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Apéry's Constant Calculation and Prime Numbers Distributions: A Matrix Approach

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Abstract. We built a matrix M made of zeta function series terms in rows. It allows matrix $M.M^T$ and $M^T.M$ to be build giving interesting characteristics. Equalling sums of terms in rows and columns in M gave a general method for $\zeta(2k+1)$ calculations. Some important results were deduced from the knowledge of all $\zeta(s)$ and related to matrix $M.M^T$ and Euler product formula. Finally, considering matrix $M^T.M$ characteristics, a method for prime numbers enrichment is proposed giving a promising tool for information technologies encoding.

Key Words: Apéry – Zeta function – Odd Integers – Prime Numbers – Hilbert's Matrix

1. INTRODUCTION.

The Riemann zeta function or Euler-Riemann zeta function $\zeta(s)$ is a function of complex variable s given by:

$$\zeta(s) = \sum_{k=1}^{+\infty} \frac{1}{k^s}$$

In 1735, Euler solved the famous Basel problem [**1**] and found $\zeta(2) = \frac{\pi^2}{6}$ using the polynomial development of $\frac{sinx}{x}$. He also brilliantly showed that $\zeta(4) = \frac{\pi^4}{90}$ and $\zeta(6) = \frac{\pi^6}{945}$. Finally, for even integers greater than 1, he established the general formula:

$$\forall k > 0 \qquad \zeta(2k) = \mid B_{2k} \mid \frac{(2\pi)^{2k}}{2(2k)!}$$

Where B_{2k} are the Bernoulli numbers ($B_2 = \frac{1}{2}$; $B_4 = -\frac{1}{30}$; …) [**2,3**].

But for odd integers, the problem appeared much more complicated [**4-6**] and Euler himself was not able to find an analytical result for $\zeta(3)$. The Indian mathematician Ramanujan [**7**] worked a lot on this problem without success. We had to wait for Apéry's work in 1978, who demonstrated that $\zeta(3)$ is an irrational number. This number was called Apéry's constant in honour of this important result. But even now, there exist no analytical or closed form expression for $\zeta(3)$ and for all others values of zeta function for odd integers i.e. $\zeta(2k+1)\ \ k \geq 1$.

This problem remains of major interest in mathematics for numbers theory [**8**] particularly for prime numbers determination. From Euler work, zeta function is related to prime numbers through Euler product formula:

$$\zeta(s) = \sum_{k=1}^{+\infty} \frac{1}{k^s} = \prod_{p\ prime} \frac{1}{1-p^{-s}}$$

Moreover, odd integers zeta function is of interest in physics particularly for quantum electrodynamics [**9**].

The objective of the present paper is to propose a general method available for $\zeta(2k+1)$ calculation and to give consequences for famous prime numbers determination problem. We based our approach on a matrix representation of zeta functions and tried to find an analytical relationship for $\zeta(2k+1)$ using known values of zeta series i.e. $\zeta(2k)$.

In the first section, we will describe this matrix M and its transpose M^T to show their interesting characteristics. We will then build the products $M.M^T$ and $M^T.M$ to show how they could be useful for both zeta function determination and prime numbers analysis.

In a second section, we will use matrix M to establish analytical calculations for $\zeta(3)$ and $\zeta(5)$ and give a general closed form for all $\zeta(2k+1)$ $k \geq 1$.

Finally, in a third section, we will discuss some consequences of these results for zeta function properties and prime numbers determination.

2. A MATRIX REPRESENTATION OF ZETA FUNCTION VALUES.

2.1 The matrix M with zeta series terms in rows.

We started our approach by representing zeta function numbers in the form of a square matrix M having an infinite number of rows and columns:

$$\begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{array} \\ M = \begin{array}{c} \zeta(1) \\ \zeta(2) \\ \zeta(3) \\ \zeta(4) \\ \zeta(5) \\ \vdots \end{array} & \begin{bmatrix} \frac{1}{1^1} & \frac{1}{2^1} & \frac{1}{3^1} & \frac{1}{4^1} & \frac{1}{5^1} & \frac{1}{6^1} & \frac{1}{7^1} & \cdots \\ \frac{1}{1^2} & \frac{1}{2^2} & \frac{1}{3^2} & \frac{1}{4^2} & \frac{1}{5^2} & \frac{1}{6^2} & \frac{1}{7^2} & \cdots \\ \frac{1}{1^3} & \frac{1}{2^3} & \frac{1}{3^3} & \frac{1}{4^3} & \frac{1}{5^3} & \frac{1}{6^3} & \frac{1}{7^3} & \cdots \\ \frac{1}{1^4} & \frac{1}{2^4} & \frac{1}{3^4} & \frac{1}{4^4} & \frac{1}{5^4} & \frac{1}{6^4} & \frac{1}{7^4} & \cdots \\ \frac{1}{1^5} & \frac{1}{2^5} & \frac{1}{3^5} & \frac{1}{4^5} & \frac{1}{5^5} & \frac{1}{6^5} & \frac{1}{7^5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

Figure 1: Matrix M made of $\zeta(s)$ numbers.

As showed in Figure 1, rows of M correspond to terms of zeta function series $\zeta(s)$:

- First row: $\zeta(1)$

- Second row: $\zeta(2)$

- Third row: $\zeta(3)$ …

Columns are numbered 1,2,3, ... from the left to the right.

