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HYBRID LINEAR MULTISTEP METHOD WITH MULTIPLE HYBRID PREDICTORS FOR STIFF DIFFERENTIAL EQUATIONS

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Abstract
In this paper, a hybrid linear multistep method with multiple hybrid predictors for stiff initial value problems (IVPs) in ordinary differential equations (ODEs) is presented in this paper. The construction of these method is based on interpolation and collocation approach. The stability of the method is investigated using the boundary locus method and numerical experiments are demonstrated with respect to the method on stiff differential equations. The results in the numerical illustrations are compared with ode15s of MATLAB CODE, and were found to be in total agreement with each other.
Keywords: *Collocation and interpolation, Linear multistep methods, Multiple hybrid predictors, Stiff differential equations..*

1. Introduction

The interest in this paper is to develop hybrid linear multistep methods that approximate the solution of the stiff initial value problems;

$$y' = f(x, y), y(x_0) = y_0 \in \mathbb{R} \qquad (1.1)$$

where $y(x) : [x_0, x_N] \to \mathbb{R}^m$ in which
$f : [x_0, x_N] \times \mathbb{R}^m \to \mathbb{R}^m$ is differentiable and continuous. Stiff initial value problems arise in areas such as classical mechanics, chemical kinematics, Biological sciences etc. Several methods have been developed for solving these class problems. These are found everywhere in literature like the work of [9], [11], [6], [5], [12] etc. Hybrid linear methods are developed to circumvent the stability and order barriers theorem imposed by [3]. These methods are obtained by incorporating off-step points in the derivative process of the linear multistep methods (LMM). Several works have been done on hybrid linear multistep such as [8] and [2] among others.
The hybrid linear multistep method considered in this paper is derived by incorporating an off-step point into the [5], second derivative linear multistep methods (SDLMM) given as;

$$y_{n+k} = \alpha_{k-1} y_{n+k-1} + h \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \gamma_k f'_{n+k} \qquad (1.2)$$

by adding an extra term hf_{n+v_m} to the right hand side of (1.2). We obtained discrete hybrid linear multistep schemes of the form;

$$M: y_{n+k} = y_{n+k-1} + h \left(\sum_{j=0}^{k} \beta_j^{(m)} f_{n+j} + \beta v_m^{(m)} f_{n+v_m} \right) + h^2 \lambda_k^{(m)} f'_{n+k} \qquad (1.3)$$

with the hybrid predictors given as;

$$H1: \; y_{n+v_{l+1}} = \sum_{j=0}^{k} \alpha_j^{(l)} y_{n+j} + h(\beta_k^{(l)} f_{n+k} + \beta^{(l)} v_l f_n + v_l + h^2 \lambda^{(l)} f'_{n+k}) \qquad (1.4)$$

$$H2: \text{where } y_{n+v_l} = y_{n+k} + h\sum_{j=0}^{k} \beta_j^{(-l)} f_{n+j} + h^2 \lambda_k^{(-l)} f'_{n+k} \quad (1.5)$$

The v_l and v_m are hybrid parameters and are chosen as $v_m = k - 1/2$, $v_l = \frac{v_{l+1+1}}{2}$, $l = 0(1)m-1$, $v_l \in (0,k)$, $v_l \neq j, 0(1)k$, to generate nested hybrid predictors. The parameter m is chosen as $m = k-1$. Where y_{n+j} is the numerical approximation to the exact solution

$$y(x_{n+j}) = y_{n+1}, f_{n+j} = f(x_{n+j}, y_{n+j}), f_{n+v_m} = f(x_{n+m}, y_{n+v_m}),$$
$$f_{n+k} = f(x_{n+k}, y_{n+k}), f'_{n+k} = f'(x_{n+k}, y_{n+k}), x_{n+1} = x_n + h \Rightarrow h = x_{n+1} - x_n,$$
$$f_{n+v_n} = f(x_{n+v}, y_{n+v_l}), f_{n+v_{l+1}} = f(x_{n+v_{l+1}}, y_{n+v_{l+1}})$$

and $\{\beta_j^{(m)}\}_{j=0}^{k}, \{\beta_j^{(-l)}\}_{j=0}^{k}, \{\alpha_k^{(l)}\}_{j=0}^{k}, \beta v_m^{(m)}, \lambda_k^{(m)}, \lambda_k^{(l)}$, $\beta_k^{(l)}, \lambda_k^{(l)}$ and $\beta v_l^{(l)}$ are continuous coefficients to be determined. The parameter v_l provides grid point collation points, $x_{n+1+v_m}, x_{n+v_l}, x_{n+v_{l+1}}$ in the open interval (x_{n+k-1}, x_{n+k}). h and k are the step-size and step length respectively.

