

Fig. 3. Toroidal geometry. (a) Axonometric view; (b) cross section of torus surface in $\varphi = 0, \pi$ plane with centerline radius R and pipe radius r .

Symmetry about the z-axis implies that the velocity field is independent on the coordinate φ . Thus,

$$\dot{\underline{x}} = U(\eta, \theta)\hat{\eta} + V(\eta, \theta)\hat{\theta} + W(\eta, \theta)\hat{\varphi}, \quad (6)$$

since the flow is described in the meridian plane, the velocity components U and V can be expressed in terms of the stream function $\Psi(\eta, \theta)$, which satisfy the continuity equation as:

$$\begin{aligned} \underline{V}_{\perp} = U\hat{\eta} + V\hat{\theta} &= \frac{1}{h^2 \sinh \eta} (\Psi_{,\theta} \hat{\eta} - \Psi_{,\eta} \hat{\theta}) \\ &= \nabla \wedge \left(\frac{\Psi \hat{\varphi}}{h \sinh \eta} \right), \end{aligned} \quad (7a)$$

so

$$\dot{\underline{x}} = \underline{V}_{\perp} + W\hat{\varphi} = \nabla \wedge \left(\frac{\Psi \hat{\varphi}}{h \sinh \eta} \right) + W\hat{\varphi}. \quad (7b)$$

Taking the divergence of (5a) then substituting in (4c), we get

$$\mu \nabla^2 \dot{\underline{x}} + \underline{\Delta} - \nabla p = 0. \quad (8a)$$

where

$$\underline{\Delta} = \nabla \cdot (\alpha_1 \underline{\underline{A}}_2 + \alpha_2 \underline{\underline{A}}_1^2), \quad (8b)$$

Since $\dot{\underline{x}} = \underline{V}_{\perp} + W\hat{\varphi}$ and $\underline{\Delta} = \underline{\Delta}_{\perp} + \Lambda_3 \hat{\varphi}$ with

$$\underline{V}_{\perp} = U\hat{\eta} + V\hat{\theta} \text{ and } \underline{\Delta}_{\perp} = \Lambda_1 \hat{\eta} + \Lambda_2 \hat{\theta}, \text{ then } (8a)$$

may be decomposed into the φ - component and the vector equation including the η - and θ - components as:

$$\nabla^2 (W\hat{\varphi}) + \Lambda_3 \hat{\varphi} = 0, \quad (9a)$$

$$\nabla^2 \underline{V}_{\perp} + \underline{\Delta}_{\perp} - \nabla p = 0. \quad (9b)$$

Applying the curl operation to (9b) and using (7a), we get:

$$\nabla^4 \left(\frac{\Psi \hat{\varphi}}{h \sinh \eta} \right) - \frac{\hat{\varphi}}{h^2} [\partial_{\eta} (h \Lambda_2) - \partial_{\theta} (h \Lambda_1)] = 0, \quad (9c)$$

We assume that, the torus is rotating about z-axis with constant velocity $W_s \hat{\varphi}$. Therefore the linear velocity at infinity ($\eta \rightarrow 0$) vanishes, $\dot{\underline{x}}(0, \theta) = 0$, while at the toroid surface is $\dot{\underline{x}}(\eta_s, \theta) = W_s \hat{\varphi}$. So the boundary conditions take the form:

$$W = \begin{Bmatrix} W_s \\ 0 \end{Bmatrix}, \Psi = \Psi_{,\eta} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{ at } \eta = \begin{Bmatrix} \eta_s \\ 0 \end{Bmatrix}. \quad (10)$$

4. Approximate Solution for the Velocity Field

The solution of the problem reduces to the determination of the scalar components W and Ψ such that boundary conditions (10) are satisfied. The solution is obtained by the perturbation method. This method may be summarized into the following step:

1. For small Ω the dynamical functions W and Ψ can be expanded into power series about $\Omega = 0$ [30] as

$$W = \sum_{k=1} \Omega^k W^{(k)} + O(\Omega^m), \quad (11a)$$

$$\Psi = \sum_{k=1} \Omega^k \Psi^{(k)} + O(\Omega^m), \quad (11b)$$

2. Substitution from (11) into (5a) gives an expression for the stress tensor into powers of Ω .

3. After carrying out the decomposition, the result can be substituted into the pair of equations (9a) and (9c).

4. Equating the coefficients of equal powers of Ω produces a set of successive partial differential equations for the determination of the velocity components and the stream functions in successive order.

The previous introduction to the expansion technique allows the expansion of (5a), (9a) and (9c) in the form

$$\sum_{k=1} \Omega^k \left[\tau_{=1}^{(k)} - \mu A_{=1}^{(k)} - \alpha_1 A_{=2}^{(k)} - \alpha_2 \left(A_{=1}^2 \right)^{(k)} \right] = 0 \quad (12a)$$

$$\sum_{k=1} \Omega^k \left[\nabla^2 \left(W^{(n)} \hat{\phi} \right) + \Lambda_3^{(k)} \hat{\phi} \right] = 0, \quad (12b)$$

$$\sum_{k=1} \Omega^k \left\{ \nabla^4 \left(\frac{\Psi^{(n)} \hat{\phi}}{h \sinh \eta} \right) - \frac{\hat{\phi}}{h^2} \left[\partial_\eta \left(h \Lambda_2^{(k)} \right) - \partial_\theta \left(h \Lambda_1^{(k)} \right) \right] \right\} = 0 \quad (12c)$$

The boundary conditions, (10), can be written as:

$$W^{(k)} = \left\{ \begin{matrix} W_s \delta_{1k} \\ 0 \end{matrix} \right\}, \Psi^{(k)} = \Psi_{,\eta}^{(k)} = \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} \text{ for } \eta = \left\{ \begin{matrix} \eta_s \\ 0 \end{matrix} \right\}. \quad (12d)$$

where δ_{ij} is the Kronecker delta function.

