International Journal of Scientific Engineering and Applied Science (IJSEAS) – Volume-2, Issue-12,December 2016
ISSN: 2395-3470
www.ijseas.com

A method to solve the equations of the second degree in the general case (with complex coefficients)

Dr. Aziz ARBAI
Department of Mathematics, laboratory of algebra and theory of operators
University of Abdelmalek Essaadi, Morocco
arbai_aziz@yahoo.com

Abstract

In this paper we will propose a direct method which is accurate and fast to solve the equations of the second degree in the general case (which means with complex coefficients). we will also propose an algorithm that will make our calculations more easy.

Keywords: *Complex, Equations of the second degree, Algorithm, The second root*

Let the equation:

$$az^2 + bz + c = 0 \tag{1}$$

with $(a, b, c) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$

I. Recall

See [1].

$$a \neq 0 \Longrightarrow z^2 + \frac{b}{a}z + \frac{c}{a} = 0$$

$$\Longleftrightarrow z^2 + 2\frac{b}{2a}z + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\Longleftrightarrow \left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{2}$$

Let $\Delta \in \mathbb{C}$ such that $\Delta = b^2 - 4ac$.
We have two cases:

1. First case: If $\Delta \in \mathbb{R}$
   If $\Delta \in \mathbb{R}^+$,
   so $z = \frac{-b \pm \sqrt{\Delta}}{2a}$.

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and if $\Delta \in \mathbb{R}^-$,
so $z = \frac{-b \pm i\sqrt{-\Delta}}{2a}$, with i such that $i^2 = -1$.
we can generalize:

$$z = \frac{-b \pm i^{\frac{1-signe(\Delta)}{2}}\sqrt{|\Delta|}}{2a} \quad \text{if } \Delta \in \mathbb{R}$$

2. Second case: If $\Delta \notin \mathbb{R}$

   $\Delta \notin \mathbb{R}$ means that $\Delta \in \mathbb{C} \backslash \mathbb{R}$
   so $\delta \in \mathbb{C}$ such that
   $\delta^2 = \Delta = b^2 - 4ac$
   In $\mathbb{C}$, δ still exists. It's the second root (or the square root) of Δ.

   If δ is a second root of Δ, then $-\delta$ is also, because $(-\delta)^2 = \delta^2 = \Delta$.
   So, δ and $-\delta$ are the only two roots of the equation $X^2 = \Delta$.
   Hence, we simply find one of the roots: δ or $-\delta$.
   Therefore,

$$(2) \iff \left(z + \frac{b}{2a}\right)^2 = \frac{\delta^2}{4a^2}$$

$$\iff z + \frac{b}{2a} = \pm\frac{\delta}{2a}$$

Conclusion:

$$z = \frac{-b \pm \delta}{2a}$$

with δ a second root of Δ.

Remark 1. *When we write Δ in the exponential or geometric form, then to determine δ as $\Delta = re^{i\theta}$ or $\Delta = r(\cos(\theta) + i\sin(\theta))$ we just take $\delta = \pm\sqrt{r}e^{i\frac{\theta}{2}}$*

Remark 2. *It is not easy to determinate δ in function of Δ especially when we cannot write it in the exponential or the geometric form.*

II. Contribution

In this section, we will solve the problem cited in Remark 2.
We suppose that $\Delta \in \mathbb{C} \backslash \mathbb{R}$ and we cannot write Δ in the exponential or the geometric form.

Hence, $\exists (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^* \ / \ \Delta = \alpha + i\beta$ (which means Δ is a complex not real with $\beta \neq 0$).

Thereafter, we write δ in the algebraic form $\delta = x + iy$ and we try to find a couple $(x, y) \in \mathbb{R}^2$ such that $(x + iy)^2 = \alpha + i\beta$.

This step is a bit long, especially with the distressing calculations that repeats every time when practical work, specially when it comes to teach students how to find δ by hand.