Proposition1.

A hybrid linear multistep method version of (1.3) is said to be of order p if the associated local truncation error is;

$$\mathcal{L}[e^z; z] = e^{kz} - \left(e^{(k-1)z} + z\left(\sum_{j=0}^{k} \beta_j^{(m)} e^{jz} + \beta v_m^{(m)} e^{vz}\right) + z^2 \lambda_k^{(m)} e^{kz}\right) \quad (1.6)$$

for $v_m = k - \frac{1}{2}$ satisfies;

$\mathcal{L}[e^z; z] = C_{p+1} z^{p+1} + O(z^{(p+2)})$ with C_{p+1} as the error constant and $z = \lambda h$.

<u>Proof</u>

Consider the scalar test problem

$$y' = \lambda y, Re(\lambda) < 0 \quad (1.7)$$

where λ is a complex constant $\lambda = x + iy$ to the IVPs in (1.1), assuming the coefficients $\lambda_k^{(m)}, \beta_v^{(m)}, \alpha_j^{(m)}, \beta_j^{(m)}$ are known and the derivatives of the hybrid solution y_{n+v} is also known then $E^j y_n = y_{n+j}$ where E is the shift operator and $E^j = e^{jz}, z = \lambda h$. Applying the method (1.4) to the scalar test problem (1.7) yields the local truncation error of the method.

$$\mathcal{L}[e^z; z] = e^k - \left(e^{(k-1)z} + z\left(\sum_{j=0}^{k} \beta_j^{(m)} e^{jz} + \beta v_m^{(m)} e^{vz}\right) + z^2 \lambda_k^{(m)} e^{kz}\right)$$

for $v_m = k - \frac{1}{2}$, Expanding $e^{vz}, e^{jz}, e^{(k-1)z}$ and e^{kz} in Taylor series, then the method of order $\mathcal{P}$ yields; $\mathcal{L}[e^z, z] = C_{p+1} z^{p+1} + O(z^{p+1})$.

The associated continuous local truncation errors for the methods in (1.3), (1.4) and (1.5) are respectively;

$$\mathcal{L}[y(x); h]M - C_{p+1} h^{p+1} y^{p+1}(x_n),$$
$$\mathcal{L}[y(x); h]H1 = C_{p+1}{}^{(1)} h^{p+1(1)} y^{p+1(1)}(x_n),$$
$$\mathcal{L}[y(x); h]H2 = C_{p+1}{}^{(2)} h^{p+1(2)} y^{p+1(2)}(x_n),$$

For $x_n \leq x \leq x_{n+1}$ and this shows that the order of (1.2), (1.3), (1.4), and (1.5) are respectively $p = k+4, p^{(1)} = k+4$, and $p^{(2)} = k+3$ where $C_{p+1}, C_{p+1}^{(1)}$ and $C_{p+1}^{(2)}$ are the respective error constants of the schemes. With the error constant $C_{p+1}^{(1)}$ multiple error constants are generated as shown in section two.

The outline of this paper is as follows; we started with the introduction of the paper in section (1). Section (2) deals with the construction of the hybrid methods with some examples. The stability of the hybrid methods are investigated in section (3). Section (4) displaces the boundary luci of the methods. Finally, graphical results of numerical experiments on two stiff problems are presented and results compared with exact solution and ode15s of MATLAB CODE suite in [10]

