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A Comparative Study between Two Interpolation Functions: Lagrange and Trigonometric Interpolation

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Abstract

Many practical problems can be solved by using finite element method like eigenvalue problems, steady state problems, plane elasticity problems and transient problems. Finite element analysis is a numerical technique to obtain the approximate solutions of integral equations and partial differential equations that arise in the various fields of science and engineering. Generally algebraic polynomial or Lagrange interpolation is used as shape function to approximate the field variable for the computation of eigenvalues of an eigenvalue problem. In this paper we have used trigonometric interpolation namely tangent interpolation instead of Lagrange interpolation for solving an eigenvalue problem by finite element method. After calculating the eigenvalues we have compared this result with those obtained by using Lagrange interpolation and this comparison shows a close similarity between them.

Keywords: *Interpolation Function, Lagrange Interpolation, Trigonometric Interpolation, Eigenvalue.*

1. Introduction

The finite element method was initially developed on a physical basis for the structural analysis problems in civil and aeronautical engineering. The term 'finite element' was first used by Clough, R.W., 1960. After its introduction it has continually developed and improved. Though in early days the contributors have been almost engineers but now a day a large of them come from the field of mathematics. It was seen that the method also equally applied to solve many other classes of problems such as fluid flow, heat flow, electric and magnetic field. Different types of problems like hyperbolic Johnson, C., Navert, U. and Pitkaranta, J., 1984, transient Kohler, W. and Pittr, J., 1974 ; Zienkiewicz, O.C., and Parekh, C.J., 1970, heat transfer Bathe, K. J., and Khoshgftaar, M. R., 1979 and nonlinear problems Bathe, K. J., and Cimento, A.P., 1980 are solved by using finite element analysis. Eigenvalue problems Fried, I, 1969; Shertzer, J, Ram-Mohan, L. R. and Dossa, D., 1989 ; Ram-Mohan, L. R., Saigal S., Dossa, D. and Shertzer, J., 1990 are to solved in connections with various applications. The problem of trigonometric interpolation was first solved by Gauss in the book Scarborough, James B., 1966, who derived several formulas similar to Hermite's. The formula usually called Gauss's formula differs from Hermite's only in having the factor ½ written in front of all the angles; thus, $\sin \tfrac{1}{2}(x-x_0)$ etc. Lagrange interpolation is a special case of Hermite interpolation. Ram-Mohan, L. R., Saigal S., Dossa, D. and Shertzer, J., 1990 solved the eigenvalue problem

$$-\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\psi}{dx}\right)-\frac{2}{x}\psi=\lambda\psi$$

where λ and ψ denotes the eigenvalue and eigenfunction respectively. Here Lagrange interpolation is used to solve the equation by finite element method. In the present work the Lagrange interpolation will be replaced by trigonometric interpolation .The main object of this work is to investigate the effect in the solution of an eigenvalue problem by using finite element method if the trigonometric interpolation is used instead of Lagrange interpolation.

2. Lagrange Interpolation and Trigonometric Interpolation Relationship

Lagrange and trigonometric interpolation shape functions have been discussed here for the linear and quadratic elements. The Trigonometric interpolation is

$$y=\frac{\tan(x-x_1)\tan(x-x_2).....\tan(x-x_n)}{\tan(x_0-x_1)\tan(x_0-x_2).....\tan(x_0-x_n)}y_0+\frac{\tan(x-x_0)\tan(x-x_2).....\tan(x-x_n)}{\tan(x_1-x_0)\tan(x_1-x_2).....\tan(x_1-x_n)}y_1+............+\frac{\tan(x-x_0)\tan(x-x_1).....\tan(x-x_{n-1})}{\tan(x_n-x_0)\tan(x_n-x_1).....\tan(x_n-x_{n-1})}y_n$$

It is evident that $y=y_0$ when $x=x_0$, $y=y_1$ when $x=x_1$, etc.

For two points (x_1,y_1) and (x_2,y_2) the trigonometric interpolation is

$$y=\frac{\tan\left(x-x_2\right)}{\tan\left(x_1-x_2\right)}y_1+\frac{\tan\left(x-x_1\right)}{\tan\left(x_2-x_1\right)}y_2=L_1y_1+L_2y_2$$

with $L_1=\frac{\tan\left(x-x_2\right)}{\tan\left(x_1-x_2\right)}$, $L_2=\frac{\tan\left(x-x_1\right)}{\tan\left(x_2-x_1\right)}$

For the purpose of integration by Gauss quadrature method the transformation is

$$x=\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi,\quad where\quad x_A=x_1,x_B=x_2$$

Then

$$L_1=\frac{\tan\left\{\frac{1}{2}\left(x_B-x_A\right)\right\}\left(1-\xi\right)}{\tan\left(x_B-x_A\right)},\quad L_2=\frac{\tan\left\{\frac{1}{2}\left(x_B-x_A\right)\right\}\left(1+\xi\right)}{\tan\left(x_B-x_A\right)}$$