5. Solution of the First-Order Approximation

As usual, the solution of the first-order, $k=1$, produces the leading terms in the expansion of W , and Ψ . The solution of these terms represent the creeping flow around the rotating torus. The lowest order in (12) are:

$$\tau_{=1}^{(1)} - \mu A_{=1}^{(1)} = 0, \quad (13a)$$

$$\nabla^2 \left(W^{(1)} \hat{\phi} \right) = 0, \quad (13b)$$

$$\nabla^4 \left(\frac{\Psi^{(1)} \hat{\phi}}{h \sinh \eta} \right) = 0, \quad (13c)$$

with the boundary conditions:

$$W^{(1)} = \left\{ \begin{matrix} W_s \\ 0 \end{matrix} \right\}, \Psi^{(1)} = \Psi_{,\eta}^{(1)} = \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\}, \text{ for } \eta = \left\{ \begin{matrix} \eta_s \\ 0 \end{matrix} \right\}, \quad (13d)$$

The boundary conditions, (13d), imposed on (13c) implies that the only solution satisfying this boundary value problem is:

$$\Psi^{(1)} = 0. \quad (14)$$

Equation (13b) takes the form

$$\left(\nabla^2 - \frac{1}{h^2 \sinh^2 \eta} \right) W^{(1)} = 0, \quad (15a)$$

or

$$W_{,\eta\eta}^{(1)} + \frac{h}{a \sinh \eta} (1 - \cosh \eta \cos \theta) W_{,\eta}^{(1)} + W_{,\theta\theta}^{(1)} - \frac{h}{a} \sin \theta W_{,\theta}^{(1)} - \frac{W^{(1)}}{\sinh^2 \eta} = 0 \quad (15b)$$

this equation is R-separable in toroidal coordinate, details of the separated differential equations are outlined in several texts [27], [28]. Let

$$W^{(1)} = \sqrt{\cosh \eta - \cos \theta} f_1(\eta) f_2(\theta), \quad (16a)$$

so, the separated equations are:

$$(\xi^2 - 1) f_{1,\eta\eta} + 2 \xi f_{1,\eta} - \left[\frac{q^2}{\xi^2 - 1} + p^2 - \frac{1}{4} \right] f_1 = 0 \quad (16b)$$

$$f_{2,\theta\theta} + p^2 f_2 = 0, \quad (16c)$$

with $\xi = \cosh \eta$. The solutions of (16b) and (16c) are:

$$f_1(\xi) = \begin{pmatrix} P_{p-\frac{1}{2}}^q \\ Q_{p-\frac{1}{2}}^q \end{pmatrix}(\xi), \quad f_2(\theta) = \begin{pmatrix} \sin \\ \cos \end{pmatrix} p\theta. \quad (16d)$$

From (16d), the solutions are the associated Legendre function (toroidal function) of the first and second kinds $P_{p-\frac{1}{2}}(\cosh \eta)$ and $Q_{p-\frac{1}{2}}(\cosh \eta)$ with a parameter p . Therefore, the solutions are products of $\sqrt{\cosh \eta - \cos \theta} P_{p-\frac{1}{2}}(\cosh \eta)$ or $Q_{p-\frac{1}{2}}(\cosh \eta)$ and $\sin p\theta$ or $\cos p\theta$. In boundary value problems involving the flow around a torus, the parameter p is determined by the requirement that the solution be periodic in θ . Therefore, (16b) has particular solutions of the form:

$$W^{(1)} = \sqrt{\cosh \eta - \cos \theta} \sum_{p=0}^{\infty} [A_p \cos p\theta + B_p \sin p\theta] \begin{pmatrix} P_{p-\frac{1}{2}} \\ Q_{p-\frac{1}{2}} \end{pmatrix}(\cosh \eta), \quad (17a)$$

where A_p and B_p are arbitrary constants. In (17a), the upper row pertains to the interior problem ($\eta_s < \eta \leq \infty$) and the lower row to the exterior problem ($0 \leq \eta < \eta_s$), [31].