2. CONSTRUCTION OF THE METHODS

The numerical solutions of (1.1) is assumed in the form of the polynomial interpolant

$$y(x) = \sum_{j=0}^{\varphi} a_j x^j \qquad 2.1)$$

where $\varphi = k + 4$ the degree of the polynomial and is equal to the order of the method, x^j is the polynomial basis function and $\{a_j\}_{j=1}^{\varphi}$ are the real parameter constant to be determined. Differentiating (2.1) yields

$$y'(x) = f(x, y) = \sum_{j=1}^{\varphi} j a_j x^{j-1} \qquad (2.2)$$

$$y''(x) = f'(x, y) = \sum_{j=2}^{\varphi} j(j-1) a_j x^{j-2} \qquad (2.3)$$

From (2.2) at $x = x_{n+v_m}$ and interpolating (2.1) and (2.3) at $x = x_{n+k}$ and $x = x_{n+j}, j = 0(1)k-1$ to obtain the system of equations

$$\begin{pmatrix} 1 & x_{n+k-1} & x_{n+k-1}^2 & x_{n+k-1}^3 & \cdots & x_{n+k-1}^{k+4} \\ 0 & 1 & 2x_n & 3x_n^2 & \cdots & (k+3)x_n^{k+3} \\ 0 & 1 & 2x_{n+1}^2 & 3x_{n+1}^2 & \cdots & (k+3)x_{n+1}^{k+3} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 1 & 2x_{n+k} & 3x_{n+k}^2 & \cdots & (k+3)x_k^{k+3} \\ 0 & 1 & 2x_{n+v_m} & 3x_{n+v_m}^2 & \cdots & (k+3)x_{n+v_m}^{k+3} \\ 0 & 0 & 2 & 6x_{n+k} & \cdots & (k+3)(k+2)x_{n+k}^{k+2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{k+2} \\ a_{k+3} \\ a_{k+4} \end{pmatrix} = \begin{pmatrix} y_{n+k-1} \\ f_n \\ f_{n+1} \\ \vdots \\ f_{n+k} \\ f_{n+v_m} \\ f_{n+k} \end{pmatrix} \qquad (2.4)$$

Solving equation (2.4) with MATHEMATICA 10.0 software package, the coefficients $a_{j's}$ $(j = 0(1)k+4)$ are obtained and substituted into (2.1) and yields the continuous coefficients of (1.2) and we obtain the scheme in $x, y_{n+j}, f_{n+j}, f_{n+v_m}, f'_{n+k}, j = 0(1)k$ for specific k. The construction of the hybrid predictors in (1.2) and (1.3) are similar to that of (1.2).

Methods applied in (1.3), (1.4), and (1.5) are of order $p = 5,\ k = 1$

$$y_{n+1} = h\left(\frac{f_n}{6} + \frac{2}{3} f_{n+\frac{1}{2}} + \frac{f_{n+1}}{6}\right) + y_n, \quad C_5 = -\frac{1}{2880}$$

with the hybrid

$$y_{n+\frac{1}{2}} = y_{n+1} + h\left(-\frac{f_n}{24} - \frac{11f_{n+1}}{24}\right) + \frac{1}{12} h^2 f'_{n+1}$$

Method of order $p = 5,\ k = 2,\ v_m = \frac{3}{2},\ m = 4$

$$y_{n+2} = h\left(-\frac{f_n}{720} + \frac{11f_{n+1}}{60} + \frac{28}{45} f_{n+\frac{3}{2}} + \frac{47f_{n+2}}{240}\right) + y_{n+1} - \frac{1}{120} h^2 f'_{n+2}, \quad C_6 = -\frac{1}{14400}$$

with hybrids

$$m = 1,\ \ l = 0,\ \ v_1 = \frac{3}{2},\ \ v_0 = \frac{7}{4}$$

For $v_1 = \frac{3}{2}$

$$y_{n+\frac{3}{2}} = h\left(-\frac{24}{53} f_{n+\frac{7}{4}} - \frac{9f_{n+2}}{3392}\right) - \frac{5y_n}{6784} + \frac{39y_{n+1}}{848} + \frac{6477y_{n+2}}{6784} - \frac{33h^2 f'_{n+2}}{3392}, \qquad C_5 = \frac{1}{50880}$$

For $v_0 = \frac{7}{4}$

$$y_{n+\frac{7}{4}} = y_{n+2} + h\left(\frac{13f_n}{12288} - \frac{29f_{n+1}}{3072} - \frac{2969f_{n+2}}{12288}\right) + \frac{49h^2 f'_{n+2}}{2048}, \qquad C_5 = \frac{-59}{184320}$$

