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TWO CYLINDRICAL SHELL TYPE CRACKS OPENED BY BODY FORCES IN A THICK PLATE

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ABSTRACT

In this paper we have analysed two shell discontinuities in a plate of thickness $2h$ and radius infinite. The closed form and stress intensity factor are obtained by using Hankel and Fourier transforms. Point body forces are also discussed as a special case. Normal stress, graphical crack shape and solution of Fredholm integral equation are obtained.

Keywords: Stress intensity factor(SIF),body forces(rivets or stiffeners),point force.

1. INTRODUCTION

In the paper we considered a crack (cylindrical shell type crack) opened by body forces. It is very obvious to know the effects caused over physical quantities due to single discontinuity due to the presence of another similar discontinuity. The physical problem is translated to mathematical language in the following way. Discontinuities occupy the region, $b<|z|<c$, with $r=a$ and $0\le\theta\le 2\pi$ see figure 1. Then we take the cross-section by $\theta=0,\ \pi$ see figure 2 and the following boundary conditions in two dimensions.

$$\sigma_{rz}(r,\pm h)=w(r,\pm h)=0,\qquad 0\le|r|<\infty \tag{1.1}$$

$$\sigma_{rz}(a,z)=0=0\le|z|\le h \tag{1.2}$$

with the mixed-boundary condition

$$U(a,z)=0,\ \ 0\le|z|\le b,\ c\le|z|\le h \tag{1.3}$$

$$\sigma_{rr}(a,z)=0,\quad b<|z|<c, \tag{1.4}$$

There are two cracks at distance $r=a$ of lengths (c – b) each. The symmetry of geometry and the presence of symmetric body forces make the boundary conditions (1) – (4) simple. These are now,

$$\sigma_{rz}(r,h)=w(r,h)=0,\ \ 0\le r<\infty, \tag{1.5}$$

$$\sigma_{rz}(a,z)=0,\ \ 0\le z\le h, \tag{1.6}$$

$$U(a,z)=0,\ \ 0\le z\le b,\ \ c\le z\le h, \tag{1.7}$$

$$\sigma_{rr}(a,z)=0,\ b<z<c, \tag{1.8}$$

we checked throughout, see [1],

$$U(a,z) > 0, \quad b < |z| < c \tag{1.9}$$

which means that cracks really open out and crack faces do not meet each other than at crack tips. The assumptions of previous chapter are also used here. The plan of chapter is as follows :

In next section we shall formulate elasticity problem and reduce to triple series equation whose solution will also be given. Section 3 will give physical quantities. The solution of Fredholm integral equation for special point force will be gives in section 4.

2. Formulation, Reduction And Solution Of Triple Series Equations.

FORMULATION

Here also we divide the problem into two namely. Body Force Problem. (It is given in chapter VI) and the Elasticity Problem.

Elasticity Problem— We solve the equations of equilibrium, in the absence of body forces, by using the solution, $U^{(e)}(r,z),$ and $W^{(e)}(r,z),$ given as,

$$U^{e}(r,z) = \frac{1}{2}U_c^{(e)}(r,0) + \frac{2}{h}\sum_{m=1}^{\infty}\cos(\beta_m z)U_c^{(e)}(r,\beta_m), \tag{2.1}$$

while $U_c^{(e)}(r,\beta_m)$ is given by (7.2.6).

$$W^{(e)}(r,z) = \frac{2}{Ph}\sum_{m=1}^{\infty}\sin(\beta_m z)[A(m)F_5(r) + B(m)F_6(r)] \tag{2.2}$$

$$\left.\begin{aligned} F_5(r) &= (\lambda+2\mu)F_3(r) + F_1(r), \\ F_6(r) &= \mu F_2(r) + F_4(r) \\ F_1(r) &= \int I_1(\beta_m r)dr, \\ F_2(r) &= \int rI_0(\beta_m r)dr \end{aligned}\right\} \tag{2.3}$$

while $F_3(r)$ and $F_4(r)$ are given by (7.2.12)

$$\sigma_{rz}^{(e)}(r,z) = -\frac{2\mu}{Ph}\sum_{m=1}^{\infty}\beta_m\sin(\beta_m z)\left[(P+K)A(m)I_1(\beta_m r) + \frac{B(m)}{\beta_m}\left\langle I_0(\beta_m r)(2P+\beta_m)\beta_m + (2+r\beta_m)I_1(\beta_m r)\right\rangle\right], \tag{2.4}$$