For the first element $x_A=0$, $x_B=1$ and therefore

$$L_1=\frac{\tan\frac{1}{2}\left(1-\xi\right)}{\tan\left(1\right)},\quad L_2=\frac{\tan\frac{1}{2}\left(1+\xi\right)}{\tan\left(1\right)}$$

The corresponding shape function from Lagrange interpolation function is

$$N_1=\frac{1}{2}(1-\xi),\ N_2=\frac{1}{2}(1+\xi)$$

For three points $\left(x_1,y_1\right),\left(x_2,y_2\right)$, $\left(x_3,y_3\right)$ the trigonometric interpolation is

$$L_1=\frac{\tan\left(x-x_2\right)\tan\left(x-x_3\right)}{\tan\left(x_1-x_2\right)\tan\left(x_1-x_3\right)},L_2=\frac{\tan\left(x-x_1\right)\tan\left(x-x_3\right)}{\tan\left(x_2-x_1\right)\tan\left(x_2-x_3\right)}$$

$$L_3=\frac{\tan\left(x-x_2\right)\tan\left(x-x_1\right)}{\tan\left(x_3-x_2\right)\tan\left(x_3-x_1\right)}$$

Putting, $x=\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi$ where

$$x_A=x_1,\quad x_B=x_3,\quad x_2=\frac{x_1+x_3}{2}=\frac{x_A+x_B}{2}$$

$$L_1=\frac{\tan\left(\frac{x_A-x_B}{2}\xi\right)\tan\left(\frac{x_B-x_A}{2}\left(\xi-1\right)\right)}{\tan\left(\frac{x_A-x_B}{2}\right)\tan\left(x_B-x_A\right)},$$

$$L_2=\frac{\tan\left(\frac{x_B-x_A}{2}\left(1+\xi\right)\right)\tan\left(\frac{x_A-x_B}{2}\left(1-\xi\right)\right)}{\tan\left(\frac{x_B-x_A}{2}\right)\tan\left(\frac{x_A-x_B}{2}\right)}$$

$$L_3=\frac{\tan\left(\frac{x_B-x_A}{2}\xi\right)\tan\left(\frac{x_B-x_A}{2}\left(1+\xi\right)\right)}{\tan\left(x_B-x_A\right)\tan\left(\frac{x_B-x_A}{2}\right)}$$

Taking the first element having length 2 for which $x_A=0$, $x_B=2$

$$L_1=\frac{\tan\xi\tan\left(\xi-1\right)}{\tan\left(1\right)\tan\left(2\right)},\quad L_2=\frac{\tan\left(1+\xi\right)\tan\left(\xi-1\right)}{\tan\left(1\right)\tan\left(-2\right)},$$

$$L_3=\frac{\tan\xi\tan\left(1+\xi\right)}{\tan\left(1\right)\tan\left(2\right)}$$

The corresponding shape function from Lagrange interpolation function is

$$N_1=\frac{1}{2}\xi(1-\xi),\ N_2=\frac{1}{2}(1-\xi^2),\ N_3=\frac{1}{2}\xi(1+\xi)$$

3. Brief Formulation

Ram-Mohan, L. R., Saigal S., Dossa, D. and Shertzer, J., 1990 solved the following eigenvalue problem

$$-\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\psi}{dx}\right)-\frac{2}{x}\psi=\lambda\psi \qquad (1)$$

that possess the domain having limit 0 to 20 taking 20 linear elements.

The equation (1) is multiplied by the weight function *w* and integrated from 0 to 20 to get

$$\int_0^{20}\left[-\frac{d}{dx}\left(x^2\frac{d\psi}{dx}\right)-2x\psi\right]wdx=\int_0^{20}\lambda x^2\psi wdx$$

Performing the integration by parts yields the equation

$$\int_0^{20}\left(x^2\frac{dw}{dx}.\frac{d\psi}{dx}-2xw\psi\right)dx=\int_0^{20}\lambda x^2w\psi dx$$

where the boundary condition $\psi(20)=0$ is applied.

Taking the linear element from $x = x_A$ to $x = x_B$ and writing $\psi = N_1\psi_1 + N_2\psi_2$, the use of Galerkin approach $w = N_1$ and $w = N_2$, gives the elements of the stiffness matrix as $k_{ij} = \int_{x_A}^{x_B} \left(x^2 \frac{dN_i}{dx}\frac{dN_j}{dx} - 2xN_iN_j \right) dx$ with $i, j = 1, 2$.

After the substitution

$$x = \frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi = x_A \frac{1}{2}(1-\xi) + x_B \frac{1}{2}(1+\xi)$$

$$dx = \frac{h}{2} d\xi \therefore \frac{d\xi}{dx} = \frac{2}{h}$$

where, $x_B - x_A = h =$ length of an element.

$$k_{ij} = \int_{-1}^{1} \left[\left(\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi \right)^2 \right) \frac{dN_i}{d\xi}\frac{d\xi}{dx}\frac{dN_j}{d\xi}\frac{d\xi}{dx} - 2\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi \right) N_i N_j \right] \frac{h}{2} d\xi$$

the first element $x_A = 0,\ x_B = 1,\ h = 1$

The elements of the mass matrix are

$$M_{ij} = \int_{x_A}^{x_B} x^2 N_i N_j dx \text{ with } i, j = 1, 2$$

$$M_{ij} = \int_{-1}^{1} \left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi \right)^2 N_i N_j \frac{h}{2} d\xi$$

Ram-Mohan, L. R., Saigal S., Dossa, D. and Shertzer, J., 1990 used the shape function N_1 and N_2 which are obtained from Lagrange interpolation. We have used S_1 and S_2 which are obtained from trigonometric interpolation, instead of N_1 and N_2. The global stiffness matrix $\boldsymbol{K}$ and the global mass matrix $\boldsymbol{M}$ will be a 21×21 matrix.