Method of order $p = 6,\ k = 3,\ m = 2,\ v_2 = \frac{5}{2}$

$$y_{n+3} = y_{n+2} + h\left(\frac{f_n}{5400} - \frac{f_{n+1}}{360} + \frac{23f_{n+2}}{120} + \frac{136}{225}f_{n+\frac{5}{2}} + \frac{223f_{n+3}}{1080}\right) - \frac{1}{90}h^2 f'_{n+3} \quad , c_7 = -\frac{13}{604800}$$

with the hybrids

$$m = 2, l = 0,1 \quad v_2 = \frac{5}{2}, v_1 = \frac{11}{4}, v_0 = \frac{23}{8}$$

For $v_2 = \frac{5}{2}$

$$y_{n+\frac{5}{2}} = h\left(-\frac{120}{281}f_{n+\frac{11}{4}} - \frac{1055f_{n+3}}{53952}\right) + \frac{y_n}{10116} - \frac{65y_{n+1}}{35968} + \frac{255y_{n+2}}{4496} + \frac{305905y_{n+3}}{323712} - \frac{125h^2f'_{n+3}}{17984} \qquad 2.16$$

$$C_7 = -\frac{331}{48340992}$$

For $v_1 = \frac{11}{4}$

$$y_{n+\frac{11}{4}} = h\left(-\frac{924f_{n+\frac{23}{8}}}{3317} + \frac{6083f_{n+1}}{190464}\right) + \frac{329y_n}{30569472} - \frac{2409y_{n+1}}{13586432} + \frac{12551y_{n+2}}{3396608} + \frac{121846417y_{n+3}}{122277888} - \frac{34727h^2f'_{n+3}}{6793216}$$

$$C_7 = \frac{46409}{52171898880}$$

For $v_0 = \frac{23}{8}$

$$y_{n+\frac{23}{8}} = y_{n+3} + h\left(-\frac{553f_n}{8847360} + \frac{281f_{n+1}}{655360} - \frac{591f_{n+2}}{327680} - \frac{2186407f_{n+3}}{17694720}\right) + \frac{19697h^2f'_{n+3}}{2949120}, C_6 = \frac{25723}{943718400}$$

Method of order $p = 7$ $k = 4, m = 3, v_3 = \frac{7}{2}$

$$y_{n+4} = -\frac{13hf_n}{282240} + \frac{hf_{n+1}}{1890} - \frac{41hf_{n+2}}{10080} + \frac{62}{315}hf_{n+3} + \frac{1312hf_{n+\frac{7}{2}}}{2205} + \frac{3659hf_{n+\frac{7}{2}}}{17280} + y_{n+3} - \frac{25h^2f'_{n+4}}{2016}$$

$$C_8 = -\frac{1}{120960}$$

with hybrids

$$m = 3, l = 0,1,2 \quad v_3 = \frac{7}{2}, \; v_2 = \frac{15}{4}, v_1 = \frac{31}{8}, v_0 = \frac{63}{16}$$

For $v_3 = \frac{7}{2}$ $y_{n+\frac{7}{2}} = h\left(-\frac{1120f_{n+\frac{15}{4}}}{2733} - \frac{560356f_{n+4}}{1863728}\right) - \frac{1695y_n}{67166208} + \frac{119y_{n+1}}{349824} - \frac{2905y_{n+2}}{932864} + \frac{68705y_{n+3}}{104947\,2} + \frac{20995695y_{n+4}}{22388736}$

$$C_8 = \frac{263}{89554944}$$

For $v_2 = \frac{15}{4}$

$$y_{n+\frac{15}{4}} = h\left(-\frac{53440f_{n+\frac{31}{8}}}{203239} + \frac{7145326835f_{n+4}}{26638942208}\right) - \frac{324863y_n}{106555768832} + \frac{200305y_{n+1}}{4994801664} - \frac{4401045y_{n+2}}{13319471104} + \frac{7648795y_{n+3}}{1664933888} + \frac{318292518005y_{n+4}}{319667306496} - \frac{61802895h^2f'_{n+4}}{13319471104}$$

$$C_8 = \frac{706057}{1704892301312}$$

For $v_1 = \frac{31}{8}$

$$y_{n+\frac{31}{8}} = h\left(-\frac{199640f_{n+\frac{63}{16}}}{1314413} + \frac{898032601445f_{n+4}}{33078286221312}\right) - \frac{33071815y_n}{132313144885248} + \frac{19973765y_{n+1}}{6202178666496} - \frac{140112343y_{n+2}}{5513047703552}$$

$$+\frac{631757215y_{n+3}}{2067392888832} + \frac{3968270462536445y_{n+4}}{3969394346555744} - \frac{9894732365h^2f'_{n+4}}{5513047703552} \qquad C_8 = \frac{304970777}{8468041272655872}$$