The solutions (2.2) and (2.4) satisfy (1.5) identically and (2.4) along with (1.6) gives

$$a_1 A(m) + a_2 B(m) = \frac{Ph}{2\mu\beta_m}\sigma_{rzs}^{(b)}(a,\beta_m) \tag{2.4}$$

where a_1 and a_2 are known functions given by

$$a_1 = (P+Q)I_1(a\beta_m)$$
$$a_2 = I_0(a\beta_m)\langle 2P+\beta_m\rangle + (2+a\beta_m)I_1(a\beta_m)/\beta_m$$

The mixed boundary conditions (1.7) - (1.8) give the following triple series relations

$$\frac{U_0}{2}+\sum_{m=1}^{\infty}\cos(\beta_m 2)a_3(m)B(m)=-\frac{h}{2}U^{(b)}_{(a,z)},0\le z\le b,\ c\le z\le h, \tag{2.5}$$

$$\sum_{m=1}^{\infty}\beta_m\beta(m)a_6(m)\cos(\beta_m z)=-\frac{h}{2}\sigma^{(b)}_{rr}(a,z),b<z<c, \tag{2.6}$$

Where U_0, a_3, a_6 etc. are given by

$$a_3(m)=aI_0(a\beta_m)+\frac{I_1(a\beta_m)}{a_1}\left\{-a_2+\frac{Ph}{2\mu\beta_m}-\sigma^b_{rzs}(a,\beta_m)\right\}$$

$$U_0=\frac{ah}{2}.B(0)$$

$$a_4(m)=\alpha_0 I_0(a\beta_m)+\frac{I_1(a\beta_m)}{a\beta_m}$$

$$a_5(m)=\alpha_1 I_0(a\beta_m)+I_1(a\beta_m)\langle 1+a\beta_m(1+Q)\rangle$$

$$a_6(m)=\frac{a_4}{a_1}\left\langle -a_2+\frac{Ph}{2\mu\beta_m}-\sigma^b_{rzs}(a,\beta_m)\right\rangle+a_5$$

The relations (2.5) - (2.6) are called triple series equation.

SOLUTION OF TRIPLE SERIES

After taking,

$$a_3(m)\ B(m)=\phi(m) \tag{2.7}$$

and the trial solution as,

$$\beta_m\phi(m)=2\left[\int_b^c g(t)\sin(\beta_m t)dt-\frac{1}{h}\left\langle\int_0^b+\int_c^h\right\rangle U_1'(t)\sin\beta_m dt\right] \tag{2.8}$$

$$\phi_0=2\left[\int_b^c t\,g(t)dt-\frac{1}{h}\left\langle\int_0^b U_1'(t)\,dt+\int_c^h t\ U_1'(t)dt\right\rangle\right]+U_1(h) \tag{2.9}$$

The substitution of (2.7) - (2.9) into (2.5) satisfy it identically if

$$\int_b^c g(t)dt=\left[U_1(b)-U_1(c)\right]/h, \tag{2.10}$$

where,

$$U_1(z)=-\frac{h}{2}U^{(b)}(a,z) \tag{2.11}$$

And the substitution of (2.7) - (2.8) into (2.6) and then inverting

$$g(t)=\frac{1}{h^2\delta(t)}\left[\Delta_0(t)t\int_b^c g(t)K(\alpha,t)dx\right], b<t<c, \tag{2.12)a}$$

$$\delta(t)=\left[\left|G(b,t)G(t,c)\right|\right]^{1/2} \tag{2.12)b}$$

$$\Delta_0(t)=-\int_b^c \frac{\sin(qz)\left\{\sigma_{rr}^{(b)}(a,z)-U_0(z)\right\}\delta(z)}{G(t,z)}dz+D \tag{2.13}$$

$$K(\alpha,t)=\int_b^c \frac{\sin(qz)\delta(z)M(\alpha,z)}{G(t,z)}dz, \tag{2.14}$$