Using this representation, matrix M exhibits some interesting properties. At first, we can see that each column corresponds to terms of a geometric series of common ratio q. From the left to the right, we have: $q = \frac{1}{1}$; $\frac{1}{2}$; $\frac{1}{3}$; $\frac{1}{4}$; ... Of course, for $q = 1$, the series diverges but for all other columns, we have $q < 1$ and then the series converges. Considering summation rule of geometric series terms, we have for $0 < q < 1$:

$$1 + q + q^2 + q^3 + \cdots = \sum_{k=0}^{+\infty} q^k = \frac{1}{1-q}$$

Then we have then for each column of M:

$$\forall n \in \mathbb{N}, n > 1 \quad \sum_{k=1}^{+\infty} \frac{1}{n^k} = \left(\sum_{k=0}^{+\infty} \frac{1}{n^k}\right) - 1 = \frac{1}{n-1}$$

For example, we have:

$$n = 2 \Rightarrow \sum_{k=1}^{+\infty} \frac{1}{2^k} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 1$$

$$n = 3 \Rightarrow \sum_{k=1}^{+\infty} \frac{1}{3^k} = \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots = \frac{1}{2}$$

It is then possible to consider the series corresponding to the sum of all columns for columns numbers n strictly greater than 1, we obtain:

$$\sum_{n=2}^{+\infty} \frac{1}{n-1} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots = \zeta(1) = +\infty$$

Considering that the sum of all numbers in rows and columns must remain the same. For example, in Figure 1, if we consider zeta functions values from $\zeta(1)$ to $\zeta(5)$ and columns numbers from 1 to 5, the sum of all numbers is a constant equal to the sum of terms in the five rows and the sum of terms in the five columns.

Applying this principle and considering the above result, we obtain the following important well known result:

$$\sum_{s=1}^{+\infty} \zeta(s) = +\infty$$

Other interesting quantities are the diagonals of matrix M. If we first consider the main diagonal corresponding to the trace: $Tr(M)$, we have:

$$Tr(M) = \sum_{i=1}^{+\infty} a_{ii} = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots = \sum_{i=1}^{+\infty} \frac{1}{i^i}$$

It is remarkable that this last series corresponds to famous Sophomore's dream discovered in 1697 by Bernoulli and more recently modernized by Dunham [**10**]:

$$Tr(M) = \sum_{i=1}^{+\infty} i^{-i} = \int_0^1 x^{-x}\, dx$$

A study of function $f(x) = x^{-x}$ for $x \in [0,1]$ gives the following graph:

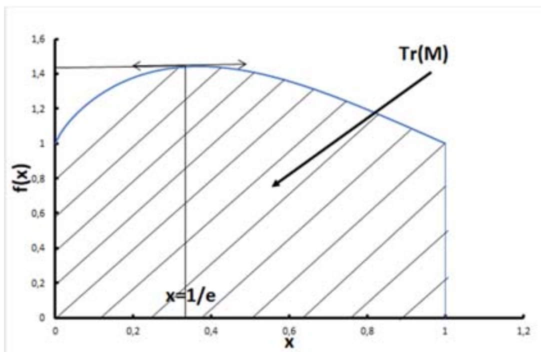


Figure 2: Graph of $f(x) = x^{-x}$ for $x \in [0,1]$.

As showed in Figure 2, function f has a maximum for $x = \frac{1}{e} \cong 0.367879441 \ldots$ and $f\left(\frac{1}{e}\right) = e^{1/e} \cong 1.44466786 \ldots$ Surface under the curve gives the value of $Tr(M)$ as a finite quantity and the series above converges very quickly as showed in the following table:

i	1	2	3	4	5	6	7	8	9
i^{-i}	1	0.25	0.037	0.004	$3.2\ 10^{-4}$	$2.1\ 10^{-5}$	$1.2\ 10^{-6}$	$6\ 10^{-8}$	$2.6\ 10^{-9}$

Table 1: Illustration of rapid convergence of $Tr(M)$ series.

With 26 terms, we obtained $Tr(M) = 1.2912859970 \ldots$

Other diagonals are also interesting, we called d_i $i = 1,2,3, \ldots$ the sum of elements in each ascending skew-diagonals taken from left to right in matrix M. We have then in Figure 1:

$$d_1 = \frac{1}{1^1}$$

$$d_2 = \frac{1}{1^2} + \frac{1}{2^1}$$

$$d_3 = \frac{1}{1^3} + \frac{1}{2^2} + \frac{1}{3^1}$$

Even if the number of rows and columns in M are infinite, the number of terms in each d_i sum is finite. The above examples can be generalized giving the following expression:

$$\forall i \in \mathbb{N} \,; i \geq 1 \quad d_i = \sum_{k=1}^{i} \frac{1}{k^{i-k+1}}$$

The following graph gives the evolution of d_i values for i varying in the range 1 to 43:

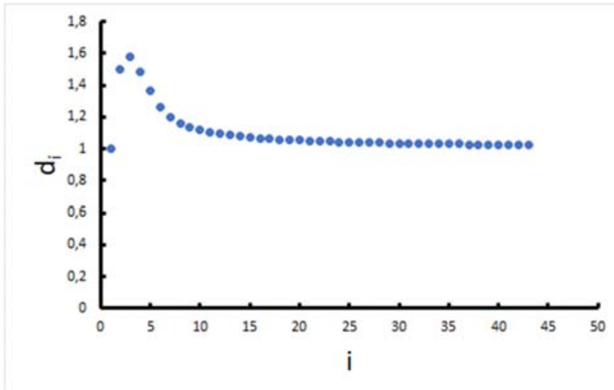


Figure 3: Graph of d_i for $1 \leq i \leq 43$.

Because d_i numbers are always the ratio of two integers, they are rational numbers. This result can easily be verified for $d_1 = 1$; $d_2 = 1.5$; $d_3 = 1.58333 \ldots$ and with more difficulty for example for d_6 =1.2658873456701234567012345670…

Finally, as showed on the graph presented in Figure 3, asymptotic value of d_i is 1 even if the sum converges slowly. This result is quite easy to understand because terms in the middle of each diagonal, i.e. far from the extremal values rapidly tend to zero. So, when number of terms increases, it only remains 1 and $\frac{1}{i}$ at the two extremities of the diagonal. Of course, we have $\lim_{i \to +\infty} \frac{1}{i} = 0$ giving a value of 1 for diagonal d_i.