Due to the boundary conditions in (13d) and the boundedness requirements, only $P_{p-\frac{1}{2}}(\cosh \eta)$ and $\cos p\theta$ survive. Therefore, the solution can be written as:

$$W^{(1)} = \sqrt{\cosh\eta - \cos\theta} \sum_{p=0}^{\infty} A_p \frac{P_{p-\frac{1}{2}}(\cosh\eta)}{P_{p-\frac{1}{2}}(\cosh\eta_s)} \cos p\theta. \quad (17b)$$

We apply the first condition in (13d) to get A_p . One may make use of the integral

$$\int_0^{2\pi} \frac{\cos p\theta}{\sqrt{\cosh\eta - \cos\theta}} d\theta = 2\sqrt{2} Q_{p-\frac{1}{2}}(\cosh\eta), \quad (17c)$$

to show that

$$A_p = \frac{2\sqrt{2}}{\pi(1+\delta_{p0})} \frac{Q_{p-\frac{1}{2}}(\cosh\eta_s)}{P_{p-\frac{1}{2}}(\cosh\eta_s)}. \quad (17d)$$

The solution of (15b), thus, is

$$W^{(1)} = \sqrt{\cosh\eta - \cos\theta} \sum_{p=0}^{\infty} \frac{2\sqrt{2}}{\pi(1+\delta_{p0})} \frac{Q_{p-\frac{1}{2}}(\cosh\eta_s)}{P_{p-\frac{1}{2}}(\cosh\eta_s)} P_{p-\frac{1}{2}}(\cosh\eta) \cos p\theta. \quad (17e)$$

Equation (17e) represents an expression for the creeping flow around the rotating torus.

6. Second-Order Approximation

This step of approximation requires the determination of the kinematics tensors up to the order $O(\Omega^3)$.

Therefore, the contributions of $\underline{\underline{A}}_2$ and $\underline{\underline{A}}_1^2$ have to be evaluated up to the needed order. The contributions of these tensors are expressed in terms of $W^{(1)}$ which is already calculated in the first-order approximation. This step of approximation leads to partial differential equations governing the second-order terms $W^{(2)}$ and $\Psi^{(2)}$. Taking $k=2$ in (12), the coefficients of Ω^2 are:

$$\underline{\underline{\tau}}^{(2)} - \mu \underline{\underline{A}}_1^{(2)} = 0, \quad (18a)$$

$$\nabla^2(W^{(2)}\hat{\phi}) + \Lambda_3^{(1)}\hat{\phi} = 0, \quad (18b)$$

$$\nabla^4 \left(\frac{\Psi^{(2)}\hat{\phi}}{h \sinh\eta} \right) - \frac{\hat{\phi}}{h^2} \left[\partial_\eta(h\Lambda_2^{(1)}) - \partial_\theta(h\Lambda_1^{(1)}) \right] = 0, \quad (18c)$$

with the boundary conditions:

$$W^{(2)} = \Psi^{(1)} = \Psi_\eta^{(2)} = \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\}, \text{ for } \eta = \left\{ \begin{matrix} \eta_s \\ 0 \end{matrix} \right\}, \quad (18d)$$

Equation of motion up to this order of approximation requires the determination the components Λ_i , $i=1, 2, 3$ of the vector $\underline{\underline{A}}$, in (8b), up to $O(\Omega^3)$.

6.1 Calculation of $\nabla \cdot \underline{\underline{A}}_2$

Let $\nabla \dot{\underline{\underline{x}}} = \underline{\underline{L}}$ and $\dot{\underline{\underline{x}}} = W^{(1)}\hat{\phi}$ then

$$\underline{\underline{A}}_1 = \underline{\underline{L}} + \underline{\underline{L}}^T,$$

$$\underline{\underline{A}}_2 = \dot{\underline{\underline{x}}} \cdot \nabla \underline{\underline{A}}_1 + \underline{\underline{L}} \cdot \underline{\underline{A}}_1 + (\underline{\underline{L}} \cdot \underline{\underline{A}}_1)^T,$$

$$\begin{aligned} \nabla \cdot \underline{\underline{A}}_2 = & \underline{\underline{L}} \cdot \nabla \underline{\underline{A}}_1 + (\dot{\underline{\underline{x}}} \cdot \nabla) \nabla \cdot \underline{\underline{A}}_1 + \nabla^2 \dot{\underline{\underline{x}}} \cdot \underline{\underline{A}}_1 \\ & + \underline{\underline{L}}^T \cdot \nabla \underline{\underline{A}}_1 + \nabla \cdot \underline{\underline{A}}_1 \cdot \underline{\underline{L}}^T + \underline{\underline{A}}_1 \cdot \nabla \underline{\underline{L}}^T \end{aligned}$$

noting that

$$\nabla \cdot \dot{\underline{\underline{x}}} = \nabla^2(W^{(1)}\hat{\phi}) = 0,$$

therefore, $\nabla \cdot \underline{\underline{A}}_2$ becomes

$$\nabla \cdot \underline{\underline{A}}_2 = \underline{\underline{L}} \cdot \nabla \underline{\underline{A}}_1 + \underline{\underline{L}}^T \cdot \nabla \underline{\underline{A}}_1 + \underline{\underline{A}}_1 \cdot \nabla \underline{\underline{L}}^T$$

or in a more convenient form

$$\nabla \cdot \underline{\underline{A}}_2 = 2\underline{\underline{L}} \cdot \nabla \underline{\underline{L}} + \frac{1}{2} \nabla (\underline{\underline{A}}_1 : \underline{\underline{A}}_1). \quad (19a)$$

6.2 Calculation of $\nabla \cdot \underline{\underline{A}}_1^2$

$$\nabla \cdot \underline{\underline{A}}_1^2 = \nabla \underline{\underline{A}}_1 : \underline{\underline{A}}_1 + \underline{\underline{A}}_1 : \nabla \underline{\underline{A}}_1$$

since $\nabla \cdot \dot{\underline{\underline{x}}} = \nabla^2(W^{(1)}\hat{\phi}) = 0$, so $\nabla \underline{\underline{A}}_1 : \underline{\underline{A}}_1 = 0$.

