A Rapidly Convergent Approximation Method for Nonlinear Ordinary Differential Equations

Prakash Kumar Das, M.M. Panja[1]
Department of Mathematics, Visva-Bharati, Santiniketan 731 235, West Bengal, India
Prakashdas.das1@gmail.com, madanpanja2005@yahoo.co.in
[1]Corresponding author

Abstract

This paper deals with an approximation scheme for getting rapidly convergent approximate solution of some nonlinear ordinary or partial differential equations appearing in the mathematical analysis of physical processes. Here the solutions are expressed in a series of exponential functions rather than the series of the independent variables. As a result the vanishing boundary conditions for localized solutions can be incorporated into the theory in rather straight forward way. For many cases, sum of the terms in the approximate solution can be found easily, thus provides exact analytic solution to the problem. To illustrate the efficiency and versatility of the method, proposed schemes developed here for bounded and unbounded region have been applied to some problems and have been compared with solutions obtained by some other approximation method now available. It appears that the present method is efficient, user-friendly and easy to implement through any symbolic computation software now readily available.

Keywords: Nonlinear ordinary/partial differential equation, travelling wave solution, non-smooth solution, rapidly convergent approximate solution.

1. Introduction

In many branches of science and engineering evolution of physical processes are described by nonlinear ordinary or partial differential equations (ODEs/PDEs). The solutions of such equations helps one visualize the nature of evolution of the processes involved. But in many cases it is not possible to find exact solution to these equations. Despite that there exists a vast amount of literature for constructing exact solutions of ODEs and PDEs [1, 2, 3, 4]. Also there are many approximation methods [5, 6, 7, 8, 9] to deal with equations for which the analytical methods do not work. In any such method the main interest is to develop a scheme that yields a physically meaningful solution with minimum computational effort. The effort expended is assessed in terms of the ease of implementation and computer resources required.

Among various approximation techniques for solving nonlinear boundary value problems(BVP), the Adomian decomposition method often abbreviated as ADM enjoys a great deal of popularity because of its conceptual simplicity and usefulness in dealing with nonlinear problems [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. In the traditional formulation of the ADM, approximate solution of any given nonlinear differential equation is expressed as a sum of functions with the leading term as a polynomial in the independent variable of the equation. These functions are often found to be slowly convergent. It appears that there is still another inherent limitation of the approach. For example, it is not straightforward to incorporate the infinite domain boundary conditions in solving the nonlinear equation by the ADM. To accommodate the boundary condition at $\pm\infty$ one needs to take recourse to the use of rational Padé approximants to express the successive terms in the approximate solution. Naturally, the question arises as to whether one can look for an improved form of the ADM which, on the one hand, will improve on the convergence of the series representing the approximate solution and, on the other hand, will incorporate the infinite domain boundary conditions of the problem. The present work is an effort in this direction.

We have, in this work derived a recursive scheme for solving nonlinear boundary-value problems by seeking a modification of the conventional ADM. The modification sought by us consists in introducing a more general form of the two-point boundary-value problem than that is used in the conventional ADM to develop the algorithms to solve any nonlinear differential equation. Here the inverse of the linear operator associated with the nonlinear equation is given by a twofold integral operator involving exponentials of the independent variable rather than being dependent on the independent variable as in the case of conventional ADM. The exponential dependence of the inverse operator is the source of rapid convergence of the method proposed. Further this dependence also allows us to incorporate into the theory the infinite domain boundary conditions in a rather natural way.

In section 2 we present brief review for the conventional ADM with a view to gain some feeling for the working principle of the method. We then introduce in section 3 a general nonlinear differential equation for the two-point boundary value problem and make use of it to derive the algorithms for the rapidly convergent approximate solution to the problem. The results presented are very suitable to deal with problems in bounded domain. To adapt these algorithms to treat problems in unbounded domains some modification is desirable. The desired modifications have been described in section 4. We present in sec.5 a number of case studies and thereby demonstrate the efficiency of our scheme developed here. In section 6 we have summarized our outlook on the present work and try to make some concluding remarks have been presented.

2. Salient steps of ADM

For the two point boundary value problem

$$y''(x) - \lambda^2 y(x) = N[y](x) + g(x) \tag{1}$$

within $\Omega = \{x: a \le x \le b\}$ subject to Dirichlet boundary conditions (DBCs)

$$y(a) = \alpha,\ y(b) = \beta \tag{2}$$

one assumes that $N[y]$ is analytic in y and contains all nonlinear terms. The inhomogeneous or source term $g(x)$ is continuous over Ω. To find the solution of (1) and (2) by using ADM [10] one recasts (1) in operator form

$$L[y](x) = \lambda^2\ y(x) + N[y](x) + g(x) \tag{3}$$

where $L[\cdot](x) = \frac{d^2}{dx^2}[\cdot](x)$ is a second-order linear differential operator so that its inverse can be found as

$$L^{-1}[\cdot](x) = \int_a^x \int_a^{x'} [\cdot](x'')\ dx''\ dx', \qquad x \in \Omega. \tag{4}$$

Operating L^{-1} on $L[y](x)$ for $y \in C^2(\Omega)$ and following integration by parts one gets

$$L^{-1}[L[y]](x) = y(x) - y(a) - y'(a)(x-a). \tag{5}$$

Applying operator L^{-1} on both sides of (3) and using the result of (5), one gets the solution

$$y(x) = y(a) + y'(a)(x-a) + \lambda^2\ L^{-1}[y](x) + L^{-1}[N[y]](x) + L^{-1}[g](x) \tag{6}$$

which involves an unknown term $y'(a)$. To eliminate this unknown we substitute $x = b$ in Eq. (6) and solve for $y'(a)$, to get

$$y'(a) = \frac{1}{b-a}\{y(b) - y(a) - \lambda^2\ L^{-1}[y](b) - L^{-1}[N[y]](b) - L^{-1}[g](b)\}. \tag{7}$$

Substitution of $y'(a)$ into (6) gives the solution $y(x)$ involving L^{-1} operator as

$$y(x) = y(a) + \frac{1}{b-a}\{y(b) - y(a) - \lambda^2\ L^{-1}[y](b) - L^{-1}\big[N[y]\big](b) - L^{-1}[g](b)\}(x-a)$$
$$+\lambda^2\ L^{-1}[y](x) + L^{-1}[N[y]](x) + L^{-1}[g](x). \tag{8}$$

To evaluate terms involving the unknown $y(x)$ in R.H.S of (8), one writes

$$y(x) = \sum_{m=0}^{\infty} y_m(x) \tag{9}$$

and recasts the nonlinear term into

$$N[y](x) = \sum_{m=0}^{\infty} A_m(y_0(x), y_1(x), \cdots, y_m(x)). \tag{10}$$

Here the terms $A_m(x) = A_m(y_0(x), y_1(x), \cdots, y_m(x))$. are known as Adomian polynomials extracted from nonlinear term by using the formula [9, 10, 11, 12, 13, 14]

$$A_m(x) = \frac{1}{m!}[\frac{d^m}{d\varepsilon^m} N(\sum_{k=0}^{\infty} y_k \varepsilon^k)]_{\varepsilon=0}, \quad m \ge 0 \tag{11}$$

proposed by Adomian and Rach in their seminal work in Ref.9. It is important to note here that the laborious calculations for getting $A_m[N[y]](x)$ successively can be easily executed in MATHEMATICA through the following statement:

```
(* Calculating Adomian polynomial A_n(x) of (11) *)
A[n_,x_]:=Module[{NL,A},

ys[z_]:=∑_{i=0}^{n} yy[i,x]z^i;

NL[y_,yp_]:=#;

Expand[1/n! (D[NL[ys[z],D[ys[z],x]],{z,n}])/.z → 0]
]&
```

In ADM, the leading approximation has been taken as the solution of $L[y](x) = g(x)$ satisfying boundary condition (2)

$$y_0(x) = y(a) + \frac{y(b)-y(a)-L^{-1}[g](b)}{b-a}(x-a) + L^{-1}[g](x). \tag{12}$$

Here the terms involving conditions on the boundry is a first order polynomial in $(x-a)$. The successive terms in the expansion (9) for $y(x)$ can then be obtained recursively by using the formula

$$y_{n+1}(x) = \lambda^2 L^{-1}[y_n](x) - \frac{\lambda^2\ L^{-1}[y_n](b)+L^{-1}[A_n](b)}{b-a}(x-a) + L^{-1}[A_n](x), \tag{13}$$

for $n \geq 0$. It is important to mention here that whenever the domain becomes infinite i.e., $[0,\infty)$ or $(-\infty,\infty)$, $y^{'}(a)$ appearing in the formula (6) cannot be determined in a straightforward way due to presence of the factor $(x-a)$ multiplying $y^{'}(a)$. In order to resolve such problem we assemble all linear terms involved in the equation into the operator L to get a more accurate and physically meaningful leading order approximation of the solution to the problem as described in the following section.