2.2 The matrices $M^T - M.M^T - M^T.M$.

It is then easy to build matrix M^T by exchanging rows and columns in M. We obtain a matrix with the terms of a geometric progression in each row often called a Vandermonde matrix:

$$M^T = \begin{bmatrix} \frac{1}{1^1} & \frac{1}{1^2} & \frac{1}{1^3} & \frac{1}{1^4} & \frac{1}{1^5} & \frac{1}{1^6} & \frac{1}{1^7} & \cdots \\ \frac{1}{2^1} & \frac{1}{2^2} & \frac{1}{2^3} & \frac{1}{2^4} & \frac{1}{2^5} & \frac{1}{2^6} & \frac{1}{2^7} & \cdots \\ \frac{1}{3^1} & \frac{1}{3^2} & \frac{1}{3^3} & \frac{1}{3^4} & \frac{1}{3^5} & \frac{1}{3^6} & \frac{1}{3^7} & \cdots \\ \frac{1}{4^1} & \frac{1}{4^2} & \frac{1}{4^3} & \frac{1}{4^4} & \frac{1}{4^5} & \frac{1}{4^6} & \frac{1}{4^7} & \cdots \\ \frac{1}{5^1} & \frac{1}{5^2} & \frac{1}{5^3} & \frac{1}{5^4} & \frac{1}{5^5} & \frac{1}{5^6} & \frac{1}{5^7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Figure 4: Transpose matrix M^T representation.

We can now calculate the product $M.M^T$. We called a_{ij} terms of this matrix and we obtain by use of classical matrix product rule:

$$a_{11} = \frac{1}{1^1}.\frac{1}{1^1} + \frac{1}{2^1}.\frac{1}{2^1} + \frac{1}{3^1}.\frac{1}{3^1} + \cdots = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \zeta(2)$$

$$a_{12} = \frac{1}{1^1}.\frac{1}{1^2} + \frac{1}{2^1}.\frac{1}{2^2} + \frac{1}{3^1}.\frac{1}{3^2} + \cdots = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = \zeta(3)$$

$$a_{21} = \frac{1}{1^2}.\frac{1}{1^1} + \frac{1}{2^2}.\frac{1}{2^1} + \frac{1}{3^2}.\frac{1}{3^1} + \cdots = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = a_{12} = \zeta(3)$$

$$a_{22} = \frac{1}{1^2}.\frac{1}{1^2} + \frac{1}{2^2}.\frac{1}{2^2} + \frac{1}{3^2}.\frac{1}{3^2} + \cdots = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \zeta(4)$$

Extending calculations to all a_{ij} gives the following $M.M^T$ matrix:

$$M.M^T = \begin{bmatrix} \zeta(2) & \zeta(3) & \zeta(4) & \zeta(5) & \zeta(6) & \zeta(7) & \zeta(8) & \cdots \\ \zeta(3) & \zeta(4) & \zeta(5) & \zeta(6) & \zeta(7) & \zeta(8) & \zeta(9) & \cdots \\ \zeta(4) & \zeta(5) & \zeta(6) & \zeta(7) & \zeta(8) & \zeta(9) & \zeta(10) & \cdots \\ \zeta(5) & \zeta(6) & \zeta(7) & \zeta(8) & \zeta(9) & \zeta(10) & \zeta(11) & \cdots \\ \zeta(6) & \zeta(7) & \zeta(8) & \zeta(9) & \zeta(10) & \zeta(11) & \zeta(12) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Figure 5: $M.M^T$ matrix representation.

We have for each term a_{ij}:

$\forall i \geq 1\ ;\ \forall j \geq 1\ \ a_{ij} = \zeta(i+j)$

It shows that a_{ij} only depends on $(i+j)$ making matrix $M.M^T$ a Hankel type matrix. Because Hankel matrices are closely related to Toeplitz matrices and Toeplitz matrices properties are of major importance in applied mathematics, this result is very important for a better understanding of zeta function behaviour.

The trace of $M.M^T$ matrix corresponds to even values of zeta function:

$$Tr(M.M^T) = \sum_{k=1}^{+\infty} \zeta(2k) = \sum_{k=1}^{+\infty} |B_{2k}| \frac{(2\pi)^{2k}}{2(2k)!}$$

We can also calculate the other product i.e. $M^T.M$. We called b_{ij} terms of this matrix and calculations gave:

$$b_{11} = \frac{1}{1^1}.\frac{1}{1^1} + \frac{1}{1^2}.\frac{1}{1^2} + \frac{1}{1^3}.\frac{1}{1^3} + \cdots = 1 + 1 + 1 + \cdots = +\infty$$

$$b_{12} = \frac{1}{1^1}.\frac{1}{2^1} + \frac{1}{1^2}.\frac{1}{2^2} + \frac{1}{1^3}.\frac{1}{2^3} + \cdots = \sum_{k=1}^{+\infty}\left(\frac{1}{2}\right)^k = \sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)^k - 1 = \frac{1}{1-\frac{1}{2}} - 1 = 1$$

$$b_{21} = \frac{1}{2^1}.\frac{1}{1^1} + \frac{1}{2^2}.\frac{1}{1^2} + \frac{1}{2^3}.\frac{1}{1^3} + \cdots = b_{12} = 1$$

$$b_{13} = \frac{1}{1^1}.\frac{1}{3^1} + \frac{1}{1^2}.\frac{1}{3^2} + \frac{1}{1^3}.\frac{1}{3^3} + \cdots = \sum_{k=1}^{+\infty}\left(\frac{1}{3}\right)^k = \sum_{k=0}^{+\infty}\left(\frac{1}{3}\right)^k - 1 = \frac{1}{1-\frac{1}{3}} - 1 = \frac{1}{2}$$

$$b_{31} = b_{13}$$

$$b_{22} = \frac{1}{2^1}.\frac{1}{2^1} + \frac{1}{2^2}.\frac{1}{2^2} + \frac{1}{2^3}.\frac{1}{2^3} + \cdots = \sum_{k=1}^{+\infty}\left(\frac{1}{2}\right)^{2k} = \sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)^{2k} - 1 = \frac{1}{1-\left(\frac{1}{2}\right)^2} - 1 = \frac{1}{3}$$

$$b_{33} = \sum_{k=1}^{+\infty}\left(\frac{1}{3}\right)^{2k} = \frac{1}{8}$$

Some other results are given in the following matrix representation.

$$M^T.M = \begin{bmatrix} +\infty & \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{1} & \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \frac{1}{11} & \frac{1}{13} & \cdots \\ \frac{1}{2} & \frac{1}{5} & \frac{1}{8} & \frac{1}{11} & \frac{1}{14} & \frac{1}{17} & \frac{1}{20} & \cdots \\ \frac{1}{3} & \frac{1}{7} & \frac{1}{11} & \frac{1}{15} & \frac{1}{19} & \frac{1}{23} & \frac{1}{27} & \cdots \\ \frac{1}{4} & \frac{1}{9} & \frac{1}{14} & \frac{1}{19} & \frac{1}{24} & \frac{1}{29} & \frac{1}{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

<u>Figure 6</u>: $M^T.M$ matrix representation.