Where $M(\alpha,z)$ is defined as

$$M(\alpha,z)=\sum_{m=1}^{\infty}\cos(\beta_m z)\sin(\alpha\beta_m)a_7(m) \quad where\, a_7(m)=\frac{a_5(m)-a_6(m)}{a_5(m)} \quad \text{and}$$

$$U_0(z)=\left(\int_0^b+\int_c^h\right)U_1^1(\alpha)M(\alpha,z)d\alpha+\left(\int_0^b+\int_c^h\right)U_1^1(\alpha)\frac{\sin(q\alpha)d\alpha}{G(\alpha,z)} \tag{2.15}$$

The equation (2.12) is called Fredholm Integral equation of second kind. D in (2.13) is an airbitrary constant which will be determined through (2.10).

3 Physical Quantities

The quantities of interest in fracture mechanics are fracture-design parameters. Two important fracture design paremeters are stress intensity-factor and crack shape.

STRESS-INTENSITY FACTORS

The stress-intensity factors at crack tips are defined as,

$$K_b=\lim_{z\to b^-}\sqrt{b-z}\,\sigma_{rr}^{(e)}(a,z), \tag{3.1}$$

$$K_c=\lim_{z\to c^+}\sqrt{z-c}\,\sigma_{rr}^{(e)}(a,z), \tag{3.2}$$

$$N_b=\lim_{z\to b^-}\sqrt{b-z}\,\sigma_{rz}^{(e)}(a,z)=0, \tag{3.3}$$

$$N_c=\lim_{z\to c^+}\sqrt{z-c}\,\sigma_{rz}^{(e)}(a,z)=0, \tag{3.4}$$

N_b and N_c are zero because in our paper we took $\sigma_{rz}(a,z)=0$. But $\sigma_{rz}^{(b)}(a,z)$ does not possess singularity at crack tips therefore N_b, N_c will be zero.

NORMAL STRESS

Normal stress, $\sigma_{rz}^{(b)}(a,z)$, in the vicinity of crack tips, (a,b), (a,c) is evaluated through the value of series on lefthand side of (2.6) while, $0\le z<b, c<z\le h$,

$$\sigma_{rr}^{(e)}(a,z)=\int_b^c \frac{g(t)\sin(qt)}{G(t,z)}dz+\Delta_1(z), \tag{3.5}$$

$$\Delta_1(z)=\int_b^c g(\alpha)M(\alpha,z)d\alpha-\frac{1}{h}\left(\int_0^b+\int_c^h\right)U_1'(\alpha)\left\{M(\alpha,z)+\frac{\sin(q\alpha)}{G(\alpha,z)}\right\}d\alpha, \tag{3.6}$$

Now, we substitute the value of $g(t)$ from (2.12) into first part of right hand side of (3.5) and evaluate the integrals,

$$\sigma_{rr}^{(e)}(a,z)=\frac{1}{\pi h}\begin{cases}\Delta_2(z)+\Delta_1(z),\\ -\Delta_2(z)+\Delta_1(z)\end{cases} \tag{3.7}$$

$$\Delta_2(z)=\frac{\sin(qz)}{\delta(z)}\left[\Delta_0(z)+\int_b^c g(\alpha)K(\alpha,z)d\alpha\right], \tag{3.8}$$

Now using (3.7) into (3.1) - (3.2) and evaluate the limit,

$$\left.\begin{aligned}K_b&=\frac{\Delta_3(b)}{\pi h},\\ K_c&=-\frac{\Delta_3(c)}{\pi h},\end{aligned}\right\} \tag{3.9}$$

$$\Delta_3(x)=\psi_0(x)\left[\Delta_0(x)+\int_b^c g(\alpha)K(\alpha,x)d\alpha\right], \tag{3.10}$$

$$\psi_0(x)=\left[q\sin(qx)G(b,c)\right]^{-1/2}\ x=b,c,q=\frac{\pi}{h} \tag{3.11}$$

CRACK SHAPE

Crack shape is the graph of crack opening displacement $U^{(e)}(a,z),\ b<z<c.$

$$U^{(e)}(a,z)=h\int_z^c g(t)dt+U_1(c),\ b\le z\le c, \tag{3.12}$$

Thus knowing $g(t)$ we can plot $U^{(e)}(a,z)$ graphically.

