylan [27] and H. G. Enjieu Kadji [28]. It is important to note that around the resonance peaks, the amplitudes and accumulate energies of the system device are higher than those received in any oscillations. In this case, this oscillator model can give more interesting applications in physical or engineering, particularly when the model is used as a MEMS device, Selkov model, Brusselator, ship rolling motion, ENSO phenomenon etc., but the model with high energies is very dangerous since it can give rise to catastrophe damage.

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