Therefore, $\nabla \cdot \underline{\underline{A}}_1^2$ reduces to $\nabla \cdot \underline{\underline{A}}_1^2 = \underline{\underline{A}}_1 : \nabla \underline{\underline{A}}_1$ or

$$\nabla \cdot \underline{\underline{A}}_1^2 = 2\underline{\underline{L}} \cdot \nabla \underline{\underline{L}} + \frac{1}{4} \nabla (\underline{\underline{A}}_1 : \underline{\underline{A}}_1). \quad (19b)$$

The substitution of (19a) and (19b) into (8b) gives

$$\begin{aligned} \underline{\underline{A}}^{(1)} = & \alpha_1 \nabla \cdot \underline{\underline{A}}_2 + \alpha_2 \nabla \cdot \underline{\underline{A}}_1^2 \\ = & 2(\alpha_1 + \alpha_2) \underline{\underline{L}} \cdot \nabla \underline{\underline{L}} + \frac{1}{4} (2\alpha_1 \\ & + \alpha_2) \nabla (\underline{\underline{A}}_1 : \underline{\underline{A}}_1) + O(\Omega^3) \end{aligned} \quad (20a)$$

The expression $\underline{\underline{L}} \cdot \nabla \underline{\underline{L}}$ possesses the final form:

$$\underline{\underline{L}} \cdot \nabla \underline{\underline{L}} = h^{-2} \left[\frac{(\cosh\eta \cos\theta - 1)}{a \sinh\eta} \hat{\eta} + \frac{\sin\theta}{a} \hat{\theta} \right] F(\eta, \theta) \quad (20b)$$

with

$$F(\eta, \theta) = (W_{,\eta})^2 + (W_{,\theta})^2 + \frac{2h(\cosh\eta \cos\theta - 1)}{a \sinh\eta} WW_{,\eta} + \frac{2h \sin\theta}{a} WW_{,\theta} + \frac{1}{\sinh^2 \eta} W^2 \quad (20c)$$

The vector \underline{A} in (20a) is entirely in $\eta\theta$ -surface, so its components are:

$$A_1^{(1)} = \hat{\eta} \cdot \underline{A}^{(1)} = h^{-2} \left[\frac{(\cosh\eta \cos\theta - 1)}{a \sinh\eta} \right] F(\eta, \theta) \quad (21a)$$

$$A_2^{(1)} = \hat{\theta} \cdot \underline{A}^{(1)} = h^{-2} \left[\frac{\sin\theta}{a} \right] F(\eta, \theta), \quad (21b)$$

$$A_3^{(1)} = \hat{\phi} \cdot \underline{A}^{(1)} = 0, \quad (21c)$$

The governing equations of the second-order approximation, (18), are decomposed into the following two boundary value problems; namely

$$\nabla^2 (W^{(2)} \hat{\phi}) = 0, \quad (22a)$$

$$\nabla^4 \left(\frac{\Psi^{(2)} \hat{\phi}}{h \sinh\eta} \right) = \frac{2(\alpha_1 + \alpha_2)}{\mu h^2} \left\{ \partial_\eta \left(\frac{\sin\theta}{ah} \right) F(\eta, \theta) - \partial_\theta \left(\frac{\cosh\eta \cos\theta - 1}{ah \sinh\eta} \right) F(\eta, \theta) \right\} \hat{\phi} = 0 \quad (22b)$$

The boundary conditions, (18d), imposed on (22a) implies that the only solution satisfying this boundary value problem is trivial solution, $W^{(2)} = 0$, and the density function on the right hand side of (22b) which includes the first-order velocity $W^{(1)}(\eta, \theta)$ and its derivatives possesses a very complicated form. In fact, this expression is of the form of a double summation since $W^{(1)}(\eta, \theta)$ is given in the form of an infinite series. Therefore, the solution of the second-order system, (22), will be the subject of the second paper.

7. Surface Traction

This section is devoted to the evaluation of the surface traction at the boundary η_s . The surface traction is defined by

$$\underline{S}(\eta, \theta)|_{\eta_s} = \underline{\tau}(\eta, \theta) \cdot \hat{\eta}|_{\eta_s} = (\mu \underline{A}_1 + \alpha_1 \underline{A}_2 + \alpha_2 \underline{A}_1^2) \cdot \hat{\eta}|_{\eta_s} \quad (23)$$

where the unit vector $\hat{\eta}$ is the normal to any arbitrary surface, $\eta = \text{const}$. Hence, the surface traction is the stress vector per unit area on the surface of a toroidal shell $\eta = \eta_s$.