The above results showed that except for b_{11} , we have the following relationship for b_{ij}:

$$b_{ij} = \frac{1}{ij-1}$$

This definition recalls well-known Hilbert matrix:

$$H_{ij} = \frac{1}{i+j-1}$$

According to these relationships, all terms are the reverse of an integer fully determined by row and column numbers. Like Hilbert matrix, $M^T.M$ exhibits interesting characteristics particularly for prime numbers distribution. Some of them will be discussed in chapter 4 of this paper but analysis will not be complete and a deep numerical study of this matrix is surely required.

3. A CALCULATION METHOD FOR $\zeta(2k+1)$.

3.1 Calculation of $\zeta(3)$.

Considering matrix M presented in Figure 1, columns are numbered $i = 1,2,3, \ldots$ we have for each of them the sum of $(n+1)$ first terms of a geometric progression given by:

$$S_n = \frac{1}{i^0} + \frac{1}{i^1} + \frac{1}{i^2} + \frac{1}{i^3} + \cdots + \frac{1}{i^n} = \frac{1-\left(\frac{1}{i}\right)^{n+1}}{1-\left(\frac{1}{i}\right)} \quad \frac{1}{i} \neq 1$$

$$\Rightarrow \frac{1}{i^2} + \frac{1}{i^3} + \frac{1}{i^4} = \frac{1-\left(\frac{1}{i}\right)^5}{1-\left(\frac{1}{i}\right)} - \frac{1}{i^0} - \frac{1}{i^1} = \frac{i^5-1}{i^5}.\frac{i}{i-1} - \frac{i+1}{i} = \frac{1}{i(i-1)} - \frac{1}{i^4(i-1)}$$

And we can write that $\zeta(2) + \zeta(3) + \zeta(4)$ is equal to the sum of all columns called C_i $i = 1,2,3, \ldots$

$$\zeta(2)+\zeta(3)+\zeta(4)=\frac{1}{1^2}+\frac{1}{1^3}+\frac{1}{1^4}+\sum_{i=2}^{+\infty}C_i=3+\sum_{i=2}^{+\infty}C_i$$

$$\Rightarrow\ \zeta(2)+\zeta(3)+\zeta(4)=3+\sum_{i=2}^{+\infty}\left(\frac{1}{i(i-1)}-\frac{1}{i^4(i-1)}\right)=3+\sum_{i=2}^{+\infty}\frac{1}{i(i-1)}-\sum_{i=2}^{+\infty}\frac{1}{i^4(i-1)}$$

As a well-known telescopic series, we have:

$$\sum_{i=2}^{+\infty}\frac{1}{i(i-1)}=1$$

Giving,

$$\zeta(2)+\zeta(3)+\zeta(4)=4-\sum_{i=2}^{+\infty}\frac{1}{i^4(i-1)} \qquad (a)$$

Using the same approach than above, we also have:

$$\zeta(2)+\zeta(3)=\frac{1}{1^2}+\frac{1}{1^3}+\sum_{i=2}^{+\infty}C_i$$

With,

$$C_i=\frac{1-\left(\frac{1}{i}\right)^4}{1-\frac{1}{i}}-\frac{1}{i^0}-\frac{1}{i^1} \qquad \frac{1}{i}\neq 1$$

$$\Rightarrow C_i=\frac{1}{i(i-1)}-\frac{1}{i^3(i-1)} \qquad i\neq 1$$

$$\Rightarrow \zeta(2)+\zeta(3)=2+\sum_{i=2}^{+\infty}\frac{1}{i(i-1)}-\sum_{i=2}^{+\infty}\frac{1}{i^3(i-1)}=3-\sum_{i=2}^{+\infty}\frac{1}{i^3(i-1)} \qquad (b)$$

Considering now $\zeta(3)+\zeta(4)$, we have:

$$\zeta(3)+\zeta(4)=\frac{1}{1^3}+\frac{1}{1^4}+\sum_{i=2}^{+\infty}C_i$$

With,

$$C_i=\frac{1-\left(\frac{1}{i}\right)^5}{1-\frac{1}{i}}-\frac{1}{i^0}-\frac{1}{i^1}-\frac{1}{i^2} \qquad \frac{1}{i}\neq 1$$

$$\Rightarrow C_i=\frac{1}{i^2(i-1)}-\frac{1}{i^4(i-1)} \qquad i\neq 1$$

$$\Rightarrow \zeta(3)+\zeta(4)=2+\sum_{i=2}^{+\infty}\frac{1}{i^2(i-1)}-\sum_{i=2}^{+\infty}\frac{1}{i^4(i-1)} \qquad (c)$$

Considering (a), (b) and (c), we have the following series to determine:

$$\sum_{i=2}^{+\infty}\frac{1}{i^2(i-1)}\ ;\ \sum_{i=2}^{+\infty}\frac{1}{i^3(i-1)}\ ;\ \sum_{i=2}^{+\infty}\frac{1}{i^4(i-1)}$$

To our knowledge, there exists no known solution for each of them and they are essential to determine $\zeta(3)$.