4. Solution of Fredholm Integral Equation

The numerical solution of Fredholm integral equation will be given for a special type of point force see figure 3, 4. Before we go for solution of Fredholm integral equation we evaluate the constant D. To evaluate constant D we take,

$$g(t)=\frac{1}{h^2\delta(t)}\Delta_0(t) \tag{4.1}$$

Then we use (2.10). The double singular integral is to be evaluated numerically.

$$D=\frac{\sin(qc/2)}{F\left(\frac{\pi}{2},\mu\right)}\left[\pi h\left\langle U_1(b)-U_1(c)\right\rangle-\frac{\left(\Delta_4(t)+\Delta_5(t)\right)}{h}\right], \tag{4.3}$$

$$\mu^2 = \frac{G(b,c)}{G(0,c)} \tag{4.4}$$

$$\Delta_3(t) = \int_b^c \frac{\sin(qz)\delta(z)\sigma_{rr}^{(b)}(a,z)}{G(z,t)} dz \tag{4.5}$$

$$\left.\begin{aligned} \Delta_4(t) &= \int_0^b U_1'(\alpha)\sin(q\alpha)\left[F_1 - \frac{h}{2}\frac{\delta(\alpha)}{G(\alpha,t)}\right]d\alpha \\ \Delta_5(t) &= \int_c^h U_1'(\alpha)\sin(qh)\left[F_1 + \frac{h}{2}\frac{\delta(\alpha)}{G(\alpha,t)}\right]d\alpha \\ F_1 &= F\left(\frac{\pi}{2},\mu\right)/\sin(qc/2) \end{aligned}\right\} \tag{4.6}$$

In $\Delta_4(t)$ $\alpha < t$ which gives $\cos(q\alpha) > \cos qt$, therefore,

$$\frac{1}{\cos(q\alpha)-\cos(qt)} = +\frac{1}{\cos(q\alpha)}\sum_{l=0}^{\infty}\left\langle\frac{\cos(qt)}{\cos(q\alpha)}\right\rangle^l, \tag{4.7}$$

And, $\alpha > t \Rightarrow \cos(q\alpha) < \cos(qt)$, then,

$$\frac{1}{\cos(q\alpha)-\cos(qt)} = -\frac{1}{\cos(qt)}\sum_{n=0}^{\infty}\left\langle\frac{\cos(q\alpha)}{\cos(qt)}\right\rangle^n, \tag{4.8}$$

Then,

$$\int_b^c \frac{dt}{\delta(t)}\Delta_4(t) = F_1^2\int_0^b U_1'(\alpha)\sin(q\alpha)d\alpha - \frac{h}{2}\sum_{l=0}^{\infty}\int_b^c \frac{\cos^l(qt)}{\delta(t)}dt\int_0^b u_1^1 \frac{\sin(q\alpha).\delta(\alpha)}{\cos^{l+1}(q\alpha)}d\alpha \tag{4.9}$$

And,

$$\int_b^c \frac{\Delta_5(t)dt}{\delta(t)} = F_1^2\int_c^h U_1'(\alpha)\sin(q\alpha)d\alpha + \frac{h}{2}\sum_{n=0}^{\infty}\int_b^c \frac{}{\cos(qt)\delta(t)}dt\int_c^h U_1'(\alpha)\delta(\alpha)\sin(q\alpha)\cos^l(q\alpha)\,d\alpha \tag{4.10}$$