7.1 Determination of the Rivlin-Ericksen Tensors

The velocity field $\underline{\dot{x}}$ is defined by

$$\underline{\dot{x}} = W^{(1)}(\eta, \theta) \hat{\phi}, \quad (24)$$

therefore,

$$\nabla \underline{\dot{x}} = h^{-1} \left(\hat{\eta} \partial_\eta + \hat{\theta} \partial_\theta + \frac{\hat{\phi}}{\sinh\eta} \partial_\phi \right) (W^{(1)} \hat{\phi}) = a_{13} \hat{\eta} \hat{\phi} + a_{23} \hat{\theta} \hat{\phi} + a_{31} \hat{\phi} \hat{\eta} + a_{32} \hat{\phi} \hat{\theta} \quad (25b)$$

where

$$\left. \begin{aligned} a_{13} &= h^{-1} W_{,\eta}^{(1)} \\ a_{23} &= h^{-1} W_{,\theta}^{(1)} \\ a_{31} &= \frac{(\cosh\eta \cos\theta - 1)}{a \sinh\eta} W^{(1)} \\ a_{32} &= \frac{\sin\theta}{a} W^{(1)} \end{aligned} \right\}, \quad (25b)$$

and

$$(\nabla \underline{\dot{x}})^T = b_{13} \hat{\eta} \hat{\phi} + b_{23} \hat{\theta} \hat{\phi} + b_{31} \hat{\phi} \hat{\eta} + b_{32} \hat{\phi} \hat{\theta} \quad (26a)$$

where

$$\left. \begin{aligned} b_{13} &= \frac{(\cosh\eta \cos\theta - 1)}{a \sinh\eta} W^{(1)} \\ b_{23} &= \frac{\sin\theta}{a} W^{(1)} \\ b_{31} &= h^{-1} W_{,\eta}^{(1)} \\ b_{32} &= h^{-1} W_{,\theta}^{(1)} \end{aligned} \right\}. \quad (26b)$$

Therefore, Rivlin-Ericksen tensor $\underline{\underline{A}}_1$ is given by the expression:

$$\underline{\underline{A}}_1 = C_{13} (\hat{\eta} \hat{\phi} + \hat{\phi} \hat{\eta}) + C_{23} (\hat{\theta} \hat{\phi} + \hat{\phi} \hat{\theta}), \quad (27a)$$

where

$$\left. \begin{aligned} C_{13} = C_{31} &= h^{-1} W_{,\eta}^{(1)} + \frac{(\cosh\eta \cos\theta - 1)}{a \sinh\eta} W^{(1)} \\ C_{23} = C_{32} &= h^{-1} W_{,\theta}^{(1)} + \frac{\sin\theta}{a} W^{(1)} \end{aligned} \right\}, \quad (27b)$$

The tensor $\underline{\underline{A}}_2$ possesses the form

$$\underline{A}_2 = \dot{x} \cdot \nabla \underline{A}_1 + \underline{A}_1 \cdot (\nabla \dot{x})^T + \left[\underline{A}_1 \cdot (\nabla \dot{x})^T \right]^T, \quad (28)$$

we calculate each term separately and the final form is

$$\underline{A}_2 = Q_{11} \hat{\eta} \hat{\eta} + Q_{12} \hat{\eta} \hat{\theta} + Q_{21} \hat{\theta} \hat{\eta} + Q_{22} \hat{\theta} \hat{\theta}, \quad (29a)$$

where

$$\left. \begin{aligned} Q_{11} &= \frac{4(\cosh \eta \cos \theta - 1) W^{(1)} C_{13}}{a \sinh \eta} \\ Q_{12} = Q_{21} &= \frac{2 \sin \theta}{a} W^{(1)} C_{13} + \frac{2(\cosh \eta \cos \theta - 1) W^{(1)} C_{23}}{a \sinh \eta} \\ Q_{22} &= \frac{4 \sin \theta}{a} W^{(1)} C_{23} \end{aligned} \right\}, \quad (29b)$$

The tensor \underline{A}_1^2 is given by using (27a) as

$$\underline{A}_1^2 = G_{11} \hat{\eta} \hat{\eta} + G_{12} \hat{\eta} \hat{\theta} + G_{21} \hat{\theta} \hat{\eta} + G_{22} \hat{\theta} \hat{\theta} + G_{33} \hat{\phi} \hat{\phi}, \quad (30)$$

where

$$\left. \begin{aligned} G_{11} &= C_{13}^2 \\ G_{12} = G_{21} &= C_{13} C_{23} \\ G_{22} &= C_{23}^2 \\ G_{33} &= C_{23}^2 + C_{13}^2 \end{aligned} \right\}, \quad (30a)$$

So, the surface traction due to the tensors $\underline{A}_1, \underline{A}_2$ and

\underline{A}_1^2 is given as:

$$\begin{aligned} S(\eta, \theta) &= (\mu \underline{A}_1 + \alpha_1 \underline{A}_2 + \alpha_2 \underline{A}_1^2) \cdot \hat{\eta} \\ &= S_\eta \hat{\eta} + S_\theta \hat{\theta} + S_\phi \hat{\phi} \end{aligned} \quad (31a)$$

where

$$\left. \begin{aligned} S_\eta &= \alpha_1 Q_{11} + \alpha_2 G_{11} \\ S_\theta &= \alpha_1 Q_{21} + \alpha_2 G_{21} \\ S_\phi &= \mu C_{13} \end{aligned} \right\}, \quad (31b)$$

8. Results and Discussion

Here, we consider the problem of determining the velocity field of a second-order viscoelastic fluid due to a steady rotation of a rigid toroidal body along its symmetry axis, which is assumed to coincide with the axis Oz of the cylindrical coordinates. The solution of (13b) which satisfies the boundary conditions (13d) is given in (17e). The creeping flow is one in which fluid particles are carried in toroids eccentric with the

rotating torus surface η_s .