For $v_0 = \frac{63}{16}$

$$y_{n+\frac{63}{16}} = h\left(\frac{225211f_n}{48318382080} - \frac{100489f_{1+n}}{3019898880} + \frac{455803f_{2+n}}{4026531840} - \frac{933979f_{3+n}}{3019898880} - \frac{3009042239f_{4+n}}{48318382080}\right)$$

$$+y_{4+n} + \frac{480249h^2f'_{4+n}}{268435456} \qquad C_7 = \frac{9971923}{40387440094720}$$

Table 2: The Error Constants For The Hybrid Methods in (1.3) - (1.5)

k	Order P (M)	p^1	$P^{(2)}$	$C_{p+1}(M)$	$C_{p+1}(H_{1a})$	$C_{p+1}(H_{1b})$	$C_{p+1}(H_{1c})$	$C_{p+1}(H_{1d})$	H_2	SDLMM[5]
1	4	4	3	$\frac{-1}{12000}$					[illegible]	$\frac{1}{72}$
2	5	5	4	$\frac{-1}{14400}$	[illegible]				[illegible]	[illegible]
3	6	6	5	[illegible]	[illegible]	[illegible]			[illegible]	$\frac{17}{7200}$
4	7	7	6	$\frac{-1}{120960}$	[illegible]	[illegible]	1η		$\frac{-9971923}{40387440094720}$	$\frac{41}{30240}$
5	8	8	7	[illegible]	[illegible]	[illegible]	2η	3η	[illegible]	[illegible]

where $1\eta = \frac{304970777}{846804127265 52}$, $2\eta = \frac{622155933063}{31968863424 06894}$, and $3\eta = \frac{27702196100781}{1929282550196475025397 76}$

Definition 1: A numerical integrator is said to be A-stable if the absolute values of the roots of the stability polynomial of the numerical integrator lie in the open half of the complex plane $\mathbb{C}$.

Definition 2: [13]. A numerical integrator is said to be A (α)-stable for some $\alpha\epsilon[0,\pi/2]$, if the wedge $S_\alpha = \{z \mid |Arg(-z)| < \alpha, z \neq 0\}$ is contained in the region of absolute stability. The largest α_{max} is the angle of absolute stability or argument of stability.

Definition 3: [11]. A numerical integrator is zero-stable if the roots of the first characteristics polynomials satisfy $|r_i| \leq 1$ with roots of $|r_i| = 1$ being simple.

3. STABILITY OF THE HYBRID SCHEMES

Applying the resulting schemes for fixed k to the scalar test problems $y' = \lambda y, Re(\lambda) < 0$ yield the stability polynomials as:

$$\pi(r,z) = r^k - r^{k-1} - z\left(\sum_{j=0}^{k}\beta_j^{(m)}r^j + \beta v_m^{(m)}\left(H(r,z)\right) - z^2\lambda_k^{(m)}\right)$$

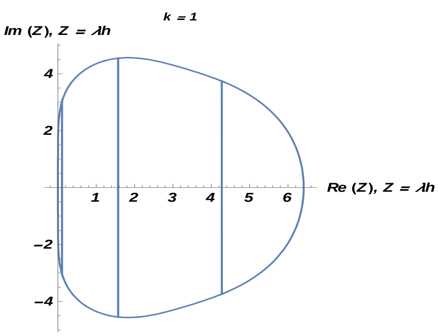
where,

$$H(r,z) = r^k - z\left(\beta_k^{(1)}f_{n+k} + \beta v_1^{(1)}\left(\ldots\left(r^k + z\sum_{j=0}^{k}\beta_j^{(-1)}r^j + z^2\lambda_k^{(-1)}r^k + z^2\lambda_k^{(1)}f_{n+k}\right)\ldots\right)\right)$$

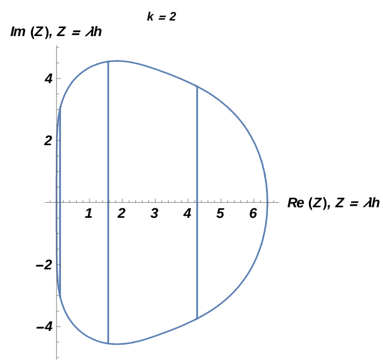
The methods were found to be A-stable for $k \leq 5$ and A(α)-stable for $k \leq 6$ to k=10 where the computer could not proceed with the computation. The figures below show the stability plot (boundary loci) of the methods.