This article aims to find once and for all δ in function of Δ and therefore result in a direct algorithm, exact and very fast for us to determine the solutions of (1).

i. Algorithm

$\rightarrow a, b, c$
$\rightarrow \Delta = b^2 - 4ac$

$\rightarrow if\ \Delta \in \mathbb{R}\ so\ z = \frac{-b \pm i^{\frac{1-signe(\Delta)}{2}} \sqrt{|\Delta|}}{2a} \quad end$

$\rightarrow else \quad \rightarrow \delta = \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}$

$\quad \rightarrow z = \frac{-b \pm \delta}{2a} \quad end$

with
$\|\Delta\| = \sqrt{Re(\Delta)^2 + Im(\Delta)^2}$ (Δ's module)
$Re(\Delta)$ (The real part of Δ)
$Im(\Delta)$ (the imaginary part of Δ).

So we have two important operations when $\Delta \in \mathbb{C} \backslash \mathbb{R}$:

1. To prove that $\delta^2 = \Delta$ when $\delta = \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}$.
2. How to find this "magic" formula? (optional but very interesting when we talk about pedagogy).

ii. Proof

We have $\delta^2 = \left(\frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}\right)^2 = \frac{(\|\Delta\|+\Delta)^2}{2(\|\Delta\|+Re(\Delta))}$

so $\delta^2 = \frac{\alpha^2+\beta^2+(\alpha+i\beta)^2+2(\alpha+i\beta)\sqrt{\alpha^2+\beta^2}}{2(\sqrt{\alpha^2+\beta^2}+\alpha}$

hence $\delta^2 = \frac{\alpha^2+\beta^2+\alpha^2+2i\alpha\beta-\beta^2+2(\alpha+i\beta)\sqrt{\alpha^2+\beta^2}}{2(\sqrt{\alpha^2+\beta^2}+\alpha)}$

$\Leftrightarrow \delta^2 = \frac{\alpha^2+i\alpha\beta+(\alpha+i\beta)\sqrt{\alpha^2+\beta^2}}{(\sqrt{\alpha^2+\beta^2}+\alpha)}$

$\Leftrightarrow \delta^2 = \frac{\alpha(\alpha+i\beta)+(\alpha+i\beta)\sqrt{\alpha^2+\beta^2}}{\sqrt{\alpha^2+\beta^2}+\alpha}$

$\Leftrightarrow \delta^2 = \frac{(\alpha+i\beta)\left[\alpha+\sqrt{\alpha^2+\beta^2}\right]}{\sqrt{\alpha^2+\beta^2}+\alpha}$

$\Leftrightarrow \delta^2 = (\alpha + i\beta)$

$\Leftrightarrow \delta^2 = \Delta$

Of course, we do not forget to check that

$\|\Delta\| + Re(\Delta) > 0$ (for $\frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}$ to be defined).

In fact,

$\|\Delta\| + Re(\Delta) = \sqrt{\alpha^2 + \beta^2} + \alpha$

We have $\sqrt{\alpha^2 + \beta^2} \geq \sqrt{\alpha^2} = |\alpha| \geq -\alpha$ (it's obvious)

so $\sqrt{\alpha^2 + \beta^2} \geq -\alpha \Longrightarrow \sqrt{\alpha^2 + \beta^2} + \alpha \geq 0$

Now, we suppose that $\sqrt{\alpha^2 + \beta^2} + \alpha = 0$

$\Longleftrightarrow \sqrt{\alpha^2 + \beta^2} = -\alpha$

$\Longrightarrow \alpha^2 + \beta^2 = \alpha^2$

$\Longrightarrow \beta^2 = 0 \Longrightarrow \beta = 0$

β is the imaginary part of Δ

so

$\beta = 0 \Longrightarrow \Delta = \alpha \in \mathbb{R}$

since $\Delta \notin \mathbb{R}$ ($\Delta \in \mathbb{C}\backslash\mathbb{R}$)

absurd!

hence $\sqrt{\alpha^2+\beta^2} + \alpha \neq 0$

Conclusion

$\|\Delta\| + Re(\Delta) = \sqrt{\alpha^2+\beta^2} + \alpha > 0.$

■

iii. The "How?"