The matrix eigenvalue equation is $\boldsymbol{K}\psi = \lambda \boldsymbol{M}\psi$. The equation is solved by Jacobi's method, to find the eigenvalues λ_1, λ_2, , λ_{20}. Eigenvalues obtained by Ram-Mohan et al, 1990 and those obtained by using trigonometric interpolation are shown in the table 1.

4. Result and Discussion

Ram-Mohan, L. R., Saigal S., Dossa, D. and Shertzer, J., 1990 used linear element and calculated the eigenvalues using Lagrange interpolation for the domain 0 to 20 taking 20 elements having length 1 for each element. We have calculated the eigenvalues using trigonometric interpolation taking same number of elements but different domain and different length for each element. The results are shown in table for comparison. The eigenvalues are calculated for the domain 0 to 20, 0 to 10 and 0 to 5 taking the length of elements 1, 0.5 and 0.25 respectively but the number of elements is 20 in each case. Ram-Mohan, L. R., Saigal S., Dossa, D. and Shertzer, J., 1990 calculated the eigenvalues using Lagrange interpolation which is shown in the second column of table 1 and the third column of table 1 shows the eigenvalues that are obtained using trigonometric interpolation. Table 2 and table 3 show the behavior of results in the case of smaller lengths of the elements.

Table 1. Eigenvalues for the domain 0 to 20 having length 1 for each element

Eigenvalues	Lagrange	Trigonometric
λ_1	12.928	16.055
λ_2	11.565	12.305
λ_3	10.786	11.824
λ_4	9.724	11.109
λ_5	8.522	10.217
λ_6	7.292	9.216
λ_7	6.110	8.165
λ_8	5.022	7.117
λ_9	4.050	6.107
λ_{10}	3.199	5.162
λ_{11}	2.462	4.298
λ_{12}	1.842	3.526
λ_{13}	1.319	2.848
λ_{14}	0.8855	2.266
λ_{15}	0.5333	1.780
λ_{16}	0.2550	1.390
λ_{17}	0.0475	1.094
λ_{18}	-0.0929	0.8961
λ_{19}	-0.2381	0.6656
λ_{20}	-0.9417	-1.334

Table 2. Eigenvalues for the domain 0 to 10 having length 0.5 for each element

Eigenvalues	Lagrange	Trigonometric
λ_1	57.483	59.868
λ_2	46.768	46.253
λ_3	43.905	43.729
λ_4	39.802	40.007
λ_5	35.072	35.602
λ_6	30.187	30.948
λ_7	25.471	26.362
λ_8	21.113	22.046
λ_9	17.206	18.114
λ_{10}	13.777	14.614
λ_{11}	10.815	11.552
λ_{12}	8.287	8.913
λ_{13}	6.155	6.668
λ_{14}	4.380	4.784
λ_{15}	2.923	3.230
λ_{16}	1.755	1.978
λ_{17}	0.851	1.007
λ_{18}	0.196	0.3061
λ_{19}	-0.220	-0.1393
λ_{20}	-0.981	-1.066

Table 3. Eigenvalues for the domain 0 to 5 having length 0.25 for each element

Eigenvalues	Lagrange	Trigonometric
λ_1	243.971	245.751
λ_2	188.434	187.462
λ_3	177.243	176.785
λ_4	160.941	161.058
λ_5	142.041	142.646
λ_6	122.491	123.439
λ_7	103.606	104.748
λ_8	86.160	87.365
λ_9	70.522	71.694
λ_{10}	56.799	57.873
λ_{11}	44.944	45.880
λ_{12}	34.826	35.607
λ_{13}	26.288	26.911
λ_{14}	19.165	19.638
λ_{15}	13.308	13.644
λ_{16}	8.585	8.805
λ_{17}	4.891	5.015
λ_{18}	2.144	2.196
λ_{19}	0.2985	0.3064
λ_{20}	-0.9623	-1.009

From table 1 it is seen that in Lagrange interpolation the first nineteen values are smaller and last value is larger than trigonometric interpolation. But from table 2 and 3 it is seen that the first value is smaller, next two values are larger and next sixteen values are smaller and last value is larger. It is noticeable that for smaller length of element the deviation between the two sets of values is comparatively small than larger length of element.

5. Conclusion

We have calculated the eigenvalues by using trigonometric interpolation for different length of elements. The result shows a better agreement for smaller values of element length. The deviation can be minimized by taking the small size element.

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