3. Proposed scheme for bounded domain $[a,b]$

Instead of considering (1), we consider a more general equation

$$y''(x) - (\lambda_1+\lambda_2)\ y'(x) + \lambda_1\lambda_2\ y(x) = N[y](x) + g(x) \tag{14}$$

with the same boundary condition as given in (2). Instead of including the linear term $\lambda^2 y(x)$ of Eq.(1) into R.H.S in ADM, we incorporate it into the operator $O[\cdot](x) = \frac{d^2}{dx^2}[\cdot](x) - (\lambda_1+\lambda_2)\frac{d}{dx}[\cdot](x) + \lambda_1\lambda_2[\cdot](x) \equiv (\frac{d}{dx} - \lambda_2)(\frac{d}{dx} - \lambda_1)[\cdot](x)$, so that Eq.(14) can now be recast into the form

$$O[y](x) = N[y](x) + g(x), \quad a \leq x \leq b. \tag{15}$$

The linear operator $O[\cdot]$ can be written in the form

$$O[\cdot](x) = e^{\lambda_2 x}\frac{d}{dx}(e^{(\lambda_1-\lambda_2)x}\frac{d}{dx}(e^{-\lambda_1 x}[\cdot])) \tag{16}$$

which plays a key role in expressing the solution in terms of rapidly convergent series of exponentials. One may reinterpret the inverse operator O^{-1} as a twofold integral operator given by

$$O^{-1}[\cdot](x) = e^{\lambda_1 x}\int_a^x e^{(\lambda_2-\lambda_1)x'}\int_a^{x'} e^{-\lambda_2 x''}[\cdot](x'')\ dx''\ dx'. \tag{17}$$

It is important to note here that representing inverse of a linear operator with variable coefficients by integrals is also possible whenever it is factorable. That operation of O^{-1} on LHS of (14) leads to

$$O^{-1}[y''(x) - (\lambda_1+\lambda_2)y'(x) + \lambda_1\lambda_2 y(x)]$$

$$\begin{aligned} &= e^{\lambda_1 x}\int_a^x e^{(\lambda_2-\lambda_1)s}\int_a^s e^{-\lambda_2 \mathrm{t}}(y''(\mathrm{t}) - (\lambda_1+\lambda_2)y'(\mathrm{t}) + \lambda_1\lambda_2 y(\mathrm{t}))\ dt\ ds \\ &= e^{\lambda_1 x}\int_a^x \{e^{-\lambda_1 s}(y'(s) - \lambda_1 y(s)) - e^{-\lambda_2 a}e^{(\lambda_2-\lambda_1)s}(y'(a) - \lambda_1 y(b))\}ds \\ &= y(x) - y(a)e^{\lambda_1(x-a)} - \frac{y\prime(a)-\lambda\ y(a)}{\lambda_2-\lambda_1}\{e^{\lambda_2(x-a)} - e^{\lambda_1(x-a)}\}. \end{aligned} \tag{18}$$

Thus, operating O^{-1} on both sides of (15) the solutions of Eq.(14) can be written in the form

$$y(x) = y(a)e^{\lambda_1(x-a)} + \frac{y'(a)-\lambda_1 y(a)}{(\lambda_2-\lambda_1)}\left(e^{\lambda_2(x-a)} - e^{\lambda_1(x-a)}\right) + O^{-1}\big[N[y]\big](x) + O^{-1}[g](x) \tag{19}$$

which involves an unknown term $y'(a)$. Note that in contrast to Eq.(6) containing $(x-a)$ as a factor in multiplying $y^{'}(a)$ in ADM, here the coefficient becomes combination of $e^{\lambda_{1,2}(x-a)}$. To eliminate $y'(a)$ we substitute $x=b$ in Eq. (19) and get

$$\frac{y'(a)-\lambda_1\ y(a)}{\lambda_2-\lambda_1} = \frac{y(b)-y(a)e^{\lambda_1(b-a)}-O^{-1}[N[y]](b)-O^{-1}[g](b)}{e^{\lambda_2(b-a)}-e^{\lambda_1(b-a)}}. \tag{20}$$

Use of this result in (19) gives the expression for $y(x)$ involving boundary conditions and inverse operator as

$$y(x) = y_0(x) - O^{-1}[N[y]](b)\frac{e^{\lambda_2(x-a)}-e^{\lambda_1(x-a)}}{e^{\lambda_2(b-a)}-e^{\lambda_1(b-a)}} + O^{-1}[N[y]](x) + O^{-1}[g](x) \tag{21}$$

where the leading order term $y_0(x)$ is given by

$$y_0(x) = y(a)e^{\lambda_1(x-a)} + \frac{y(b)-y(a)e^{\lambda_1(b-a)}-O^{-1}[g](b)}{e^{\lambda_2(b-a)}-e^{\lambda_1(b-a)}}\{e^{\lambda_2(x-a)} - e^{\lambda_1(x-a)}\}. \tag{22}$$

One can now follow the relevant steps of ADM for evaluating terms involving nonlinear operator $N[y](x)$ and get successive corrections recursively by using

$$y_{n+1}(x) = O^{-1}[A_n](x) - \frac{O^{-1}[A_n](b)}{(e^{\lambda_2(b-a)}-e^{\lambda_1(b-a)})}\{e^{\lambda_2(x-a)} - e^{\lambda_1(x-a)}\}, \quad n \geq 0. \tag{23}$$

Whenever the roots λ_1, λ_2 are equal in magnitude but of opposite sign, the operator O and the solution become simpler. Their expressions involving the operator $O[\cdot](x) \equiv (\frac{d^2}{dx^2} - \lambda^2)[\cdot](x) = (\frac{d}{dx} - \lambda)(\frac{d}{dx} + \lambda)[\cdot](x)$ is found to be

$$y(x) = y_0(x) - \frac{O^{-1}[N[y]](b)}{e^{\lambda\, b} - e^{-\lambda\, b}}\{e^{\lambda\, x} - e^{-\lambda\, x}\} + O^{-1}[N[y]](x)(\equiv \textstyle\sum_{i=0}^{\infty} y_i(x)), \tag{24}$$

where

$$y_0(x) = y(a)\ e^{-\lambda(a-x)} + \frac{y(b) - y(a)e^{-\lambda(a-b)} - O^{-1}[g](b)}{e^{\lambda\, b} - e^{-\lambda\, b}}\{e^{\lambda\, x} - e^{-\lambda\, x}\} + O^{-1}[g](x), \tag{25}$$

$$y_{n+1}(x) = O^{-1}[A_n](x) - \frac{O^{-1}[A_n](b)}{(e^{\lambda\, b} - e^{-\lambda\, b})}(e^{\lambda\, x} - e^{-\lambda\, x}), \quad n \geq 0. \tag{26}$$

The laborious calculations for getting successive terms $y_n(x)$ by using their recursion relations (26) with (11) can be easily executed in MATHEMATICA through the following statement:

$$\mathbf{HOTermsBD[mxi\_,\ \lambda\_List, xB\_List, yB\_List] := Module[\{oinv, yset\},}$$

$$\mathbf{oinv} := \mathbf{e}^{\lambda[[1]]\mathbf{x}}\left(\int_{\mathbf{xB}[[1]]}^{\mathbf{x}} \mathbf{e}^{(\lambda[[2]]-\lambda[[1]])\mathbf{x}'}\left(\int_{\mathbf{xB}[[1]]}^{\mathbf{x}'} \mathbf{e}^{-\lambda[[2]]\mathbf{x}''}\#\mathbf{dx}''\right)\mathbf{dx}'\right)\&;$$

$$\boldsymbol{yset} = \left\{\mathbf{yB}[[1]]\mathbf{e}^{\lambda[[1]](\mathbf{x}-\mathbf{xB}[[1]])} + \frac{(\mathbf{yB}[[2]] - \mathbf{yB}[[1]]\mathbf{e}^{\lambda[[1]](\mathbf{xB}[[2]]-\mathbf{xB}[[1]])})}{(\mathbf{e}^{\lambda[[2]](\mathbf{xB}[[2]]-\mathbf{xB}[[1]])} - \mathbf{e}^{\lambda[[1]](\mathbf{xB}[[2]]-\mathbf{xB}[[1]])})}\right.$$
$$\left.(\mathbf{e}^{\lambda[[2]](\mathbf{x}-\mathbf{xB}[[1]])} - \mathbf{e}^{\lambda[[1]](\mathbf{x}-\mathbf{xB}[[1]])})\right\};$$

$$\mathbf{yy[i\_, x\_] := yset[[\boldsymbol{i}+1]];}$$

$$\mathbf{Table[AppendTo[yset, Simplify[oinv[A[n-1, x][\#]/.x \to x'']] -}$$
$$\mathbf{N}\left[\frac{(\mathbf{oinv[A[n-1,x][\#]/.x \to x''])/.x \to xB[[2]]}}{(\mathbf{e}^{\lambda[[2]](\mathbf{xB}[[2]]-\mathbf{xB}[[1]])} - \mathbf{e}^{\lambda[[1]](\mathbf{xB}[[2]]-\mathbf{xB}[[1]])})}\right]$$
$$(\mathbf{e}^{\lambda[[2]](\mathbf{x}-\mathbf{xB}[[1]])} - \mathbf{e}^{\lambda[[1]](\mathbf{x}-\mathbf{xB}[[1]])})], \{\mathbf{n, 1, mxi}\}][[\mathbf{mxi}]]$$
$$]\&$$

$$\boldsymbol{seq}[\boldsymbol{\lambda\ \_List}] := \mathbf{HOTermsBD[10,\lambda][\#]\&}$$
$$\boldsymbol{gen}[\ \boldsymbol{\lambda\_List}, \mathbf{x\_}] := \mathbf{FindSequenceFunction[seq[\lambda][\#], \boldsymbol{n}]\&}$$
$$\boldsymbol{funx}[\ \boldsymbol{\lambda\_List}, \mathbf{x\_}] := \textstyle\sum_{n=1}^{\infty} \mathbf{gen[\lambda, x][\#]\&}$$