At first, we can do $(a)-(c)$ leading to:

$$\sum_{i=2}^{+\infty}\frac{1}{i^2(i-1)}=2-\zeta(2)=2-\frac{\pi^2}{6}$$

Which appeared an original result. But until now we were unable to find solutions for the other series essential to the determination of a closed form for $\zeta(3)$.

Considering (a), we decided to write:

$$\zeta(3)=4-\zeta(2)-\zeta(4)-\sum_{i=2}^{+\infty}\frac{1}{i^4(i-1)}=4-\frac{\pi^2}{6}-\frac{\pi^4}{90}-\sum_{i=2}^{+\infty}\frac{1}{i^4(i-1)}$$

It allows to suppose that $\zeta(3)$ could take the following closed form:

$$\zeta(3)=4+\frac{\pi}{\alpha}-\frac{\pi^2}{6}+\frac{\pi^3}{\beta}-\frac{\pi^4}{90} \qquad (d)$$

Using the values of $\zeta(3)$ given by OEIS [**11**], we decided to try to find values for α and β considered as rational numbers i.e. as the ratio of two integers. At first, we supposed that $\frac{1}{\alpha}$ or $\frac{1}{\beta}=0$ and we tried to find the number λ such as:

$$\lambda=\sum_{i=2}^{+\infty}\frac{1}{i^4(i-1)}=0.07068579\ldots$$

And λ is the product of a rational number by respectively π,π^2,π^3,π^4. It means that in equation (d) we considered that the maximum power of π involved is 4 and this power is an integer.

Using OEIS [**11**] value for $\zeta(3)$, we calculated numbers k_n as followed:

$$k_n=\frac{\zeta(3)-4+\frac{\pi^2}{6}+\frac{\pi^4}{90}}{\pi^n}\ with\ n=1,2,3,4$$

The best solution was found for $n=1$ giving $k_1=-0.02249998776902\ldots$ and $\frac{1}{k_1}=-44.4444686044\ldots$ which is very close to rational number $-\frac{400}{9}=-44.44444\ldots$

We have then the following approximative expression for $\zeta(3)$:

$$\zeta(3) = 4 - \frac{9\pi}{400} - \frac{\pi^2}{6} - \frac{\pi^4}{90} = 1.202056864734865\ldots$$

Giving a difference of 3.842…10^{-8} with OEIS [**11**] value.

This result, obtained using 50 decimals for $\zeta(3)$ and π, is accurate enough to say that, in equation (d), only one parameter we called α or β is not sufficient to obtain the right value for Apery's constant $\zeta(3)$.

Considering equation (d), we must then consider than two parameters are necessary and we search for (α, β) such as:

$$\frac{\pi^3}{\beta} = \zeta(3) - 4 + \frac{\pi^2}{6} + \frac{\pi^4}{90} - \frac{\pi}{\alpha} = A - \frac{\pi}{\alpha}$$

$$\Rightarrow \beta = \frac{\pi^3}{A - \frac{\pi}{\alpha}}$$

After several tests, we found $\alpha = -9$ and $\beta = 111.38110012428$ as the two rational numbers giving $\zeta(3)$ value in agreement with OEIS value. We can then write that Apery's constant is:

$$\zeta(3) = 4 - \frac{\pi}{9} - \frac{\pi^2}{6} + \frac{\pi^3}{111.38110012428} - \frac{\pi^4}{90}$$

The difference found with OEIS value is 10^{-15} which is very difficult to avoid considering numerical erosion phenomena.

3.2 Calculation of $\zeta(5)$.

It is now simple to apply the same method to the following rows of matrix M: $\zeta(2), \zeta(3), \zeta(4), \zeta(5), \zeta(6)$.

Using the same notations than above, we have:

$$\zeta(2) + \zeta(3) + \zeta(4) + \zeta(5) + \zeta(6) = C_1 + \sum_{i=2}^{+\infty} C_i = 5 + \sum_{i=2}^{+\infty} C_i$$

With,

$$C_i = \frac{1 - \left(\frac{1}{i}\right)^7}{1 - \frac{1}{i}} - 1 - \frac{1}{i} = \frac{1}{i(i-1)} - \frac{1}{i^6(i-1)}$$

We can now express $\zeta(5)$ as followed:

$$\zeta(5) = 5 + 1 - \sum_{i=2}^{+\infty} \frac{1}{i^6(i-1)} - \big(\zeta(2) + \zeta(3) + \zeta(4) + \zeta(6)\big)$$

Because $\zeta(2), \zeta(3), \zeta(4), \zeta(6)$ are determined, the only remaining undetermined quantity is:

$$\sum_{i=2}^{+\infty} \frac{1}{i^6(i-1)}$$

Comparable to the series

$$\sum_{i=2}^{+\infty} \frac{1}{i^4(i-1)}$$

Found for $\zeta(3)$ determination.

We numerically found:

$$\mu = \sum_{i=2}^{+\infty} \frac{1}{i^6(i-1)} = 0.016414979 \ldots$$

Considering that:

$$\zeta(2) + \zeta(3) + \zeta(4) + \zeta(6) = 4 - \frac{\pi}{9} + \frac{\pi^3}{111.38110012428} + \frac{\pi^6}{945}$$

We have for $\zeta(5)$:

$$\zeta(5) = 6 - \mu - 4 + \frac{\pi}{9} - \frac{\pi^3}{111.38110012428} - \frac{\pi^6}{945}$$

$$\Rightarrow \ \zeta(5) = 2 + \frac{\pi}{9} - \frac{\pi^3}{111.38110012428} - \frac{\pi^6}{945} - \mu$$

As showed for $\zeta(3)$ calculation, $\zeta(5)$ can be expressed as:

$$\zeta(5) = 2 + \frac{\pi}{9} + \frac{\pi^2}{\alpha} - \frac{\pi^3}{111.38110012248} + \frac{\pi^4}{\beta} + \frac{\pi^5}{\gamma} - \frac{\pi^6}{945}$$

But in that case, we have 3 parameters α, β, γ to determine which makes the problem unsolvable.