As we said in the beginning, to find δ, $\delta^2 = \Delta$ we have to find $(x, y) \in \mathbb{R}^2 /\ (x+iy)^2 = \alpha + i\beta$

$\Longleftrightarrow x^2 + 2ixy - y^2 = \alpha + i\beta$

$$\Longleftrightarrow \begin{cases} x^2 - y^2 & = \alpha \\ 2xy & = \beta \end{cases}$$

$\Longrightarrow 4x^2y^2 = \beta^2$ and $y^2 = x^2 - \alpha$

$\Longrightarrow 4x^2(x^2 - \alpha) = \beta^2$

$$\Longrightarrow x^4 - \alpha x^2 - \frac{\beta^2}{4} = 0 \tag{3}$$

(3) is a polynomial equation of degree 4 but we quickly notice that it also falls within the second degree by taking as variable $X = x^2$ (called Bi-square equation).

Hence,

(3) $\Longleftrightarrow X^2 - \alpha X - \frac{\beta^2}{4} = 0$ and $X = x^2$

so we have

$\Delta_X = \|\Delta\|^2 = \alpha^2 + \beta^2$ $(= (-\alpha)^2 - 4(-\frac{\beta^2}{4}))$

$(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^* \Longrightarrow \Delta_X \in \mathbb{R}_+^*$

$\Longrightarrow X = \frac{\alpha \pm \sqrt{\Delta_X}}{2} = \frac{\alpha \pm \sqrt{\alpha^2+\beta^2}}{2}$

$\Longrightarrow X = \frac{Re(\Delta) \pm \|\Delta\|}{2}.$

we have already shown that

$Re(\Delta) + \|\Delta\| > 0$

As the same way we show that

$$Re(\Delta) - \|\Delta\| < 0 \tag{4}$$

because on one hand we have:

$Re(\Delta) = \alpha \leq |\alpha| = \sqrt{\alpha^2} \leq \sqrt{\alpha^2+\beta^2} = \|\Delta\|,$

On the other hand,
If
$Re(\Delta) - \|\Delta\| = 0$,
then
$\alpha = \sqrt{\alpha^2 + \beta^2}$

$\Longrightarrow \alpha^2 = \alpha^2 + \beta^2 \Longrightarrow \beta = 0 \Longrightarrow \Delta \in \mathbb{R}$ absurd!
Therefore
$Re(\Delta) - \|\Delta\| \neq 0$

hence
$Re(\Delta) - \|\Delta\| < 0$

As $X = x^2 \geq 0$,
then
$X = \frac{Re(\Delta)+\|\Delta\|}{2}$
hence

$$x = \pm\sqrt{\frac{Re(\Delta) + \|\Delta\|}{2}}$$

Now we have to determine y by respecting our constraints:

$$\begin{cases} x^2 - y^2 &= \alpha \\ 2xy &= \beta \end{cases} \Longleftrightarrow \begin{cases} (3) \\ y^2 = x^2 - \alpha \\ signe(xy) = signe(\beta) \end{cases}$$

$\Longrightarrow y^2 = X - \alpha = \frac{Re(\Delta)+\|\Delta\|}{2} - Re(\Delta)$

$\Longrightarrow y^2 = \frac{\|\Delta\|-Re(\Delta)}{2} > 0$

and $\frac{\|\Delta\|-Re(\Delta)}{2} > 0$ (according to (4))
$\Longrightarrow$

$$y = \pm\sqrt{\frac{\|\Delta\| - Re(\Delta)}{2}}$$

Therefore, we have the two couples of solution:
$\delta^2 = \Delta$

$$\begin{cases} x = \sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}} \\ y = signe(\beta)\sqrt{\frac{\|\Delta\|-Ret(\Delta)}{2}} \end{cases}$$

Or

$$\begin{cases} x = -\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}} \\ y = -signe(\beta)\sqrt{\frac{\|\Delta\|-Re(\Delta)}{2}} \end{cases}$$

hence,

$$\delta = \pm(\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}} + signe(\beta)i\sqrt{\frac{\|\Delta\|-Re(\Delta)}{2}})$$

$$\Longleftrightarrow$$

$$\delta = \pm\frac{(\frac{\|\Delta\|+Re(\Delta)}{2} + signe(\beta)i\sqrt{\frac{\|\Delta\|^2-Re(\Delta)^2}{4}})}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}$$