4. Proposed scheme for unbounded domain

Whenever the domain of interest of Eq. (14) become infinite, we write the inverse operator O^{-1} as

$$O^{-1}[\cdot](x) = e^{\lambda_1 x} \int e^{(\lambda_2 - \lambda_1)x'} \int e^{-\lambda_2 x''}[\cdot](x'')\ dx''\ dx'. \tag{27}$$

In this case operation of O^{-1} on LHS of (14) gives

$$O^{-1}(y''(x) - (\lambda_1 + \lambda_2)y'(x) + \lambda_1\lambda_2 y(x)) = y(x) - \frac{c}{(\lambda_2 - \lambda_1)}\ e^{\lambda_2 x} - d\ e^{\lambda_1 x}. \tag{28}$$

Operating O^{-1} on both sides of $O[y](x) = N[y](x) + g(x)$ and using (28), we get

$$y(x) = \frac{c}{(\lambda_2 - \lambda_1)}\ e^{\lambda_2 x} + d\ e^{\lambda_1 x} + O^{-1}[N[y]](x) + O^{-1}[g](x)(\equiv \textstyle\sum_{i=0}^{\infty} y_i(x)) \tag{29}$$

involving two arbitrary constants c and d. The correction to the leading order due to presence of nonlinearities are obtained by executing steps followed in ADM such that

$$y_0(x) = \frac{c}{(\lambda_2 - \lambda_1)}\ e^{\lambda_2 x} + d\ e^{\lambda_1 x} + O^{-1}[g](x) \tag{30}$$

and

$$y_{n+1}(x) = O^{-1}[A_n(x)], \quad n \geq 0. \tag{31}$$

Note that when $\lambda_2 = -\lambda_1 = \lambda$, solution in (29) reduces to

$$y(x) = \frac{c}{2\lambda}\ e^{\lambda x} + d\ e^{-\lambda x} + O^{-1}[N[y]](x) + O^{-1}[g](x). \tag{32}$$

Consequently, the leading order and successive corrections to the solution (29) are given by

$$y_0(x) = \frac{c}{2\lambda}\ e^{\lambda x} + d\ e^{-\lambda x} + O^{-1}[g](x) \tag{33}$$

$$y_{n+1}(x) = O^{-1}[A_n](x), \quad n \geq 0. \tag{34}$$

Assuming $\lambda > 0$, use of the vanishing boundary condition $y(\infty) = 0$ in (32) for the localized solution of Eq.(24) within $[0, \infty)$ recommends $c = 0$ so that the leading and higher order correction terms of the solution are given by (34) with

$$y_0(x) = de^{-\lambda x} + O^{-1}[g](x). \tag{35}$$

For $\lambda < 0$, one has to proceed in the same way by retaining the term involving $e^{\lambda x}$.

5. Illustrative examples

In order to check the efficiency of our method presented in previous two sections, we will now apply recursion formula (25) and (26) in case of problem in finite domain and (35) and (34) for problems in infinite domain to obtain approximate solutions of some physically interesting nonlinear differential equations.

5.1 Bounded domain

Example 1. Here we consider the nonlinear BVP involving quadratic nonlinearity [10]

$$y''(x) - 12y(x) = 6(y(x)^2 + 1), \quad 0 \le x \le 1 \tag{36}$$

satisfying DBCs

$$y(0) = 0, \quad y(1) = -\frac{3}{4}. \tag{37}$$

The exact solution to this problem can be found as [10]

$$y(x) = -\frac{x(x+2)}{(1+x)^2}. \tag{38}$$

Comparing (36) with (14) one gets

$$\lambda = \sqrt{12}, \quad N[y] = 6(y^2 + 1), \quad g(x) = 0. \tag{39}$$

Using (39) in (11), (25) and (26), the leading and higher order terms for the solution $y(x)$ of Eq.(36) can be found as

$$y_0(x) = -\frac{3\left(e^{-2\sqrt{3}x} - e^{2\sqrt{3}x}\right)}{4\left(e^{-2\sqrt{3}} - e^{2\sqrt{3}}\right)}, \tag{40}$$

and

$$y_i(x) = 6\, O^{-1}[A_{i-1}](x) - 6\frac{(e^{\sqrt{12}x} - e^{-\sqrt{12}x})}{(e^{\sqrt{12}} - e^{-\sqrt{12}})}[O^{-1}[A_{i-1}](x)]_{x=1}, \; i \ge 1. \tag{41}$$

The explicit expressions for first few correction terms are given by

$$y_1(x) = 0.237416\left(e^{-2\sqrt{3}x} - e^{2\sqrt{3}x}\right) + \frac{\left(3e^{4\sqrt{3}} + (8 - 22e^{4\sqrt{3}} + 8e^{8\sqrt{3}})e^{2\sqrt{3}x} + 3e^{4\sqrt{3}(x+1)}\right)\left(e^{-2\sqrt{3}x} - 1\right)^2}{32\left(e^{4\sqrt{3}} - 1\right)^2} \tag{42}$$

$$\begin{aligned} y_2(x) = {} & 0.011158 + 2.7033 \times 10^{-7} e^{-6\sqrt{3}x} + 0.00381502 e^{-4\sqrt{3}x} \\ & + (0.0024934 + 0.0203319x)e^{-2\sqrt{3}x} - (0.0173707 - 0.0203319x)e^{2\sqrt{3}x} \\ & - 9.568 \times 10^{-5} e^{4\sqrt{3}x} - 2.7033 \times 10^{-7} e^{6\sqrt{3}x}. \end{aligned} \tag{43}$$

The leading and higher order correction terms for (36) and (37) obtained by using ADM [10] can be found as

$$y_0(x) = -\frac{3}{4}x, \tag{44}$$

$$y_1(x) = -\frac{57}{32}x + \frac{3}{32}x^2(32 - 16x + 3x^2), \tag{45}$$

$$y_2(x) = \frac{873}{896}x - \frac{3}{896}x^3(1064 - 1295x + 672x^2 - 168x^3 + 18x^4). \tag{46}$$

In order to exhibit the efficiency of rapidly converent approximation scheme (RCAS) proposed here we compute the Sup- and L^2-error defined by

$$SupE_n = Sup_{x \in \Omega} E_n(x) \tag{47}$$

and

$$L^2 E_n = \sqrt{\int_\Omega E_n(s)^2 ds} \tag{48}$$

in terms of absolute error

$$E_n(x) = |y(x) - \textstyle\sum_{i=0}^{n} y_i(x)|. \tag{49}$$

A comparison of numerical values of the absolute, Sup- and L^2-errors for approximation with first few terms are presented in Fig.1a and Table 1. The results presented in Table 1 and Fig.1a indicate approximate results obtained by RCAS developed here is better than that obtained by ADM at each level of approximations.

Table 1: Comparison of errors in ADM and present mathod(PM) for Ex.1

Error	$SupE_1$	L^2E_1	$SupE_2$	L^2E_2	$SupE_3$	L^2E_3	$SupE_4$	L^2E_4
ADM	.13	.09	.12	.08	.13	.09	.15	.10
PM	.09	.06	.06	.04	.02	.01	.01	.01

Example 2. We consider the nonlinear BVP involving cubic nonlinearity

$$y''(x) - 2\ y(x) = 2y(x)^3,\ 0 \le x \le \tfrac{\pi}{4};\ y(0) = 0,\ y(\tfrac{\pi}{4}) = -1. \tag{50}$$

Exact solution to this equation can be found as $y(x) = -tan(x)$. Comparing (50) with (15)one gets,

$$\lambda = \sqrt{2}, N[y] = 2y^3, g(x) = 0. \tag{51}$$

The leading order term for $y(x)$ can be found as

$$y_0(x) = -\frac{sinh(\sqrt{2}x)}{sinh(\sqrt{2}\frac{\pi}{4})}. \tag{52}$$

Using results of (51) and (52) into the Eq. (11) and Eq. (26), the first two correction terms in $\sum_{i=0}^{\infty} y_i(x)$ for $y(x)$ can be found as

$$y_1(x) = 0.0347465\ sinh(\sqrt{2}x) - \frac{1}{32} cosech(\frac{\pi}{2\sqrt{2}})^3\{-12\sqrt{2}\,x\,cosh(\sqrt{2}x) + 9sinh(\sqrt{2}x)\ + sinh(3\sqrt{2}x)\}$$

$$\begin{aligned} y_2(x) = &-0.00347899\ sinh(\sqrt{2}x) + 2e^{-\sqrt{2}x}\{0.000053724\ e^{-4\sqrt{2}x} - 0.00005372\ e^{6\sqrt{2}x} + \\ &+10^{-3}e^{4\sqrt{2}x}(-2.13429 + 2.73518\,x) + 10^{-3}e^{-2\sqrt{2}x}(2.13429 + 2.73518\,x) \\ &-0.00773626\ e^{\sqrt{2}x}(-2.16616 + x)(-0.630093 + x) \\ &+\ 0.00773626\ (0.630093 + x)(2.16616 + x)\} \end{aligned} \tag{53}$$

