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Estimation of the parameters of the three-parameter distribution

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Abstract

In this paper, we will see some alternative estimation methods for the multidimensional lognormal distribution three-parametric as the method of maximum likelihood, the median method, the method of the straight of Wicksell and two different method of the maximum of likelihood amended. In the case of stochastic lognormal three-parametric univariate process, we will see two alternative methods of estimation: The method of moments and the estimation of maximum likelihood revised.

Keywords: *Three-parameter lognormal diffusion process; maximum likelihood; alternative methods*

I. Introduction

Given the importance of the lognormal distribution as to its applications, the problem of estimating the parameters of the lognormal distribution from a given sample is a large problem that has been addressed by several investigators. This has gone through the theoretical and computer problems that appeared during the application of the estimation method of the maximum likelihood. Therefore these difficulties, several alternative estimation methods have appeared. Most of them have been discussed by Aitchison and Brown [1],Calitz [6] and Cohen [7] [8] [9] and others more recent between which, we will mention Giesbrecht and Kempthorne [12], Wingo [21] [22], Lifson and Bhattacharyya [17], Kappenman [15] and Arbai [2]. The one-dimensional case was investigated by Brown and Hewitt [5]. The case of multidimensional diffusion was studied by Basawa and Prakasa Rao [3]. These authors have used the continuous sample , and by method of the maximum likelihood they found the corresponding estimators. The case of lognormal process with exogenous factors was considered by Molina [18].

II. Alternative methods of estimation in multivariate distribution with three-parameters

Let $X = (X_1, \cdots, X_k)'$ a random vector of multivariate three-parametric lognormal $\Lambda_k(\gamma, \mu, B)$.
The density of X is:

$$f(x;\gamma,\mu,B) = ((2\pi)^{\frac{k}{2}} \prod_{i=1}^{k}(x_i - \gamma_i) \mid B \mid^{\frac{1}{2}})^{-1} \times exp\{-\frac{1}{2}(ln(x-\gamma)-\mu)'B^{-1}(ln(x-\gamma)-\mu)\},$$

for $x_i > \gamma_i$ and B a symmetric matrix positive definite such that $\sigma_{ii} > 0$

i. The method of maximum likelihood

Let $\{x^j\}$ a sample formed by n observations , $1 \leq j \leq n$ such that: $x^j = (x_{1j}, \cdots, x_{kj})'$
The likelihood function is:

$$L(x^1, \cdots x^n, \gamma, \mu, B) = \prod_{j=1}^{n} f(x^j; \gamma, \mu, B)$$

with

$$f(x^j;\gamma,\mu,B) = ((2\pi)^{\frac{k}{2}} \prod_{i=1}^{k}(x_{ij} - \gamma_i) \mid B \mid^{\frac{1}{2}})^{-1} \times exp\{-\frac{1}{2}(ln(x^j-\gamma)-\mu)'B^{-1}(ln(x^j-\gamma)-\mu)\}$$

then

$$\begin{aligned} L(x^1, \cdots x^n, \gamma, \mu, B) &= \prod_{j=1}^{n}((2\pi)^{\frac{k}{2}} \prod_{i=1}^{k}(x_{ij} - \gamma_i) \mid B \mid^{\frac{1}{2}})^{-1} \times exp\{-\frac{1}{2}(ln(x^j-\gamma)-\mu)'B^{-1}(ln(x^j-\gamma)-\mu)\} \\ &= (2\pi)^{-\frac{nk}{2}} \mid B \mid^{-\frac{n}{2}} \prod_{j=1}^{n}\prod_{i=1}^{k}(x_{ij}-\gamma_i)^{-1} \times \prod_{j=1}^{n} exp\{-\frac{1}{2}(ln(x^j-\gamma)-\mu)'B^{-1}(ln(x^j-\gamma)-\mu)\} \end{aligned}$$

hence,

$$lnL(x^1, \cdots x^n, \gamma, \mu, B) = -\frac{n}{2}kln(2\pi) - \frac{n}{2}ln \mid B \mid - \sum_{j=1}^{n}\sum_{i=1}^{k} ln(x_{ij}-\gamma_i) - \frac{1}{2}tr\{\sum_{j=1}^{n}(ln(x^j-\gamma)-\mu)'B^{-1}(ln(x^j-\gamma)-\mu)\}$$

We derive:

$$\begin{aligned} dlnL(x^1, \cdots x^n, \gamma, \mu, B) = &-\frac{n}{2}tr(B^{-1}dB) + \sum_{j=1}^{n}\sum_{i=1}^{k}\frac{1}{x_{ij}-\gamma_i}d\gamma_i + \frac{1}{2}tr\{\sum_{j=1}^{n}[(ln(x^j-\gamma)-\mu)'B^{-1}dBB^{-1}(ln(x^j-\gamma)-\mu) \\ &+ 2(ln(x^j-\gamma)-\mu)'B^{-1}d\mu - 2(ln(x^j-\gamma)-\mu)'B^{-1}W_j d\gamma]\} \end{aligned}$$

where

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$$W_j = \frac{dln(x^j-\gamma)}{d\gamma} = \begin{pmatrix} -\frac{1}{x_{1j}-\gamma_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\frac{1}{x_{kj}-\gamma_k} \end{pmatrix}$$