Non of the integral is singular. Now for $\Delta_3(t): z > t, \cos(qz) < \cos(qt)$

$$\int_b^c \frac{dt}{\delta(t)}\int_b^c \frac{\sin(qz)\delta(z)\sigma_{rr}^{(b)}(a,z)}{G(z,t)}dz = F_1\int_b^c \frac{\sin(qz)G(b,z)}{\delta(z)}\sigma_{rr}^{(b)}(a,z)d_2 - E\int_b^c \frac{\sin(qz)\sigma_{rr}^{(b)}(a,z)}{\delta(z)} + E_2 \tag{4.11}$$

Where $E = E(\pi/2,\mu)$, complete elliptic integral of second type.

$$E_2 = \sum_{n=0}^{\infty}\int_b^c \frac{\delta(t).dt}{\cos(qt)^{n+1}}\int_b^c \frac{\sin(qz).\sigma_{rr}^{(b)}(a,z)\cos^n(qz)}{\delta(z)}dz \tag{4.12}$$

If $z < t$, then in (4.11) replace E_2 by E_2 and given as,

$$E_2^1 = -\sum_{l=0}^{\infty}\int_b^c \delta(t)\cos^l(qt)dt\int_b^c \frac{\sin(qz)\sigma_{rr}^{(b)}(a,z)d_2}{\delta(z)\langle\cos(qz)\rangle^{l+1}} \tag{4.13}$$

Thus, the evaluation of D does not involve any singular integral. The same logic we shall apply when we will be solving Fredholm integral equation numerically. The quantities $U^{(b)}(a,z), \sigma_{rz}^{(b)}(a,z)$ and $\sigma_{rr}^{(b)}(a,z)$ are given by (7.5.1), (7.5.3) and (7.5.5), respectively. $M(t,z)$ is given by (7.5.8) which will be used.

$$U^{(b)}(a,z) = \frac{aP_0 d_3(0)}{2\mu h^2}\left[\frac{4h^2+z^2}{\left(4h^2-z^2\right)^2} + \frac{16h^2+z^2}{\left(16h^2-z^2\right)^2} - \left\langle \frac{16h^4+24h^2z^2+z^4}{\left(4h^2-z^2\right)^4} + \frac{256h^4+96h^2z^2+z^4}{\left(16h^2-z^2\right)^4}\right\rangle d_4\right]$$

d_1, d_2 : distance of point force from r=0.

$$d_3(k) = \frac{d_1^{\,2+2k} - d_2^{\,2+2k}}{2^{2k}k!\left(\mathrm{K}+1\right)!}$$

$$d_4 = \frac{3}{2}\left(a^2 + \frac{d_1^{\,2} - d_2^{\,2}}{16}\right)$$

$$d_5 = \left(\lambda + 2\mu\right)d_3(0)$$

$$d_6 = \frac{3}{16}\left[32\left(\lambda+\mu\right) + \left(\lambda+2\mu\right)\left(d_1^{\,2} + d_2^{\,2}\right)\right]$$

$$d_7 = 12a^2\left(\lambda+2\mu\right)d_3(1)$$

$$d_8 = \frac{a}{8}\mu_1\left(d_3(0) - \frac{d_3(1)}{16}\right)$$

$$d_9 = \frac{a}{16}\left(\frac{\pi\mu_2}{h} - \frac{d_3(1)}{16}\right)$$

$$d_{10} = \frac{\pi a \mu_2 d_3(1)}{256h}$$

$$d_{11} = \left(1+10a^2\right)\frac{\pi a}{16h}\left(\lambda+\mu\right)d_3(1)$$

$$\sigma_{rz}^{(b)}(a,z) = \frac{aP_0}{2\mu h\left(\lambda+2\mu\right)}\left[d_5\left\langle\frac{4h^2+z^2}{\left(4h^2-z^2\right)^2} + \frac{16h^2+z^2}{\left(16h^2-z^2\right)^2}\right\rangle - \right.$$

$$d_6\left\langle\frac{16h^4+24h^2z^2+z^4}{\left(4h^2-z^2\right)^4} + \frac{256h^4+96h^2z^2+z^4}{\left(16h^2-z^2\right)^4}\right\rangle -$$

$$\left. d_7\left\langle\frac{6h^3z+hz^2}{\left(4h^2-z^2\right)^4} + \frac{48h^3z+2hz^3}{\left(16h^2-z^2\right)^4}\right\rangle\right]$$

$$\sigma_{rr}^{(b)}(a,z) = -\frac{2P_0}{\mu h(\lambda+2\mu)}\Big[d_8 + d_8\theta_1(h,z)$$
$$-d_{10}\langle\theta_2(h,z)+\theta_3(h,z)\rangle - d_{11}\theta_4(h,z)\Big]$$