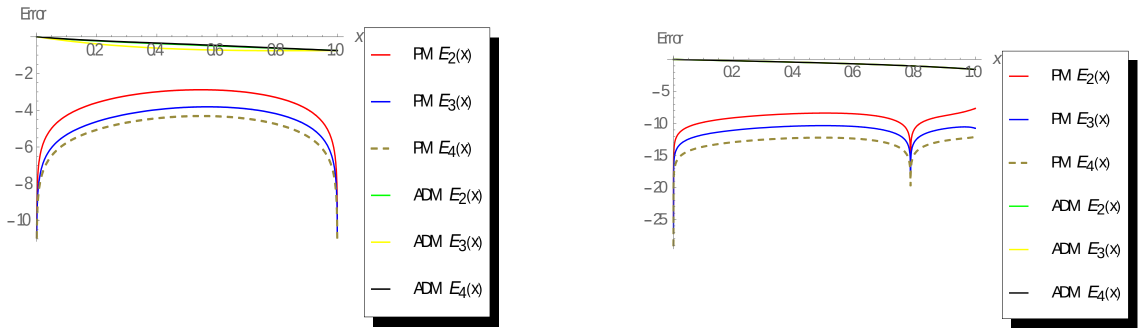
and so on. To compare the efficiency of proposed, the leading and first two corrections terms for $y(x)$ obtained by ADM are presented here.

$$\begin{aligned} y_0(x) &= -\frac{4}{\pi}x, \\ y_1(x) &= \frac{13\pi}{120}x - \frac{4}{15\pi^3}x^3(5\pi^2 + 24x^2), \\ y_2(x) &= -\frac{2273\pi^3}{806400}x + 2x^3(\frac{13\pi}{720} + \frac{29}{150\pi}x^2 - \frac{176}{105\pi^3}x^4 - \frac{64}{15\pi^5}x^6), \cdots. \end{aligned} \tag{54}$$

The Sup- and L^2-errors of approximate solutions obtained by ADM and present method have been calculated and compared in Table 2. Their point-wise absolute errors have been graphically described in figure 1b. From the careful analysis of the results presented in Table 2 and Fig.1b it appears that approximate solutions obtained by present method for BVPs within bounded domain involving cubic nonlinearity are better than the results obtained by ADM.

Table 2. Errors sup E_n and L^2E_n in ADM and present method of Ex. 2.

Error	sup E_1	L^2E_1	sup E_2	L^2E_2	sup E_3	L^2E_3	sup E_4	L^2E_4
ADM	.02	.01	.006	.004	.002	.001	.0009	.0006
PM	.002	.001	.0002	.0002	.00003	.00002	4.9×10^{-6}	$3.3\ \times 10^{-6}$



a) b)

Figure 1: Point-wise absolute errors $E_2(x), E_3(x), E_4(x)$ in present method (PM) and ADM a) Ex. 1 and b) Ex.2.Errors have been presented in $\log_{10}[\cdot]$ scale.

5.2. Unbounded domain

Here we will study the applicability of approximation scheme proposed in section 4 for getting rapidly convergent approximate solution of some nonlinear ODEs which have relation with the symmetry reduction of various nonlinear PDEs of physical interest. We have divided our study into few cases according to the nature of the problem viz. degree of nonlinearity/number of equations/nature of the solution as described below. The systematic steps of section 4 to get approximate solution have been implemented through the program in MATHEMATICA

HOTermsUBD[mxi_,λ_List]:=Module[{oinv,yset},

$\mathbf{oinv} := \mathbf{e}^{\lambda[[1]]\mathbf{x}} \int \mathbf{e}^{(\lambda[[2]]-\lambda[[1]])\mathbf{x}} \left(\int \mathbf{e}^{-\lambda[[2]]\mathbf{x}} \#\mathbf{dx}\right)\mathbf{dx}\&;\ \mathbf{yset} = \left\{\mathbf{d}\mathbf{e}^{\lambda[[2]]\mathbf{x}}\right\};$
$\mathbf{yy}[\mathbf{i\_}, \mathbf{x\_}] := \mathbf{yset}[[\boldsymbol{i} + \mathbf{1}]];$
$\mathbf{Table}\left[\mathbf{AppendTo}\left[\mathbf{yset}, \mathbf{Simplify}\left[\mathbf{oinv}[\mathbf{A}[\mathbf{n}-\mathbf{1}, \mathbf{x}][\#]]\right]\right], \{\mathbf{n}, \mathbf{1}, \mathbf{mxi}\}\right][[\mathbf{mxi}]]$
]&
$\boldsymbol{seq}[\boldsymbol{\lambda\_List}] := \mathbf{HOTermsUBD}[\mathbf{10}, \boldsymbol{\lambda}][\#]\&$
$\boldsymbol{gen}[\boldsymbol{\lambda\ \_List}, \mathbf{x\_}] := \mathbf{FindSequenceFunction}[\mathbf{seq}[\boldsymbol{\lambda}][\#], \boldsymbol{n}]\&$
$\boldsymbol{funx}[\boldsymbol{\lambda\ \_List}, \mathbf{x\_}] := \sum_{n=1}^{\infty} \mathbf{gen}[\boldsymbol{\lambda}, \boldsymbol{x}][\#]\&$
(*Calculating corrections $\{y_1(x), y_2(x), \cdots\cdots, y_m(x)\}$ by using formula (27) in (34) *)

5.2.1. Smooth solution

5.2.1.a Equations with single nonlinear term

In this section we consider some partial differential equations involving monomial nonlinearities e.g., KdV equation, Boussinesq equation, Burger equation, Regular long wave equation, Klein-Gordon equation, Complex version of Boussinesq equation which appear in the mathematical analysis of a wide range of physical processes. All these equations do not involve with space-time coordinates explicitly. Consequently, those equations remain invariant under translation in space-time variables and are reducible to nonlinear ordinary differential equations in the similarity variable $\xi = c(x - vt)$ [1]. The unknown quantities c, v involved in the similarity variable ξ will be appear as paraeters in the reduced nonlinear ODEs. The order of the transformed ODEs can be reduced by integrating as many times as required. The value of the integration constants appearing in the process of reduction of order are chosen to be zero to maintain sharp (exponential) decay of the solution at $\pm\infty$. We are presenting similarity variable and their reduced form (ODE) for each of the equations mentioned above in Tables 3-5. We have then carried out systematic steps of RCAS discussed in section 4 to the reduced ODEs to get the group invariant (traveling wave) solution to the problem. In most of the cases, sum of the terms $U_i(\xi)$ reduces to simple function which can be recast into the solution to the problem in their conventional form obtained earlier.

The symbol d appearing in the column sum in each row in the Tables 3b & 4b is the integration constant. The other symbol $m\_$ present in the column Refs. of the same table depends on d as well as on the parameters of the equation concerned. For example for KdV Equation $m_{KdV} = -\frac{1}{2}log(\frac{d}{2v})$. In all cases leading order term $U_0 = de^{-\lambda\ \xi}$.

5.2.1.b System of equations

In this section we consider some system of coupled PDEs e.g., generalised Zakharov, Devey-Stewartson, Klein-Gordon-Zakharov, coupled Higgs, Maccari equations which can be reduced to single ODE whenever field variables are expressed as the product of phase-amplitude factors and independent variables are expressed in terms of similarity variables as required by their symmetry transformations viz., space-time translations. Before solving those reduced ODEs by using method proposed here we first demonstrate the reduction of system of PDEs to ODEs with an example viz., Generalized Zakharov equation

$$iE_t + E_{xx} + 2\lambda|E|^2 E + \delta\ \eta E = 0 \quad (55)$$

$$\eta_{tt} - c^2\eta_{xx} + \mu(|E|^2)_{xx} = 0. \quad (56)$$

Transformations of field variables $E(x,t)$ to $E(x,t) = e^{i(\alpha x+\beta t)}u(\xi)$, and $\eta(x,t)$ to $\eta(x,t) = \eta(\xi)$, with similarity variables $\xi = k(x - 2\alpha t)$ of above equations lead to the system of ODEs

$$k^2 u''(\xi) - (\alpha^2 + \beta)u(\xi) + \delta u(\xi)\eta(\xi) + 2\lambda u(\xi)^3 = 0, \quad (57)$$

$$(4\alpha^2 - c^2)\eta''(\xi) + \mu(u(\xi)^2)'' = 0. \quad (58)$$

One can now express the field variable $\eta(\xi)$ in terms of $u(\xi)$ by integrating (58) twice and taking integration constants to zero (assuming departure of η from its equilibrium configuration at infinity is negligible) and get

Table 3a: Similarity variables and reduction of some PDEs involving quadratic nonlinearity.