(Multiplying and dividing by $\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}$ which is different from zero)

$$\Longleftrightarrow$$

$$\delta = \pm \frac{(\frac{\|\Delta\|+Re(\Delta)}{2} + signe(\beta) i \sqrt{\frac{\alpha^2+\beta^2-\alpha^2}{4}})}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}$$

$$\Longleftrightarrow$$

$$\delta = \pm \frac{1}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}} (\frac{\|\Delta\| + Re(\Delta)}{2} + signe(\beta) i \frac{|\beta|}{2})$$

but

$$signe(\beta)|\beta| = \beta = Im(\Delta)$$

then,

$$\delta = \pm \frac{1}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}} (\frac{\|\Delta\| + Re(\Delta) + iIm(\Delta)}{2})$$

Conclusion:

$$\delta = \pm \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}$$

■

III. Examples

i. Example 1:

To solve in $\mathbb{C}$ the equation:

$$4iz^2 + 2(1+6i)z + 2(-1+7i) = 0 \quad (5)$$

We will simplify the equation (5):
$2iz^2 + (1+6i)z + (-1+7i) = 0$

We have $\Delta = (1+6i)^2 - 8i(-1+7i)$

$\Delta = 1 - 36 + 12i + 8i + 56$

$\Delta = 21 + 20i$ (It is impossible to write it in the exponential or the geometric form).

By applying our formula:

$$\delta = \pm \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}$$

we have

$\|\Delta\| = \sqrt{(21)^2 + (20)^2} = \sqrt{441+400} = \sqrt{841}$

then

$\|\Delta\| = 29$

therefore,

$$\delta = \pm \frac{\frac{29+21+20i}{2}}{\sqrt{\frac{29+21}{2}}} = \pm \frac{25+10i}{\sqrt{25}}$$

$\Longrightarrow$

$$\delta = \pm(5+2i)$$

$\Longrightarrow$

$$z = \frac{-(1+6i) \pm (5+2i)}{4i}$$

Conclusion:

$$S = \left\{-1-i, -2+\frac{3}{2}i\right\}$$

We can verify that:

$4i\,[z-(-1-i)]\,[z-(-2+\frac{3}{2}i)] = 4iz^2 + 2(1+6i)z + 2(-1+7i)$.

ii. Example 2:

To solve in $\mathbb{C}$ the equation:

$$z^2 - (2+i)z + 2i = 0$$

We have $\Delta = (2+i)^2 - 4 \times 2i = 4 - 1 + 4i - 8i$

$\Longrightarrow$

$$\Delta = 3 - 4i$$

$\Longrightarrow \|\Delta\| = \sqrt{9+16}$

$\Longrightarrow \|\Delta\| = 5$

so

$$\Delta = \pm\frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}} = \pm\frac{\frac{5+3-4i}{2}}{\sqrt{\frac{5+3}{2}}}$$

$\Longrightarrow$

$$\Delta = \pm\frac{4-2i}{2} = \pm(2-i)$$

$\Longrightarrow$

$$z = \frac{(2+i) \pm (2-i)}{2}$$

$\Longrightarrow$

$$S = \{i, 2\}$$

We can verify that

$$(z-i)(z-2) = z^2 - (2+i)z + 2i$$

.

IV. Conclusion

1. Ultimately,

- We gave all the fast and efficient methods and formulas to solve an equation with complex coefficients.
- We found the general formula that gives us the second roots (square root) of a complex number.

So, let $\alpha \in \mathbb{C} \backslash \mathbb{R}$,
then

$$X^2 = \alpha \Longleftrightarrow X = \pm \frac{\frac{\|\alpha\|+\alpha}{2}}{\sqrt{\frac{\|\alpha\|+Re(\alpha)}{2}}}$$

2. It remains to find δ in function of the coefficients a, b, c (directly from the coefficients of the equation like the usual $\Delta = b^2 - 4ac$).

References

[1] A. Arbai. *Principes d'algèbre et de Géométrie.* Alkhalij Al Arabi- Tetouan - 2éme édition, 2015.