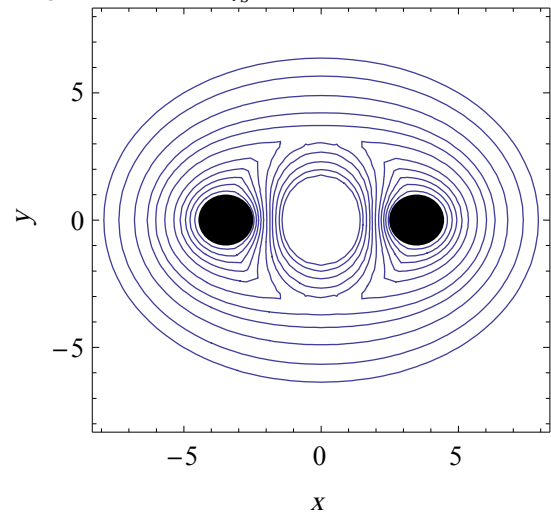


Fig. 4a. Axial velocity contours at $r = 1, R = \frac{7}{2}$.

Figures 4a, 4b and 4c display the patterns of axial velocity $W^{(1)}$ contours in the vicinity of a torus for $r = 1$ and $R = \frac{7}{2}, \frac{5}{2}$ and $\frac{3}{2}$, respectively. The contours near the torus surface, $\eta \rightarrow \eta_s$, are completely different from that in the core flow, $\eta \rightarrow 0$, (the velocity approaches zero in the core flow). It depends on η only in the core and on η and θ in the neighborhood of the wall. Therefore, such fluid exhibits boundary layer effects.

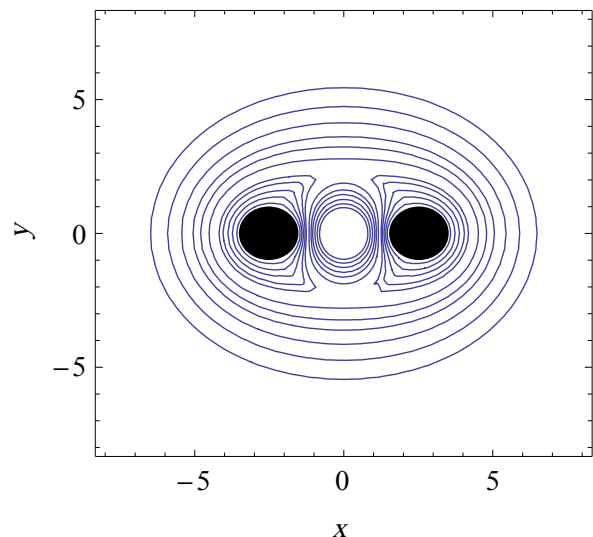


Fig. 4b. Axial velocity contours at $r = 1, R = \frac{5}{2}$.

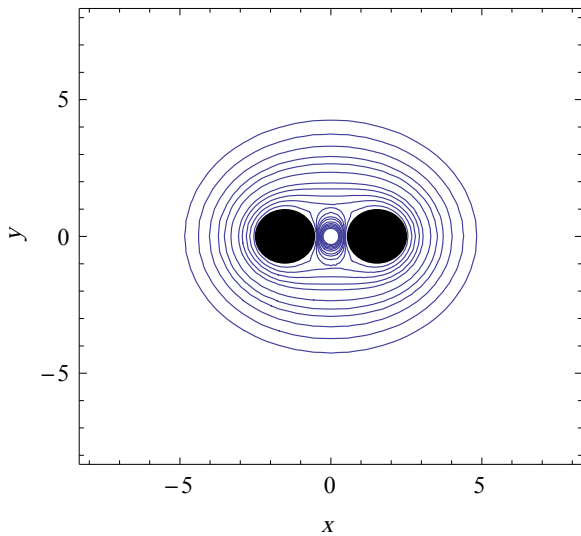


Fig. 4c. Axial velocity contours at $r = 1$, $R = \frac{3}{2}$.

The behavior of $W^{(1)}$ as a function of the coordinate η at $R = \frac{7}{2}$, $\frac{5}{2}$ and $\frac{3}{2}$ is shown in Figs. 5a, 5b and 5c respectively. It is clear that, all curves satisfy the boundary condition at the torus surface $\eta = \eta_s$. As $\eta \rightarrow 0$, the core velocities are everywhere reduce with increasing θ .

Figures 6 show the behavior of $W^{(1)}$ as a function of θ for different values of η . It is clear that, $W^{(1)}$ increases with increasing θ until reaching maxima at $\theta = \pi$ then decreases as $\theta \rightarrow 2\pi$. As the value of η decreases (core flow), the maxima decrease. The same behavior is observed for different values of toroidal geometrical parameter R . To give more insight about the velocity fields around the torus, the axial velocity $W^{(1)}$ is plotted in three dimensional graphs as in Fig.7.