Eq.	PDE	ξ	Reduced ODE
KdV	$u_t + 6uu_x + u_{xxx} = 0$	$c(x - vt)$	$u'' - \frac{v}{c^2}u + \frac{3}{c^2}u^2 = 0$
Boussinesq(B)	$u_{tt} + au_{xx} + b(u^2)_{xx} + cu_{xxxx} = 0$	$k(x - vt)$	$u'' - (\frac{av^2}{-ck^2})u + \frac{b}{ck^2}u^2 = 0$
RLW	$u_t + u_x + auu_x - bu_{xxt} = 0$	$c(x - vt)$	$u'' - (\frac{(v-1)}{c^2bv})u + \frac{a}{2bvc^2}u^2 = 0$
Klein-Gordon(KG)	$u_{tt} - \alpha^2 u_{xx} + \beta u - \gamma u^2 = 0$	$k(x - ct)$	$u'' - \frac{\beta}{k^2(\alpha^2 - c^2)}u - \frac{\gamma}{k^2(\alpha^2 - c^2)}u^2 = 0$
Complex Boussinesq(CB)	$u_{tt} - \gamma u_{xx} - \alpha u_{xxxx} - \beta(u^2)_{xx} = 0$	$ik(x - ct)$	$u'' - (\frac{\gamma - c^2}{\alpha k^2})u - \frac{\beta}{\alpha\ k^2}u^2 = 0$
Zakharov-Kuznetsov(ZK)	$u_t + \delta u_x + \alpha uu_x + \beta u_{xxx} + \gamma u_{xxy} = 0$	$kx + ly - ct$	$u'' - \frac{(c-\delta k)}{(\beta k^3 + \gamma k^2 l)} + \frac{\alpha k}{2(\beta k^3 + \gamma k^2 l)}u^2 = 0$
Kadomtsev-Petviashvili(KP)	$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} + 3u_{zz} = 0$	$ax + by + cz - \delta t$	$u'' + \frac{(3b^2 + 3c^2 - a\delta)}{a^4}u + \frac{3}{a^2}u^2 = 0$

Table 3b: Leading order and correction terms of the Approx. Soln. In (29) and their sum.

Eq.	λ	U_0	U_n	sum	Refs.
KdV	$\frac{\sqrt{v}}{c}$	$de^{-\lambda\ \xi}$	$-2^{1-n}nv\left(-\frac{\delta e^{-\lambda\ \xi}}{v}\right)^n$	$\frac{4e^{\lambda\ \xi}v^2 d}{(2e^{\lambda\ \xi}v + d)^2}$	$\frac{v}{2}Sech^2(\frac{1}{2}\lambda\ \xi + m_{KdV})$ [5]

B	$\sqrt{-\frac{a+v^2}{ck^2}}$	$de^{-\lambda\,\xi}$	$\frac{6^{1-n}n(a+v^2)\left(\frac{bde^{-\lambda\,\xi}}{a+v^2}\right)^n}{b}$	$\frac{36d(a+v^2)^2e^{\lambda\,\xi}}{(6v^2e^{\lambda\,\xi}+6ae^{\lambda\,\xi}-bd)^2}$	$-\frac{3(a+v^2)}{2b}\operatorname{sech}^2\left(\frac{1}{2}\lambda\,\xi+m_{Bou}\right)$ [22]
RLW	$\sqrt{\frac{v-1}{bc^2v}}$	$de^{-\lambda\,\xi}$	$-\frac{12n(v-1)}{a}\left(\frac{ade^{-\lambda\,\xi}}{12-12v}\right)^n$	$\frac{144d(v-1)^2e^{\lambda\,\xi}}{(ad-12e^{\lambda\,\xi}+12ve^{\lambda\,\xi})^2}$	$\frac{3(v-1)}{a}\operatorname{sech}^2(\frac{\lambda}{2}\xi+m_{RLW})$ [23]
KG	$\sqrt{\frac{\beta}{k^2(\alpha^2-c^2)}}$	$de^{-\lambda\,\xi}$	$-\frac{6^{1-n}n\beta}{\gamma}\left(-\frac{d\gamma}{\beta}e^{-\lambda\,\xi}\right)^n$	$\frac{36d\beta^2e^{\lambda\,\xi}}{(6\beta e^{\lambda\,\xi}+d\gamma)^2}$	$\frac{3(v-1)}{a}\operatorname{sech}^2\left(\frac{\lambda}{2}\xi+m_{KG}\right)$ [24]
CB	$\sqrt{\frac{\gamma-c^2}{k^2\alpha}}$	$de^{-\lambda\,\xi}$	$-\frac{6n(c^2-\gamma)}{\beta}\left(-\frac{d\beta e^{-\lambda\,\xi}}{6c^2-6\gamma}\right)^n$	$\frac{36d(c^2-\gamma)^2e^{\lambda\,\xi}}{(6c^2e^{\lambda\,\xi}-6\gamma e^{\lambda\,\xi}+d\beta)^2}$	$\frac{3(c^2-\gamma)}{2\beta}sech^2(\frac{i\lambda\xi}{2}+m_{CB})$ [25]
ZK	$\sqrt{\frac{c-k\delta}{k^3\beta+k^2l\gamma}}$	$de^{-\lambda\,\xi}$	$\frac{12^{1-n}n(k\delta-c)}{k\alpha}\left(\frac{dk\alpha e^{-\lambda\,\xi}}{k\delta-c}\right)^n$	$\frac{144d(c-k\delta)^2e^{\lambda\,\xi}}{(12ce^{\lambda\,\xi}-12k\delta e^{\lambda\,\xi}+dk\alpha)^2}$	$\frac{3(c-k\delta)}{k\alpha}sech^2(\frac{\lambda}{2}\xi+m_{ZK})$ [26]
KP	$\sqrt{\frac{(3b^2+3c^2-a\delta)}{-a^4}}$	$de^{-\lambda\,\xi}$	$-\frac{2n(a\delta-3b^2-3c^2)\left(a^2de^{-\lambda\xi}\right)^n}{a^2(-2a\delta+6b^2+6c^2)^n}$	$\frac{4d(-a\delta+3b^2+3c^2)^2e^{\lambda\,\xi}}{((-2a\delta+6b^2+6c^2)e^{\lambda\,\xi}-a^2d)^2}$	$\frac{(a\delta-3b^2-3c^2)}{2a^2}sech^2(\frac{\lambda\,\xi}{2}+m_{ZP})$ [27]

$$\eta(\xi)=\frac{\mu}{(c^2-4\alpha^2)}u(\xi)^2,\ c^2\neq 4\alpha^2. \tag{59}$$

Now Eqs. (57) and (58) can be decoupled by using (59) into (57) to get equation for $u(\xi)$ as

$$u''(\xi)-\frac{(\alpha^2+\beta)}{k^2}u(\xi)-\frac{(\delta\mu-2\lambda(4\alpha^2-c^2))}{k^2(4\alpha^2-c^2)}u(\xi)^3=0. \tag{60}$$

This equation is suitable for the application of RCAS developed in section 4 to obtain approximate or exact solution to the problem. Substitution of that solution into (59) provides approximate/exact solution of $\eta(\xi)$. Further substitution of expression for ξ into those formula will give space-time dependent solution of the Generalized Zakharov system of equations. Following similar procedure one can find solutions of other such systems of equations summarized in Table 6a & 6b.

5.2.2 Non smooth solution

In 1993, Camassa and Holm [40, 41] derived an equation

$$v_t+2kv_x-v_{xxt}+3vv_x-2v_xv_{xx}-vv_{xxx}=0 \tag{61}$$

for shallow water wave with $v(x,t)$ related to a layer-mean horizontal velocity at the spatial coordinate x and time t. This equation now known as Camassa−Holm (CH) equation was derived as an asymptotic model for long gravity waves at the surface of shallow water. Because the dispersive term is nonlinear this equation has peaked solitary wave solutions of the form

$$v(x,t)=ce^{-|x-ct|} \tag{62}$$

for $k=0$, which travels with wave speed c. The solitary wave solution (62) known as *peakon*, has a distinct feature like discontinuous first derivative at the wave peak in contrast to smoothness of conventional solitary waves. However, we have applied RCAS proposed here to the ODE associated to the CH equation for $k\neq 0$ and obtained peakon solution as given in the Table 7a,b.

A snap of the wave at some instant is given by the function $v(x,t)$ is displayed in Fig.2 for $d=15$ and $k=.001$. From last expression it appears that although the traveling wave solution $v(x,t)$ of CH equation maintains soliton-like behaviour throughout the motion continuous at $x=ct$, its derivative is not continuous there. As a result the solution maintains a sharp peak at $x=ct$ throughout the motion.

Table 4a: Similarity variables and reduced ODEs of some PDEs involving cubic nonlinearity.