4. STABILITY PLOTS OF THE HYBRID LINEAR METHODS

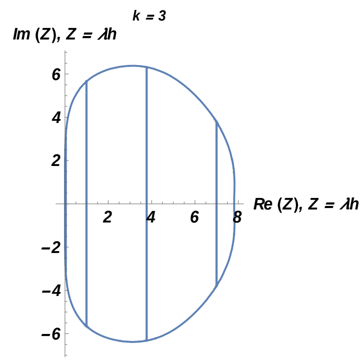
Figure



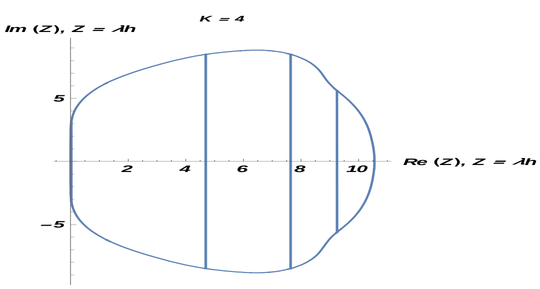
k = 1



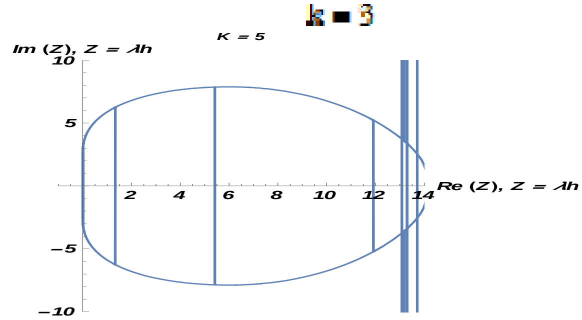
k = 2



k = 3



K=4



k=5

The above figures show the region of stability of the hybrid schemes.

5. NUMERICAL IMPLEMENTATION

[5], [11] noted that linear multistep methods for stiff ODEs must be implicit and A-stable and therefore require a scheme to resolve the implicitness. Therefore Newton- Raphson iterative scheme is used to resolved the implicitness of the methods given as;

$y_{n+k}^{(s+1)} = y_{n+k}^{(s)} - J\left\{y_{n+k}^{(s)}\right\}^{-1} F\left(y_{n+k}^{(s)}\right)$, S=0, 1, 2 …Where $J\left(y_{n+k}^{[s]}\right)$ is the Jacobian matrix of the vector system of the method.

$F\left(y_{n+k}^{[s]}\right) = y_{n+k}^{(s)} - \left[y_{n+k-1} + h\left(\sum_{j=0}^{k}\beta_j^{(m)}f_{n+j} + \beta v_m^{(1)}f_{n+v_m}^{[s]}\right)\right] + h^2\lambda_k^{(m)}f_{n+k}^{[s]}$ the starting value for the Newton - Raphson scheme is generated from modified explicit one-step formulas;

$$y_{n+1}^{0} = \begin{cases} y_n + hf_n, & p = 1 \\ y_n + \frac{h}{2}(f_{n-1} + f_n), & p = 2 \end{cases}$$

We considered two test problems in this paper to test the hybrid methods derived.

Example 1.

Consider the nonlinear moderately stiff problems in formula [1978]

$$y'_1 = -0.1y_1 - 199.9y_2,$$
$$y'_2 = -200y_2,$$
$$y_1(0) = 2 \quad y_2(0) = 1$$

The exact solution is $y_1(x) = e^{-0.1x} + e^{-200x}$ and $y_2(x) = e^{-200x}$

Using equations (2.1) – (2.4) with h=0.0001, we have the result of the example as shown graphically in Figure 1.