The numerical evaluation of singular integral

$$\int_b^c \frac{\sin(qz)F(z)d_2}{\delta(z)G(z,t)} = \sum_{n=0}^{\infty}\left[\int_b^t \frac{\cos^n(qt)\sin(qz)F(z)dz}{\delta(z)\cos(qz)^{n+1}}\int_t^c \frac{\cos^n(qz)\sin(qz)F(z)dz}{\delta(z).\cos(qt)^{n+1}}\right]$$

Now we substitute,

$$2\sin^2(qz/2) = 2\sin^2(qc/2) - G(b,c)\sin^2\theta \tag{4.14}$$

Then

$$\int_b^c \frac{\sin(qz)F(z)dz}{\delta(z)\,G(z,t)} = \sum_{n=0}^{\infty}\left[\cos^n(qt)\int_{\theta_1}^{\pi/2}\frac{\phi(\theta)d\theta}{\langle\phi_1(\theta)\rangle^{n+1}} - \frac{1}{\cos^{n+1}(qt)}\int_0^{\theta_1}\phi(\theta)\langle\phi_1(\theta)\rangle^n\,d\theta\right], \tag{4.15}$$

Where $F(z)$ changes to $\phi(\theta)$ after using (4.14) and

$$\phi_1(\theta) = \cos(qc) + G(b,c)\sin^2\theta \tag{4.16}$$

None of the integrals is singular, there fore can easily be evaluated numerically. The Fredholm integral equation (2.12) will reduce to n linear algebraic equations in n variables which are n values of $g(t)$, $i = 1,\ldots,n$, giving, $g_i = g(t_i)$ *i.e., n* g_i's.

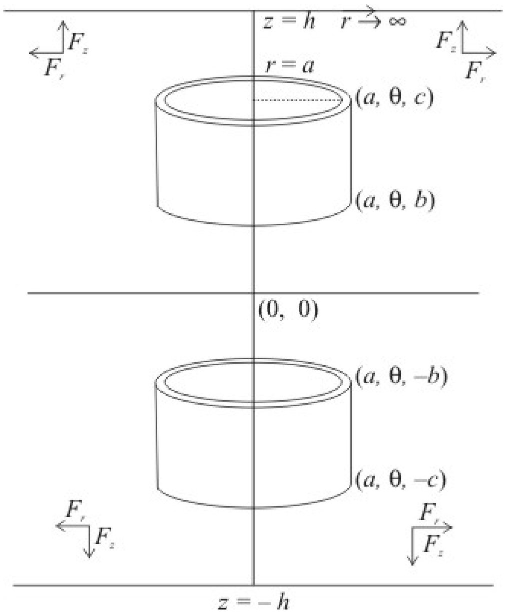

Fig-1. Geometry of two shell type discontinuities in a plate of thickness 2*h* and radius Infinite.