Eq.	PDE	Transformation	Reduced ODE

Dispersive(Dis)	$u_t - \delta\ u^2 u_x + u_{xxx} = 0$	$\xi = k(x - \lambda\ t)$	$u'' - \frac{\lambda}{k^2} u - \frac{\delta}{3\ k^2} u^3 = 0$
ϕ^4	$u_{tt} - \alpha u_{xx} - u + u^3 = 0$	$\xi = c(x - v\ t)$	$u'' - \frac{1}{c^2(v^2-\alpha)} u + \frac{1}{c^2(v^2-\alpha)} u^3 = 0$
Cubic Klein-Gordon(CKG)	$u_{tt} - \alpha^2 u_{xx} + \beta u - \gamma u^3 = 0$	$\xi = k(x - c\ t)$	$u'' - \frac{\beta}{k^2(\alpha^2-c^2)} u - \frac{\gamma}{k^2(\alpha^2-c^2)} u^3 = 0$
The generalised ZK-BBM(GZK)	$u_t + u_x + a(u^3)_x + b(u_{xt} + u_{yy})_x = 0$	$\xi = rt + px + qy$	$u'' + \frac{(p+r)}{bp(q^2+pr)} u + \frac{a}{b(q^2+pr)} u^3 = 0$
Schrödinger(Sch)	$iE_t + E_{xx} - \mu \|E\|^2 E = 0$	$E = e^{i(\alpha x + \gamma t)} u(\xi)$ $, \xi = k(x - 2ct)$	$u'' - (\frac{\gamma - \alpha^2}{k^2}) u + \frac{\mu}{k^2} u^3 = 0$
Hamiltonian Amplitude(HA)	$iE_x + E_{tt} + 2\sigma \|E^2\| E - \varepsilon E_{xt} = 0$	$E = e^{i(\alpha x - \beta t)} u(\xi)$ $\xi = (2\beta + \alpha\varepsilon)x + (1 + \beta\varepsilon)t$	$u'' - \frac{(\beta^2 + (1+\beta\varepsilon)\alpha)}{(1 - \beta^2\varepsilon^2 - (\varepsilon^2 + \beta\varepsilon^3)\alpha)} u = \frac{2\sigma}{((\varepsilon^2 + \beta\varepsilon^3) + \beta^2\varepsilon^2 - 1)} u^3$

Table 4b: Leading order and correction terms of (29) and their sum.

Eq.	λ	U_0	U_n	sum	Refs.
Dis.	$\frac{\sqrt{\lambda}}{k}$	$de^{-\lambda\ \xi}$	$\frac{24^{1-n}\lambda e^{-\lambda\ \xi}}{d\delta}\left(\frac{d^2\delta e^{-2\lambda\ \xi}}{\lambda}\right)^n$	$\frac{24 d\lambda e^{\lambda\ \xi}}{24\lambda e^{2\lambda\ \xi} - d^2\delta}$	$\sqrt{-\frac{6\lambda}{\delta}}\ \operatorname{sech}(\lambda\xi + m_{Dis})$ [28]
ϕ^4	$\sqrt{\frac{1}{c^2(v^2-\alpha)}}$	$de^{-\lambda\ \xi}$	$-\frac{8^{1-n}e^{\lambda\ \xi}(-d^2e^{-2\lambda\ \xi})^n}{d}$	$\frac{8de^{\lambda\ \xi}}{8e^{2\lambda\ \xi}+d^2}$	$\sqrt{2}\ \operatorname{sech}(\lambda\ \xi + m_{\Phi})$ [29]
CKG	$\sqrt{\frac{\beta}{k^2(\alpha^2-c^2)}}$	$de^{-\lambda\ \xi}$	$-\frac{8^{1-n}\beta e^{\lambda\ \xi}}{d\gamma}\left(-\frac{d^2\gamma e^{-2\lambda\ \xi}}{\beta}\right)^n$	$\frac{8d\beta e^{\lambda\ \xi}}{8\beta e^{2\lambda\ \xi}+d^2\gamma}$	$\sqrt{\frac{2\beta}{\gamma}}\ \operatorname{sech}(\lambda\ \xi + m_{CKG})$ [24]
GZK	$\sqrt{-\frac{p+r}{bp(pr+q^2)}}$	$de^{-\lambda\ \xi}$	$\frac{8^{1-n}(p+r)e^{\lambda\ \xi}}{adp}\left(\frac{ad^2pe^{-2\lambda\ \xi}}{p+r}\right)^n$	$\frac{8d(p+r)e^{\lambda\ \xi}}{-ad^2p+8pe^{2\lambda\ \xi}+8re^{2\lambda\ \xi}}$	$\sqrt{-\frac{2(p+r)}{ap}}\ \operatorname{sech}(\lambda\ \xi + m_{GZK})$ [30]
Sch.	$\frac{\sqrt{\alpha^2+\gamma}}{k}$	$de^{-\lambda\ \xi}$	$-\frac{8^{1-n}(\alpha^2+\gamma)e^{\lambda\ \xi}}{d\mu}\left(-\frac{d^2\mu e^{-2\lambda\ \xi}}{\alpha^2+\gamma}\right)^n$	$\frac{8d(\alpha^2+\gamma)e^{\lambda\ \xi}}{d^2\mu+8\alpha^2e^{2\lambda\ \xi}+8\gamma e^{2\lambda\ \xi}}$	$\sqrt{\frac{2(\alpha^2+\gamma)}{\mu}}\ \operatorname{sech}(\lambda\ \xi + m_{Sch})$ [31]

HA	$\sqrt{\frac{\alpha(\beta\varepsilon+1)+\beta^2}{1-\alpha(\beta\varepsilon^3+\varepsilon^2)-\beta^2\varepsilon^2}}$	$de^{-\lambda\ \xi}$	$-\frac{4^{1-n}\left(-d^2\sigma e^{-2\lambda\ \xi}\right)^n e^{-\lambda\ \xi}}{d\sigma(\alpha\beta\varepsilon+\alpha+\beta^2)^{n-1}}$	$\frac{4d\left(\alpha\beta\varepsilon+\alpha+\beta^2\right)e^{\lambda\ \xi}}{d^2\sigma+(4\beta^2+4\alpha+4\alpha\beta\varepsilon)e^{2\lambda\ \xi}}$	$\sqrt{\frac{\alpha\beta\varepsilon+\alpha+\beta^2}{\sigma}}\ \mathrm{sech}(\lambda\ \xi+m_{HA})$ [30]

Table 5a: Similarity reduction of PDEs involving biquadratic nonlinearities.

Eq.	PDEs	Transformation	Reduced ODE
gKdV	$u_t+\delta\ u_{xxx}+u^3u_x=0$	$\xi=c(x-vt)$	$u''-\frac{v}{\delta c^2}u+\frac{a}{4\delta c^2}u^4=0$

Table 5b: Leading and general term of the series in (29) and their sum for reduced ODE ($(a)_n \rightarrow$ Pochhammer symbol).

Eq.	λ	U_n	sum	Refs.
gKdV	$\sqrt{\frac{v}{\delta c^2}}$	$-\frac{40^{1-n}ve^{2\lambda\ \xi}\left(-\frac{ad^3e^{-3\lambda\ \xi}}{v}\right)^n}{ad^2}\times\frac{(\frac{2}{3})_{n-1}}{(1)_{n-1}}$	$\frac{4d\lambda e^{-\lambda\ \xi}}{\left(8+\frac{ad^3}{5v}e^{-3\lambda\ \xi}\right)^{\frac{2}{3}}}$	$(\frac{10v}{a})^{\frac{1}{3}}sech^{\frac{2}{3}}(\frac{3}{2}\lambda\xi+m)$ [32]

Table 6a: Similarity reduction of system of PDEs involving cubic nonlinearities.

Eq.	PDEs	Transformation	Reduced ODE
Generalised Zakharov(GZ)	$iE_t+E_{xx}+2\lambda\|E\|^2E+\delta\ \eta E=0$ $\eta_{tt}-c^2\eta_{xx}+\mu(\|E\|^2)_{xx}=0$	$E=e^{i(\alpha x+\beta t)}u(\xi)$ $\eta=\eta(\xi)$ $\xi=k(x-2\alpha t)$	$u''-\frac{(\alpha^2+\beta)}{k^2}u-\frac{(\delta\mu-2\lambda(4\alpha^2-c^2))}{k^2(4\alpha^2-c^2)}u^3=0$
Devey-Stewartson(DS)	$iE_t+E_{xx}-E_{yy}-2\|E\|^2E-2E\eta=0$ $\eta_{xx}+\eta_{xx}+2(\|E\|^2)_{xx}=0$	$E=e^{i(px+qy+rt)}u(\xi)$ $\eta=\eta(\xi)$ $\xi=kx+cy+2(cq-kp)t$	$u''+\frac{(q^2-p^2-r)}{(k^2-c^2)}u+\frac{2}{(k^2+c^2)}u^3=0$
Klein-Gordon-Zakharov(KGZ)	$E_{tt}-E_{xx}+E-\alpha\eta E=0$ $\eta_{tt}-\eta_{xx}-\beta(\|u\|^2)_{xx}=0$	$E=e^{i(kx+wt)}u(\xi)$ $\eta=\eta(\xi)$ $\xi=wx+kt$	$u''-\frac{(w^2-k^2-1)}{(k^2-w^2)}u-\frac{\alpha\beta w^2}{(k^2-w^2)^2}u^3=0$