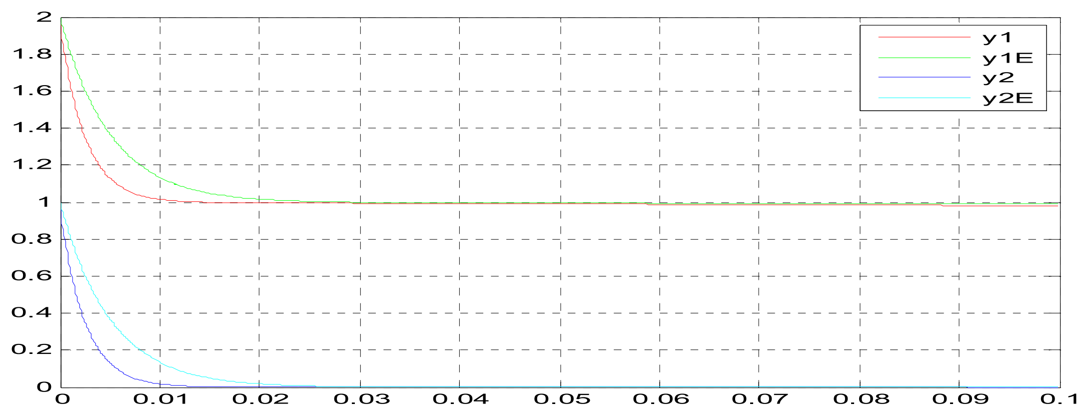


Figure 1: Graphical result of Example 1

Example 2.

Consider also the Vander pol equation in Hairer and Warner (1996)

$$y'_1 = y_2$$
$$y'_2 = \left((1 - y_1^2)y_2 - y_1\right)/\varepsilon \quad y_1(0) = 2,\ y_2(0) = 0$$

Similarly, using equations (2.1) through (2.4) with $h = 0.001$ and $\varepsilon = 10^{-1}$, we have the result of this example as shown graphically in Figure 2.

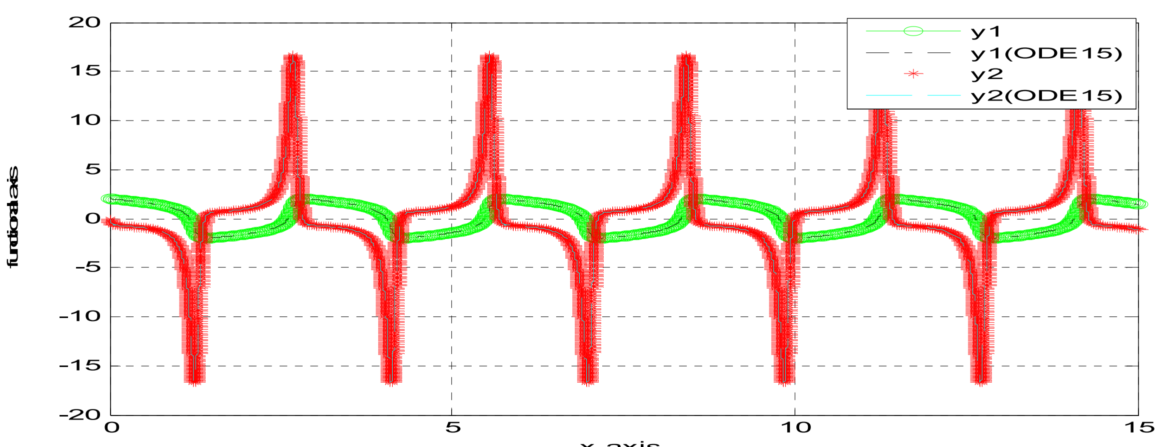


Figure 2: Graphical result of Example 2

CONCLUSION

In this paper, we have presented new class of higher order A-stable hybrid linear multistep methods for stiff initial value problems in ODEs. The schemes are found to be A-stable for $k \leq 5$ and $A(\alpha)$-stable for k=6, 7, 8, 9 to where the laptop computer could not proceed with the computation due to the rigorous computations involved. The new scheme has smaller error constants than the SDLM [5] as shown in Table (1). The methods have been demonstrated on two stiff problems and the numerical results of the scheme coincide with the exact solution as in figure (1) and ODE15s of MATHLAB as in figure (2) hence, compares favorably with ODE15s of MATHLAB. This makes the new scheme suitable for stiff initial value problems in ordinary differential equations.

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