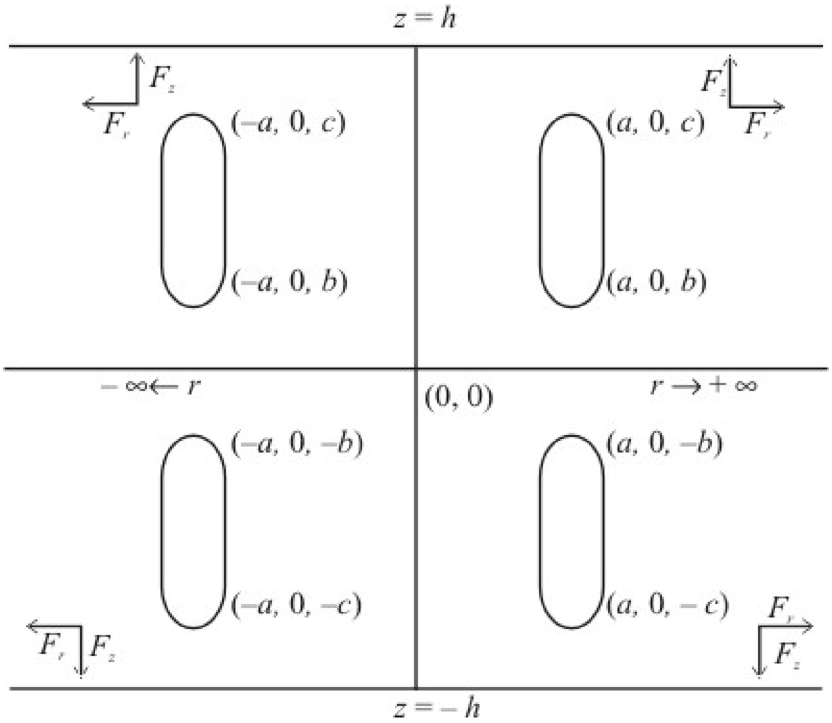

Fig.2. : Cross Section of Figure 1 by $\theta = 0, \pi$ **.**

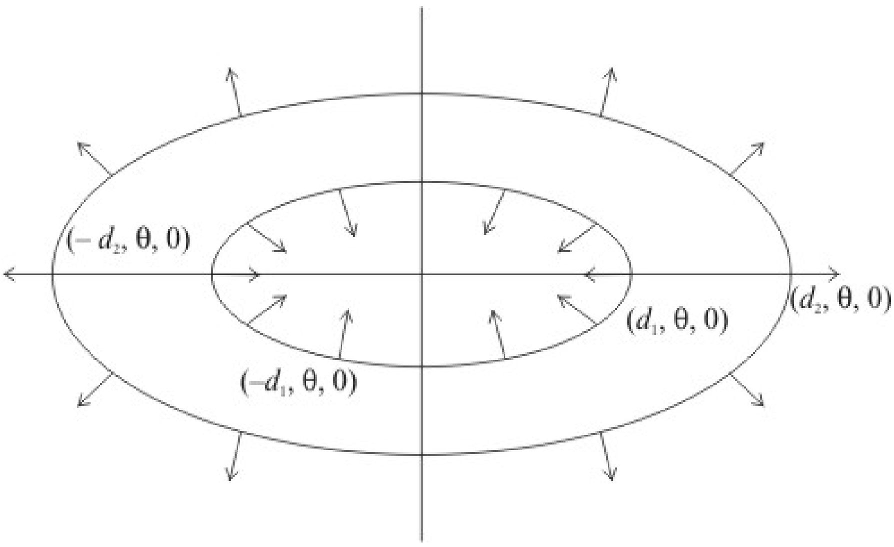

Fig. 3. Point forces in ring shape which are of opening mode.

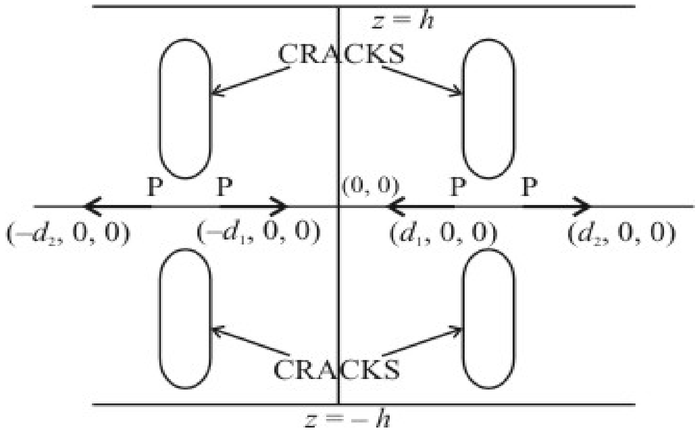

Fig. 4. Point forces with crack in two dimension.

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