Coupled Higgs(CHs)	$u_t - u_{xx} + \|u\|^2 u - 2\eta u = 0$ $\eta_{tt} + \eta_{xx} - (\|u\|^2)_{xx} = 0$	$E = e^{i(px+rt)}u(\xi)$ $\eta = \eta(\xi)$ $\xi = rx + pt$	$u'' + u + \frac{u^3}{(p^2+r^2)} = 0$
Maccari systems(Mc)	$iE_t + E_{xx} + \eta E = 0$ $\eta_t + \eta_y - (\|E\|^2)_x$	$E = e^{i(px+qy+rt)}u(\xi)$ $\eta = \eta(\xi)$ $\xi = k(x-2pt) + qy$	$u'' - \frac{1}{k^2}(p^2 + r - \frac{c}{q-2kp})u - \frac{u^3}{k(q-2kp)} = 0$

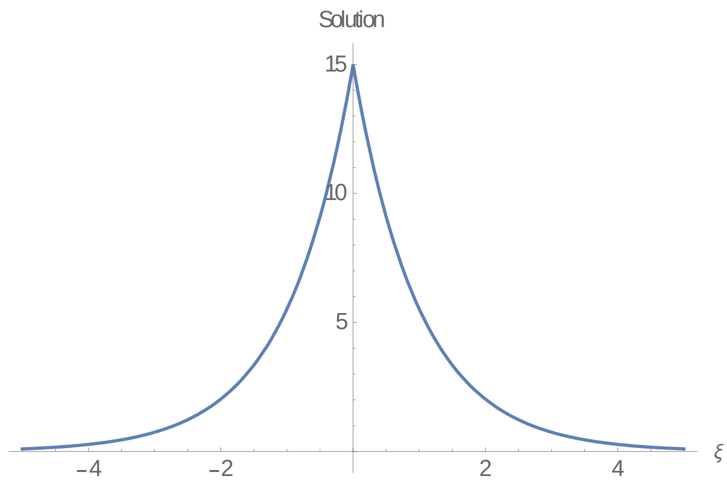


Figure 2: Peakon Soln. Of CH Eq. Obtained by present method.

Table 6b: Leading and correction terms in (29) and their sum.

Eq.	λ	U_n	sum	Refs.
GZ	$\frac{\sqrt{\alpha^2+\beta}}{k}$	$-\frac{8^{1-n}(4\alpha^2-c^2)(\alpha^2+\beta)e^{\lambda\xi}}{d(-2c^2\lambda_1+8\alpha^2\lambda_1-\delta\mu)} \times \left(\frac{-d^2 e^{-\lambda\xi}(2c^2\lambda_1-8\alpha^2\lambda_1+\delta\mu)}{(c^2-4\alpha^2)(\alpha^2+\beta)}\right)^n$	$\frac{d\lambda e^{-\lambda\xi}}{1+\frac{d^2(2c^2\lambda-8\alpha^2\lambda+\delta\mu)}{8(c^2-4\alpha^2)(\alpha^2+\beta)}e^{-2\lambda\xi}}$	$\sqrt{\frac{2(c^2-4\alpha^2)(\alpha^2+\beta)}{(2c^2\lambda-8\alpha^2\lambda+\delta\mu)}} \times$ $\mathrm{sech}(\lambda\xi + m_{GZ})$ [33, 34]
DS	$\sqrt{\frac{(p^2-q^2+r)}{(k^2-c^2)}}$	$\frac{4^{1-n}(c^2+k^2)(p^2-q^2+r)e^{\lambda\xi}}{d(c^2-k^2)} \times \left(\frac{d(c^2-k^2)e^{-2\lambda\xi}}{(c^2+k^2)(p^2-q^2+r)}\right)^n$	$\frac{de^{-\lambda\xi}}{1-\frac{d^2(c^2-k^2)e^{-2\lambda\xi}}{4(c^2+k^2)(p^2-q^2+r)}}$	$-\frac{2k^2(p^2-q^2+r)}{(c^2-k^2)} \times$ $\mathrm{sech}(\lambda\xi + m_{DS})$ [35, 36]

KGZ	$\sqrt{\frac{(w^2-k^2-1)}{(k^2-w^2)}}$	$\frac{8^{1-n}(k^4-2k^2w^2+k^2+w^4-w^2)e^{\lambda\xi}}{dw^2\alpha\beta}\times\left(\frac{d^2w^2\alpha\beta e^{-2\lambda\xi}}{k^4-2k^2w^2+k^2+w^4-w^2}\right)^n$	$\frac{de^{\lambda\ \xi}}{1-\frac{d^2w^2\alpha\ \beta e^{2\lambda\ \xi}}{8(k^2+k^4-w^2-2w^2k^2+w^4)}}$	$\sqrt{\frac{2(w^2-k^2-k^4+2w^2k^2-w^4)}{\alpha\beta w}}\times$ $\mathrm{sech}(\lambda\ \xi+m_{KGZ})$ [37]
CHs	i	$\frac{8^{1-n}(p^2+r^2)e^{\lambda\ \xi}}{d}\times\left(\frac{d^2e^{-2\lambda\ \xi}}{p^2+r^2}\right)^n$	$\frac{de^{-\lambda\ \xi}}{1-\frac{e^{-2\lambda\ \xi}}{8(p^2+r^2)}}$	$\sqrt{-2(p^2+r^2)}\times$ $\mathrm{sech}(\lambda\xi+m_{CH})$ [38, 39]
Mc	$\sqrt{\frac{(p^2+r-\frac{c}{q-2kp})}{k^2}}$	$-\frac{8^{1-n}e^{\lambda\xi}(c+2kp^3+2kpr-p^2q-qr)}{dk}\times\left(-\frac{d^2ke^{-2\lambda\xi}}{c+(p^2+r)(2kp-q)}\right)^n$	$\frac{de^{-\lambda\ \xi}}{1+\frac{d^2ke^{-2\lambda\ \xi}}{8(c+2kp^3+2kpr-p^2q-qr)}}$	$\sqrt{\frac{2(c+2kp^3+2kpr-p^2q-qr)}{k}}\times$ $\mathrm{sech}(\lambda\ \xi+m_{Mc})$ [38, 39]

Table 7a: Similarity reduction of CH Eq.

Eq.	PDEs	Transformation	Reduced ODE
Camassa-Holm	$v_t+2kv_x-v_{xxt}+3vv_x$ $-2v_xv_{xx}-vv_{xxx}=0$	$v(x,t)=u(\xi)-k$ $\xi=x-ct$	$u''-u-\frac{1}{2(k+c)}(u'^2+2uu''-3u^2)=0$

Table 7b: Leading order and correction terms of series in (29) and their sum.

Condition	λ	U_0	U_1	U_n	sum	Soln. $v(x,t)$ in [42]
when $x-c\,t>0$	1	$de^{-\lambda\xi}$	0	0	$de^{-\lambda\xi}$	$d\,e^{-(x-ct)}-k$
$when\ x-c\,t<0$	1	$de^{\lambda\xi}$	0	0	$de^{\lambda\xi}$	$d\,e^{+(x-ct)}-k$

6 Conclusion

This work deals with an approximation scheme to obtaining a rapidly convergent approximate solution of nonlinear ODEs with DBCs defined over bounded/ unbounded domain. An advantage of the present method is that the boundary condition at $\pm\infty$ (BVP in unbounded domain) can be incorporated into the solution in a natural way rather than taking recourse to the use of Padé approximant of the higher order corrections terms as desired in case of ADM. From the theory developed here and case studies presented we have arrived at the following conclusion.

1. When the linear part of the nonlinear equation involves only the highest derivative $(\lambda_1=\lambda_2=0)$, the inverse operator of the RCAS became identical with that of the ADM. In that case the algorithms of the proposed scheme coincide with those of ADM.
2. For $\lambda_1\neq\lambda_2\neq 0$, the the proposed schemes for both bounded and unbounded domains provide a rapidly convergent series solution (S_1) in comparision to the same (S_2) found by using ADM. More significantly, partial sum of terms in S_1 is more accurate result than the corresponding sum of terms in S_2. As a result, one can use the present scheme to obtain exact solutions in localized within finite space of nonlinear differential equations.
3. The algorithm presented here can evaluate successive correction terms more efficiently in comparison to ADM not only for the two-point boundary value problems but are also equally effective for solving initial-boundary value problems involving PDEs exhibiting translational symmetry in both space and time variables.
4. The formulae for successive correction terms presented here can be easily implemented through the symbolic computation software viz. Mathematica, Maple, Matlab etc. The programmes in MATHEMATICA for calculating Adomian polynomials, successive correction terms and their general form, getting sum of the series have been presented here for their use.
5. The travelling wave solutions of several nonlinear scalar or system of coupled PDEs appearing in the mathematical analysis of varieties of physical processes have been obtained by using RCAS developed here and summarized in the

tables for their easy access.