We can also use approximative $\zeta(3)$ value determined above:

$$\zeta(3) = 4 - \frac{9\pi}{400} - \frac{\pi^2}{6} - \frac{\pi^4}{90}$$

Giving,

$$\zeta(5) = 2 + \frac{9\pi}{400} - \frac{\pi^3}{1889} - \frac{\pi^6}{945} = 1.03692865 \ldots$$

Giving a difference of 8.954…10^{-7} with OEIS [**12**] value: $\zeta(5) = 1.03692775 \ldots$

3.3 A general relationship for $\zeta(2k+1)$.

From above results obtained for $\zeta(3)$ and $\zeta(5)$, using the same method based on matrix M i.e. by equalling the sum of terms of rows and columns, it is self-evident that zeta function for odd integers takes the following form:

$$\forall k \geq 1 \quad \zeta(2k+1) = (2k+2) - \sum_{i=2}^{+\infty} \frac{1}{i^{2k+2}(i-1)} - \sum_{\substack{i=2 \\ i \neq 2k+1}}^{2k+2} \zeta(i)$$

Values $k = 1$ and $k = 2$ giving respectively $\zeta(3)$ and $\zeta(5)$ as demonstrated above and for $k = 3$ we have:

$$\zeta(7) = 8 - \sum_{i=2}^{+\infty} \frac{1}{i^8(i-1)} - \big(\zeta(2) + \zeta(3) + \zeta(4) + \zeta(5) + \zeta(6) + \zeta(8)\big)$$

The result will always depend on the series:

$$\sum_{i=2}^{+\infty} \frac{1}{i^{2k+2}(i-1)}$$

But we showed that these series converge rapidly to a number we called λ and μ for respectively $\zeta(3)$ and $\zeta(5)$. Moreover, we found that in that cases, λ and μ can be approximated by a power of π with an excellent accuracy.

Of course, results are based on the matrix representation of zeta function which allowed to combine properties of rows and columns to find a relationship between zeta function and geometric series. This approach remains in the same way of thinking than Euler original work for $\zeta(2k)$ determination.

4. DISCUSSION OF THE RESULTS.

4.1 Some uses of $\zeta(2k+1)$ results.

Knowledge of zeta function for even and odd integers allows simple quantities calculations. Using approximative analytical forms for $\zeta(2k+1)$, we can calculate the following quantities:

$$\zeta(2) + \zeta(3) = 4 - \frac{9\pi}{400} - \frac{\pi^4}{90} = 2.846990 \ldots$$

$$\zeta(2) + \zeta(3) + \zeta(4) = 4 - \frac{9\pi}{400} = 3.929314 \ldots$$

$$\zeta(2) + \zeta(3) + \zeta(4) + \zeta(5) = 6 - \frac{\pi^3}{1889} - \frac{\pi^6}{945} = 4.966242 \ldots$$

$$\zeta(2)+\zeta(3)+\zeta(4)+\zeta(5)+\zeta(6)=6-\frac{\pi^3}{1889}=5.983585\ldots$$

$$\zeta(2)+\zeta(3)+\zeta(4)+\zeta(5)+\zeta(6)+\zeta(7)=8-\frac{\pi^5}{76724}-\frac{\pi^8}{9450}=6.991934\ldots$$

$$\zeta(2)+\zeta(3)+\zeta(4)+\zeta(5)+\zeta(6)+\zeta(7)+\zeta(8)=8-\frac{\pi^5}{76724}=7.996011\ldots$$

$$\zeta(2)+\zeta(3)+\zeta(4)+\zeta(5)+\zeta(6)+\zeta(7)+\zeta(8)+\zeta(9)=10-\frac{\pi^7}{3040000}-\frac{\pi^{10}}{93555}$$
$$=8.998019$$

$$\zeta(2)+\zeta(3)+\zeta(4)+\zeta(5)+\zeta(6)+\zeta(7)+\zeta(8)+\zeta(9)+\zeta(10)=10-\frac{\pi^7}{3040000}$$
$$=9.999006\ldots$$

These calculations which can easily be continued clearly show that:

$$\lim_{N\to+\infty}\sum_{s=2}^{N}\zeta(s)=N$$

This result agrees with well-known result given in chapter 2.1:

$$\sum_{s=1}^{+\infty}\zeta(s)=+\infty$$

Removing then all values of 1 in each $\zeta(s)$, we also have:

$$\lim_{N\to+\infty}\sum_{s=2}^{N}\sum_{k=2}^{+\infty}\frac{1}{k^s}=1$$

Giving an important result for matrix M analysis.

It is also interesting to consider ratios of $\zeta(s)$ and $\zeta(s+1)$ and their relationship to Euler product formula given in the introduction of this paper. Values of $\zeta(s)$ are ordered because ζ is a strictly decreasing function of integers s. For $s=2,3,4,5,\ldots$ we always have $\zeta(s)>\zeta(s+1)$ and $\zeta(2)=\frac{\pi^2}{6}=1.644934\ldots$ is the highest value.

Let us consider the first ratio: $\zeta(2)/\zeta(3)$, we can write, using Euler product formula:

$$\frac{\zeta(2)}{\zeta(3)}=\frac{\prod_p\frac{p^2}{p^2-1}}{\prod_p\frac{p^3}{p^3-1}}$$

With p for the product over all prime numbers.

$$\Rightarrow \frac{\zeta(2)}{\zeta(3)} = \prod_p \frac{(p-1).(p^2+p+1)}{p.(p-1).(p+1)} = \prod_p \frac{(p^2+p)+1}{(p^2+p)}$$

The same approach for ratios $\zeta(3)/\zeta(4)$ and $\zeta(4)/\zeta(5)$ gives:

$$\frac{\zeta(3)}{\zeta(4)} = \prod_p \frac{(p^3+p^2+p)+1}{(p^3+p^2+p)}$$

$$\frac{\zeta(4)}{\zeta(5)} = \prod_p \frac{(p^4+p^3+p^2+p)+1}{(p^4+p^3+p^2+p)}$$