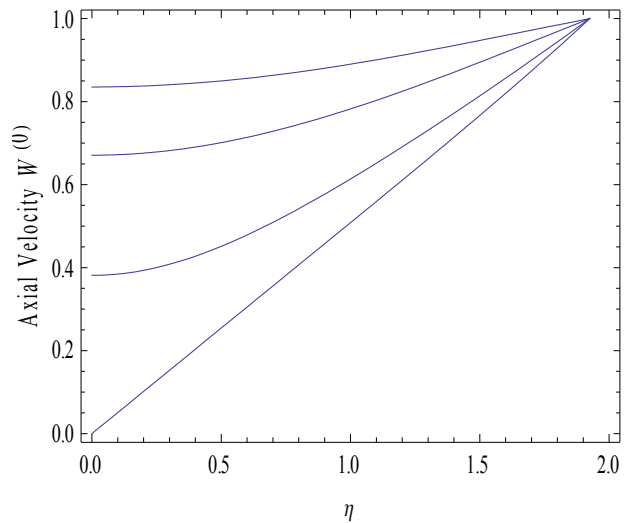


Fig. 5a. Axial velocity $W^{(1)}(\eta, \theta)$ versus η at $R = 7/2$, where θ is taken as a parameter $\theta = 0, \pi/4, \pi/2, 3\pi/4$ (bottom to top).

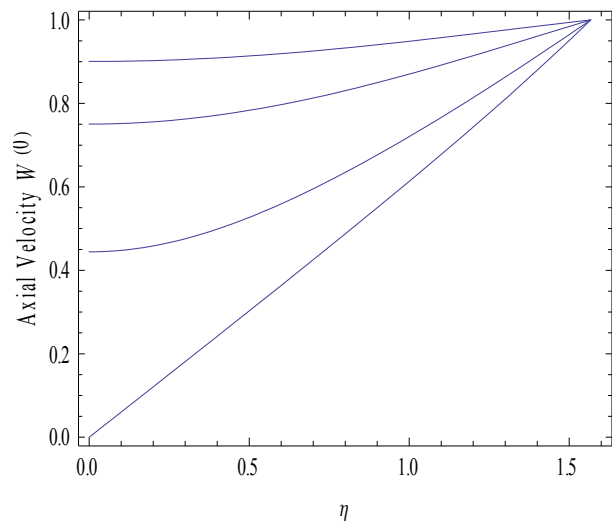


Fig. 5b. Axial velocity $W^{(1)}(\eta, \theta)$ versus η at $R = 5/2$, where θ is taken as a parameter $\theta = 0, \pi/4, \pi/2, 3\pi/4$ (bottom to top).

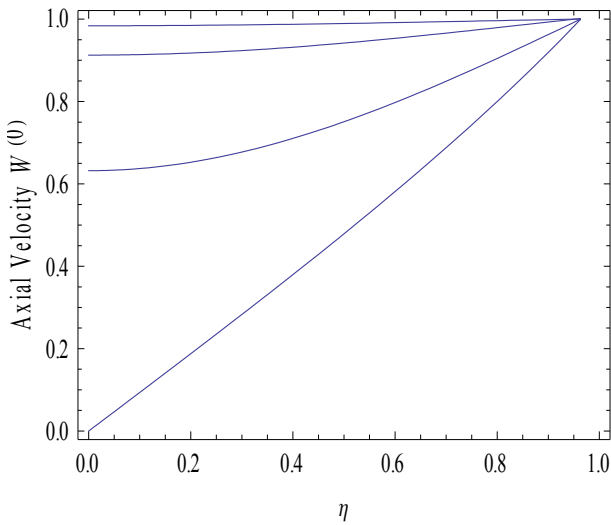


Fig. 5c. Axial velocity $W^{(1)}(\eta, \theta)$ versus η at $R = 3/2$, where θ is taken as a parameter $\theta = 0, \pi/4, \pi/2, 3\pi/4$ (bottom to top).

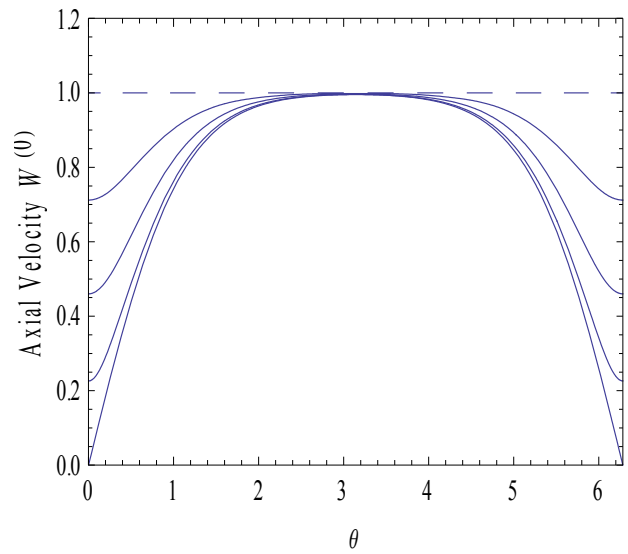


Fig. 6c. Axial velocity $W^{(1)}(\eta, \theta)$ versus θ at $R = 3/2$, where η is taken as a parameter $\eta = \eta_s, 3\eta_s/4, \eta_s/2, \eta_s/4, 0$ (top to bottom).