We conclude by noting that it remains an interesting curiosity to treat the initial boundary value problems modeled by partial differential equations involving variable coefficients which apparently do not exhibit space-time translational symmetry. In this context we note that very often one can look for change of dependent and independent variables to convert the partial differential equations with variable coefficients to similar equations with constant coefficient and thereby regain the appropriate symmetry. Works in these directions are in progress and will be reported in due course.

Acknowledgement
This work is supported by UGC assisted SAP(DRS Phase-II) program grant No. F.510/4/DRS/2009(SAP-I) through Department of Mathematics, Visva-Bharati, Santiniketan-731 235, W.B., India

References

[1] P. J. Olver, Applications of Lie groups to differential equations, Vol. 107, Springer Science & Business Media, 2000.
[2] N. H. Ibragimov, CRC Handbook of Lie group analysis of differential equations, Vol. 1,2,3, CRC press, 1994, 1994, 1996.
[3] A. G. Choudhury, P. Guha, and B. Khanra, "Solutions of some second order odes by the extended prelle-singermethod and symmetries", J. Nonlin. Math. Phys., Vol. 15, No. 4, 2008, pp. 365–382.
[4] M. Nucci, and P. Leach, "Jacobis last multiplier and the complete symmetry group of the ermakov-pinney equation", J. Nonlin. Math. Phys., Vol. 12, No. 2, 2005, pp. 305–320.
[5] W. Malfliet, "Solitary wave solutions of nonlinear wave equations", Am. J. Phys., Vol. 60, No. 7, 1992, pp. 650–654.
[6] W. Malfliet, and W. Hereman, "The tanh method:1 exact solutions of nonlinear evolution and wave equations", Phys. Scr., Vol. 54, 1996, pp. 563–568.
[7] S. Abbasbandy, E. Magyari, and E. Shivanian, "The homotopy analysis method for multiple solutions of nonlinear boundary value problems", Comm. Nonlin. Sc. Num. Simul., Vol. 14, No. 9, 2009, pp. 3530–3536.
[8] S. Xinhui, Z. Liancun, Z. Xinxin, and S. Xinyi, "Homotopy analysis method for the asymmetric laminar flow and heat transfer of viscous fluid between contracting rotating disks", Appl. Math. Model., Vol. 36, No. 4, 2012, pp. 1806–1820.
[9] G. Adomian, Solving frontier problems of physics: the decomposition method: *The Decomposition Method* Kluwer, 1994.
[10] J.-S. Duan and R. Rach, "A new modification of the adomian decomposition method for solving boundary value problems for higher order nonlinear differential equations", Appl. Math. Comput., Vol. 218, No. 8, 2011, pp. 4090–4118.
[11] G. Adomian, and R. Rach," Inversion of nonlinear stochastic operators", J. Math. Anal. Appl., Vol. 91, No. 1, 1983, pp. 39–46.
[12] G. Adomian, and R. Rach, "Analytic solution of nonlinear boundary-value problems in several dimensions by decomposition", J. Math. Anal. Appl., Vol. 174 No. 1, 1993, pp. 118–137.
[13] G. Adomian, and R. Rach, "A new algorithm for matching boundary conditions in decomposition solutions", Appl. math. comput., Vol. 57 , No. 1, 1993, pp. 61–68.
[14] G. Adomian, and R. Rach, "Modified decomposition solution of linear and nonlinear boundary-value problems", Nonlin. Anal. Th. Meth. Appl., Vol. 23, No. 5, 1994, pp. 615–619.
[15] A.-M. Wazwaz, "Approximate solutions to boundary value problems of higher order by the modified decomposition method", Comput. Math. Appl., Vol. 40, No. 6, 2000, pp. 679–691.
[16] A.-M. Wazwaz, "The modified adomian decomposition method for solving linear and nonlinear boundary value problems of tenth-order and twelfth-order", Int. J. Nonlin. Sci. Num. Simul., Vol. 1, No. 1, 2000, pp. 17–24.
[17] A. Wazwaz, "A reliable algorithm for obtaining positive solutions for nonlinear boundary value problems", Comput. Math. Appl., Vol. 41, No. 10, 2001, pp. 1237–1244.
[18] A.-M. Wazwaz, "The numerical solution of fifth-order boundary value problems by the decomposition method", J. Comput. Appl. Math., Vol. 136 , No. 1, 2001, pp. 259–270.
[19] A.-M.Wazwaz, "The numerical solution of sixth-order boundary value problems by the modified decomposition method", Appl. Math. Comput., Vol. 118 , No. 2, 2001, pp. 311–325.

[20] A.-M. Wazwaz, "A reliable algorithm for solving boundary value problems for higher-order integro-differential equations", Appl. Math. Comput., Vol. 118, No. 2, 2001, pp. 327–342.
[21] A.-M. Wazwaz, "The numerical solution of special fourth-order boundary value problems by the modified decomposition method", Int. J. comput. math., Vol. 79, No. 3, 2002, pp. 345–356.
[22] P. A. Clarkson, and M. D. Kruskal," New similarity reductions of the boussinesq equation", J. Math. Phys., Vol. 30, No. 10, 1989, pp. 2201–2213.
[23] T. S. El-Danaf, M. A. Ramadan, and F. E. A. Alaal, "The use of adomian decomposition method for solving the regularized long-wave equation", Chaos, Solitons & Fractals, Vol. 26, No. 3, 2005, pp. 747–757.
[24] H. Zhang, "New exact travelling wave solutions for some nonlinear evolution equations, part ii", Chaos, Solitons & Fractals, Vol. 37, No. 5, 2008, pp. 1328–1334.
[25] H. Zhang, "A new sub-equation method applied to obtain exact travelling wave solutions of some complex nonlinear equations", Chaos, Solitons & Fractals, Vol. 42, No. 2, 2009, pp. 911–915.
[26] J.-H. He, "Application of homotopy perturbationmethod to nonlinear wave equations", Chaos, Solitons & Fractals, Vol. 26, No. 3, 2005, pp. 695–700.
[27] M. A. Akbar, N. Hj, M. Ali, and S. T. Mohyud-din, "Some new exact traveling wave solutions to the (3+ 1)-dimensional kadomtsev-petviashvili equation", World Appl. Sci. J, Citeseer, 2012.
[28] A. J. Mohamad-Jawad, "The sine-cosine function method for exact solutions of nonlinear partial differential equations", IJRRS, Vol. 13, No. 1.
[29] M. Younis, and A. Zafar, "The modified simple equation method for solving nonlinear phi-four equation", Inter. J. Innovation and Applied Studies , Vol. 2, No. 4, 2013, pp. 661–664.
[30] M. Abdou, "The extended f-expansion method and its application for a class of nonlinear evolution equations", Chaos, Solitons & Fractals, Vol. 31, No. 1, 2007, pp. 95–104.
[31] W.-X. Ma, and M. Chen, "Direct search for exact solutions to the nonlinear schrodinger equation", Appl. Math. Comput., Vol. 215, No. 8, 2009, pp. 2835–2842.
[32] W. Hereman, and M. Takaoka, "Solitary wave solutions of nonlinear evolution and wave equations using a direct method and macsyma", J. Phys. A: Mathematical and General, Vol. 23, No. 21, 1990, pp. 4805.
[33] V. E. Zakharov, "Collapse of langmuir waves", Soviet Journal of Experimental and Theoretical Physics, Vol. 35, 1972, pp. 908.
[34] W.-H. Steeb, N. Euler, and W. Hereman," A note on the zakharov equation and lie-symmetry vector fields", Il Nuovo Cimento B, Vol. 107 , No. 10, 1992, pp. 1211–1213.
[35] M. Abdou, "The extended tanh method and its applications for solving nonlinear physical models", Appl. Math. Comput., Vol. 190, No. 1, 2007, pp. 988–996.
[36] S. El-Wakil, M. Abdou, and A. Elhanbaly, " New solitons and periodic wave solutions for nonlinear evolution equations", Phys. Lett. A, Vol. 353, No. 1, 2006, pp. 40–47.
[37] Y. Shang, Y. Huang, and W. Yuan, "New exact traveling wave solutions for the klein–gordon–zakharov equations", Comput. Math. Appl., Vol. 56 , No. 5, 2008, pp. 1441–1450.
[38] A. Bekir, "New exact travelling wave solutions of some complex nonlinear equations", Comm. Nonlin. Sc. Num. Simul., Vol. 14, No. 4, 2009, pp. 1069–1077.
[39] H. Zhao, "Applications of the generalized algebraic method to special-type nonlinear equations", Chaos, Solitons & Fractals, Vol. 36, No. 2, 2008, pp. 359–369.
[40] R. Camassa, and D. D. Holm, "An integrable shallow water equation with peaked solitons", Phys. Rev. Lett., Vol. 71, No. 11, 1993, pp. 1661.
[41] A.-M.Wazwaz, "A class of nonlinear fourth order variant of a generalized camassa–holm equation with compact and noncompact solutions", Appl. Math. Comput., Vol. 165, No. 2, 2005, pp. 485–501.
[42] Z. Liu, and T. Qian, "Peakons of the camassa–holm equation", Appl. Math. Model., Vol. 26, No. 3, 2002, pp. 473–480.