We obtain then $\forall\ s \geq 2$:

$$\frac{\zeta(s)}{\zeta(s+1)} = \prod_p \left\{1 + \frac{1}{p+p^2+p^3+\cdots+p^s}\right\}$$

When the value of p increases, the quantity in brackets aims towards 1 giving for ratio $\zeta(s)/\zeta(s+1)$ an expression having the following form:

$$\frac{\zeta(s)}{\zeta(s+1)} = a.b.c \ldots 1.1.1 \ldots$$

With

$$a = \left\{1 + \frac{1}{2+2^2+2^3+\cdots+2^s}\right\}$$

$$b = \left\{1 + \frac{1}{3+3^2+3^3+\cdots+3^s}\right\} \ldots$$

And as s value increases, quantities $a, b, c, \ldots$ aims also towards 1 giving the important following result:

$$\lim_{s\to+\infty} \frac{\zeta(s)}{\zeta(s+1)} = 1$$

Considering matrix $M.M^T$ introduced in chapter 2, it means that values of $\zeta(s)$ does not change when a sufficiently high value of s is reached.

As an example, we numerically obtained:

$$\frac{\zeta(4)}{\zeta(5)} = 1.043778 \ldots$$

With,

$$\left\{1 + \frac{1}{29+29^2+29^3+29^4}\right\} = 1.000001 \ldots$$

This result shows the rapid convergence of ratio $\zeta(s)/\zeta(s+1)$. Considering that matrix $M.M^T$ is of Hankel type, ascending skew-diagonals will rapidly become identical.

4.2 Prime numbers distribution.

As proposed in chapter 2.2, we will study matrix $M^T.M$ presented in Figure 6 more deeply. As previously showed, we have for each term b_{ij}:

$$b_{ij} = \frac{1}{ij-1}$$

Matrix is symmetric and all terms are the reverse of an integer fully determined by row and column numbers. An interesting thing is that all prime numbers appear in the denominator of matrix terms. For example, we have:

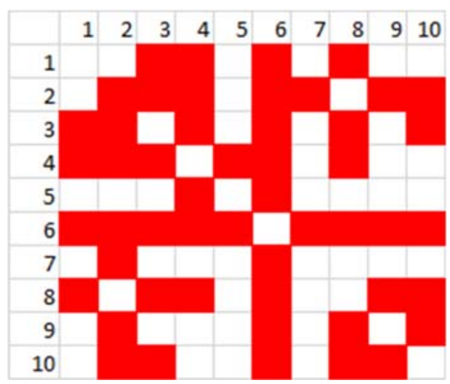
- For row number 2: $\frac{1}{1};\frac{1}{3};\frac{1}{5};\frac{1}{7};\frac{1}{9};\frac{1}{11};\frac{1}{13};\frac{1}{17};\ldots$

Gives prime numbers: 3;5;7;11;13;17…

- For row number 3: $\frac{1}{2};\frac{1}{5};\frac{1}{8};\frac{1}{11};\frac{1}{14};\frac{1}{17};\frac{1}{20};\frac{1}{23}\ldots$

Gives prime numbers: 2;5;11;17;23…

The following table shows for a square 10x10 matrix the (i,j) giving a prime number (in red).



<u>Figure 7</u>: Prime numbers (red) in a 10x10 $M^T.M$ matrix.

According to definition, even rows give the highest prime numbers density, always greater than 50%, but row or column number 6 exhibits a particularly high concentration reaching 90%. Denominators obtained for this row are:

5; 11; 17; 23; 29; 35; 41; 47; 53; 59

All these numbers are prime except of course 35.

If we consider now the square 20x20 matrix $M^T.M$, we have:

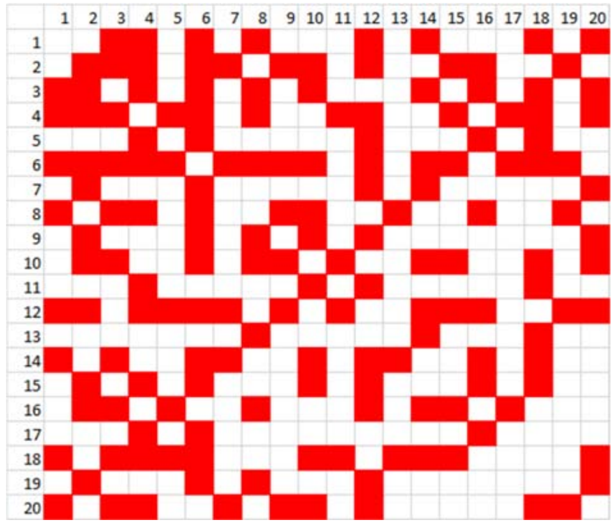


Figure 8: Prime numbers (red) in a 20x20 $M^T.M$ matrix.

This table made of 400 values clearly shows the symmetric distribution of prime numbers. Ascending skew-diagonals number 3,4,8 and 20 are only made of prime numbers. Largest one, i.e. number 20 contains prime numbers: 19; 37; 53; 67; 79; 89; 97; 103; 107; 109.

At this time, we have not found other diagonals made of only prime numbers. The highest prime number value in this table is: 379.

We can compare these tables with classical ones made of integers presented in rows. We have for the 10x10 matrix:

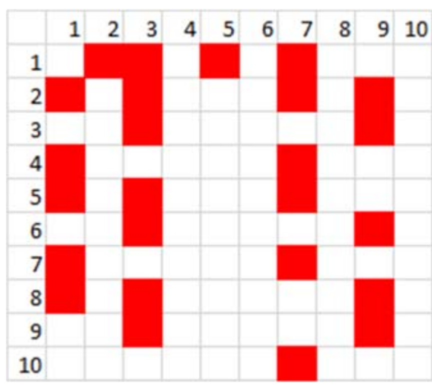


Figure 9: Prime numbers for 100 integers presented in rows.

Of course, prime numbers in table of Figure 9 are all different with a maximum of 97 but density of 25% is much lower than in Figure 7 giving 51% with a maximum value of 89. Considering the 20x20 matrices, density is 18.5% for classical table and reach 41.25% for $(ij - 1)$ representation.