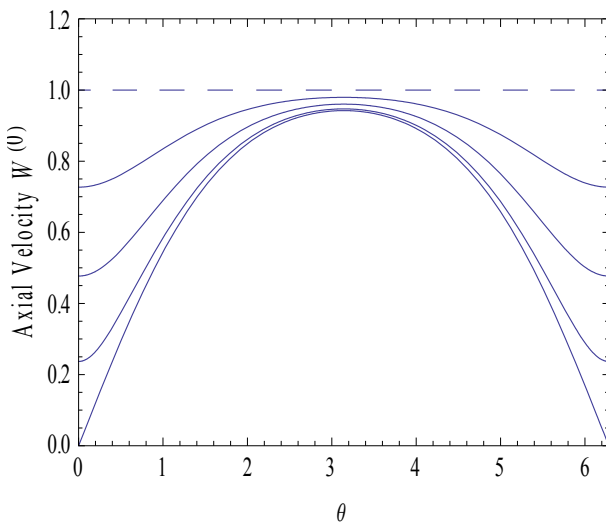


Fig. 6b. Axial velocity $W^{(1)}(\eta, \theta)$ versus θ at $R = 5/2$, where η is taken as a parameter $\eta = \eta_s, 3\eta_s/4, \eta_s/2, \eta_s/4, 0$ (top to bottom).

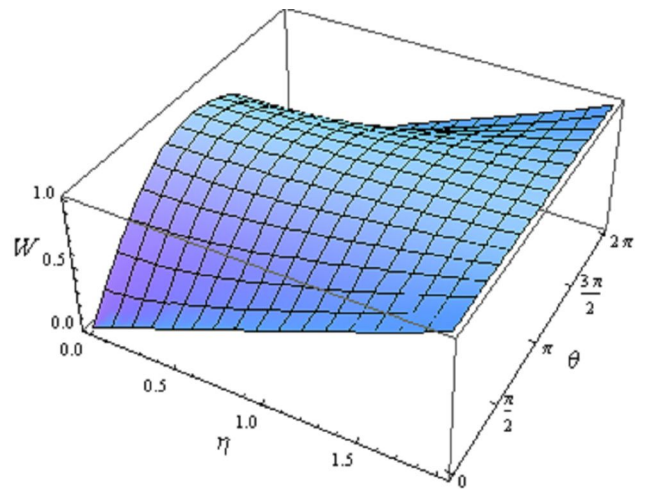


Fig. 7. Three dimensional view for the axial velocity $W^{(1)}(\eta, \theta)$.

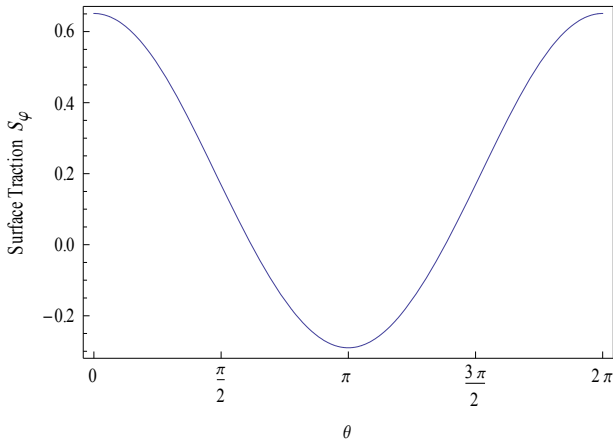


Fig. 8. ϕ -component of the surface traction S_ϕ versus θ at $\eta = \eta_s$ and $R = 7/2$.

8. Conclusion

The viscoelastic flow around axisymmetric rotation of a rigid torus in an unbounded second order viscoelastic fluid is investigated. The problem is formulated and solved within the frame of slow flow approximation using retarded motion approximation. The equations of motion using the bipolar toroidal coordinate system are formulated. The first order velocity field are determined. The first order velocity component $W^{(1)}(\eta, \theta)$ which lies in the direction of the ϕ -coordinate representing the Newtonian flow is obtained but the first order stream function $\psi^{(1)}$ is a vanishing term. The equations of motion of a second order are formulated. The solution of the second order indicates that the axial velocity is vanished while the only nonvanishing term is the second order stream function which will be the subject of a second paper of this series of papers. Laplace's equation of first order velocity $W^{(1)}(\eta, \theta)$ is solved via the usual method of separation of variables. This method shows that, the solution is given in a form of infinite sums over Legendre functions of the first kind. From the obtained solution it is found that, the leading term, $W^{(1)}(\eta, \theta)$ of the velocity. The second-order term shows to be a stream function, $\Psi^{(2)}(\eta, \theta)$, which describes a secondary flow in $\eta\theta$ -plane superimposed on the primary flow. The distribution of surface

traction which represent the stress vector per unit area at the toroid surface are calculated and discussed.

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