This observation gave us the idea of an enrichment method for a given matrix. Let us consider row number 6 in 10x10 matrix presented in Figure 6. Each number is given by $(6j - 1)$, as showed above, for $j = 6$, we obtain 35 which is not prime. But if we add condition $(6j - 1)/5$, we obtain number 7 which is prime.

The use of the two conditions: $(6j-1)$ or $(6j-1)/5$ gives the following sixth row:

5; 11; 17; 23; 29; 7; 41; 47; 53; 59 which is complete and made of 10 different prime numbers.

If we continue sixth row with the same conditions we obtain:

5; 11; 17; 23; 29; 7; 41; 47; 53; 59; 13; 71; 77; 83; 89; 19; 101; 107; 113; 119…

It remains only two non-prime numbers: 77 and 119 giving a density of 18/20=90%.

And of course, if we add another condition with prime number 7, we obtain 100% but with repetitions for 11 and 17.

The above method illustrated through row number six case appears particularly interesting for enrichment of matrices made of terms only depending on rows and columns numbers.

Finally, to illustrate how powerful this method is, we decided to apply it to 20x20 matrix presented in Figure 8. Using twelve conditions: $(ij-1)$; $(2ij-1)$; …; $(12ij-1)$, we obtained the following enrichment of matrix $M^T.M$ in prime numbers.

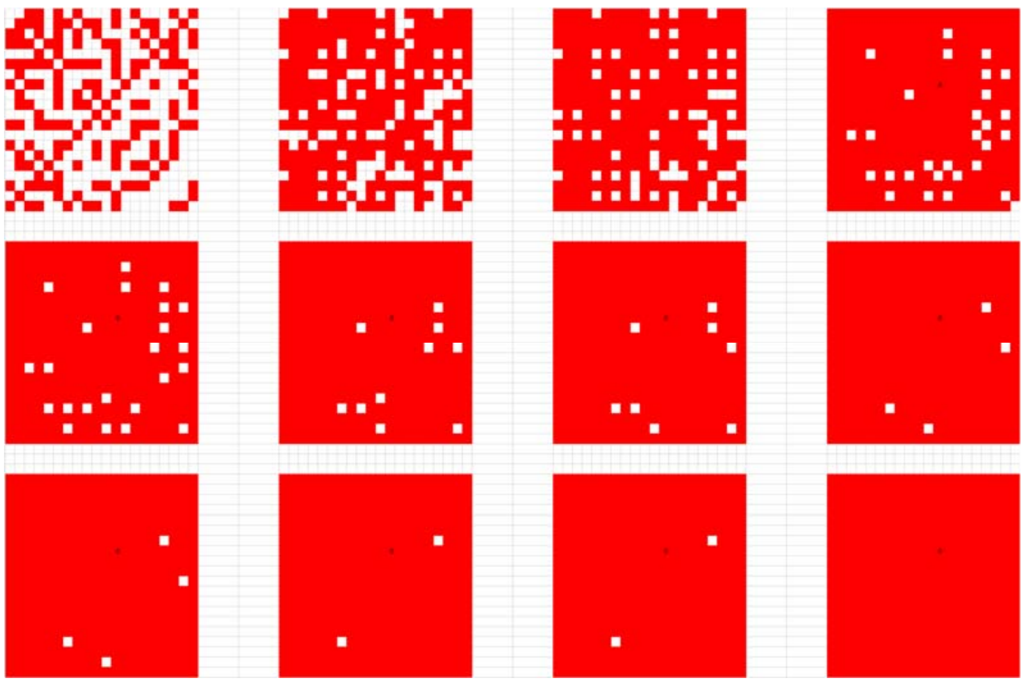


Figure 10: Enrichment process of matrix $M^T.M$.

The two last values eliminated by condition $(12ij-1)$ correspond to $(7,17)$ giving prime number $12.17.7 = 1427$.

Matrix obtained at the end this process (fully red) contains only prime numbers and could be used as a code for information technologies. Moreover, algorithm used is easy to build and to use in a computer.

Matrices having terms only depending on row and column numbers appear very interesting for prime numbers distribution study. As showed in Figure 8, matrix built with b_{ij} terms gives an interesting prime numbers density. If we consider Hilbert matrix, prime numbers distribution for 20x20 square table gives the following pattern:

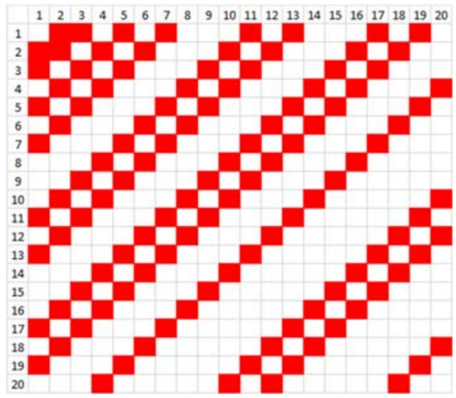


Figure 11: Prime numbers distribution in Hilbert matrix.

Skew-ascending diagonals only correspond to prime numbers giving: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37. All other numbers in this matrix are not prime. Patterns given by Hilbert matrices appear then of major interest for prime numbers distribution study. Moreover, because prime numbers appear well ordered in Hilbert matrices, patterns recognition approaches using images processing methods could be very interesting to apply. This type of method using a camera could also be useful for data encoding.

5.CONCLUSION.

In this paper, we proposed a matrix approach to study Euler-Riemann zeta function. By considering that the sum of numbers in rows and columns must be equal, we proposed a simple method available to calculate zeta function values for odd integers i.e. $\zeta(2k+1)$.

From this result, important consequences for the sums and ratios of zeta functions were deduced.

Finally, this approach produced matrix $M^T.M$ analogous to Hilbert matrix i.e. with terms only depending on rows and columns numbers. Considering this matrix as a way to represent prime numbers distributions, we found an enrichment method able to give a table only containing prime numbers.

This approach could be useful for encoding confidential information in computing technologies.

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