$$\begin{aligned} dlnL(x^1,\cdots x^n,\gamma,\mu,B) &= \frac{1}{2}tr\{\sum_{j=1}^{n}[B^{-1}(ln(x^j-\gamma)-\mu)(ln(x^j-\gamma)-\mu)'-I_k]B^{-1}dB + 2\sum_{j=1}^{n}d\mu(ln(x^j-\gamma)-\mu)'B^{-1}\} \\ &- \sum_{j=1}^{n}(ln(x^j-\gamma)-\mu)'B^{-1}W_j d\gamma + \sum_{j=1}^{n}\sum_{i=1}^{k}\frac{1}{x_{ij}-\gamma_i}d\gamma_i \\ &= \frac{1}{2}Vec(B^{-1}\sum_{j=1}^{n}[(ln(x^j-\gamma)-\mu)(ln(x^j-\gamma)-\mu)'B^{-1}-I_k])'dVec(B) \\ &+ [B^{-1}\sum_{j=1}^{n}(ln(x^j-\gamma)-\mu]'d\mu - \sum_{j=1}^{n}(ln(x^j-\gamma)-\mu)'B^{-1}W_j d\gamma + \sum_{j=1}^{n}\sum_{i=1}^{k}\frac{1}{x_{ij}-\gamma_i}d\gamma_i \end{aligned}$$

Thereafter , the maximum likelihood equations are:

$$\widehat{B}^{-1}\sum_{j=1}^{n}[(ln(x^j-\widehat{\gamma})-\widehat{\mu})(ln(x^j-\widehat{\gamma})-\widehat{\mu})'\widehat{B}^{-1}-I_k] = 0$$

$$\widehat{B}^{-1}\sum_{j=1}^{n}(ln(x^j-\widehat{\gamma})-\widehat{\mu}) = 0$$

$$\sum_{j=1}^{n}[\widehat{W}_j\widehat{B}^{-1}(ln(x^j-\widehat{\gamma})-\widehat{\mu})+u_j] = 0$$

where $u_j = \left(\frac{1}{x_{1j}-\widehat{\gamma}_1},\cdots,\frac{1}{x_{kj}-\widehat{\gamma}_k}\right)'$

Finally , maximum likelihood estimators are determined from:

$$\widehat{\mu} = \frac{1}{n}\sum_{j=1}^{n}ln(x^j-\widehat{\gamma}) \tag{1}$$

$$\widehat{B} = \frac{1}{n}\sum_{j=1}^{n}ln(x^j-\widehat{\gamma})-\widehat{\mu})(ln(x^j-\widehat{\gamma})-\widehat{\mu})' \tag{2}$$

$$\sum_{j=1}^{n}\widehat{W}_j\widehat{B}^{-1}(ln(x^j-\widehat{\gamma})-\widehat{\mu}) + \sum_{j=1}^{n}u_j = 0 \tag{3}$$

with

$$\widehat{W}_j = \begin{pmatrix} -\dfrac{1}{x_{1j}-\widehat{\gamma_1}} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\dfrac{1}{x_{kj}-\widehat{\gamma}_k} \end{pmatrix}$$

ii. Other alternative methods

We suppose that X is $\Lambda_k(\gamma, \mu, B)$, so $ln(X-\gamma)$ is $N_k(\mu, B)$.
Thus $ln(X_i - \gamma_i)$ is $N_k(\mu_i, \sigma_{ii})$ and X_i is $\Lambda_k(\gamma_i, \mu_i, \sigma_i)$
with $\sigma_i^2 = \sigma_{ii}$

ii.1 First method of maximum likelihood amended

In this method, we will replace the third equation of the maximum likelihood (3) by a statistical function of rank 1 . The order statistic of rank 1 contains more information on the threshold parameter.
For this, we will consider the following statistic function:

$$E\left[ln(X_{i_{(1)}} - \gamma_i)\right] = ln(X_{i_{(1)}} - \gamma_i) \ for \ i = 1, \cdots, k$$

thus

$$\gamma_i + exp(\mu_i + \sigma_i E(Z_{1,n})) = x_{i_{(1)}}$$

with

$$X_{i_{(1)}} = \min_{1\le j\le n}\{X_{ij}\}$$

and

$$x_{i_{(1)}} = \min_{1\le j\le n}\{x_{ij}\}$$

$Z_{1,n}$ is the order statistic of rank 1 of n independent random variables according to the normal distribution $N(0,1)$.
Finally , the estimators can be determined from the following system:

$$\begin{cases} \widehat{\mu} = \frac{1}{n}\sum_{j=1}^{n} ln(x^j - \widehat{\gamma}) \\ \widehat{B} = \frac{1}{n}\sum_{j=1}^{n}(ln(x^j - \widehat{\gamma}) - \widehat{\mu})(ln(x^j - \widehat{\gamma}) - \widehat{\mu})' \\ \widehat{\gamma}_i + exp(\widehat{\mu}_i + \widehat{\sigma}_i E(Z_{1,n})) = x_{i_{(1)}} \ for \ i = 1\cdots, k \end{cases}$$

with $x^j = (x_{1j}, \cdots, x_{kj})'$ and $x_{i_{(1)}} = \min_{1\le j\le n}\{x_{ij}\}$

ii.2 Second method of maximum likelihood amended

Same as the previous method, we will replace the third equation of the maximum likelihood (3) by the following statistical function of rank 1:

$$E\left[F(X_{i_{(1)}})\right] = F(x_{i_{(1)}}) \; for \; i = 1, \cdots, k$$

we obtain

$$E\left[F(X_{i_{(1)}})\right] = \frac{1}{n+1}$$

and

$$F(x_{i_{(1)}}) = \Phi\left(\frac{ln(x_{i_{(1)}} - \gamma_i) - \mu_i}{\sigma_i}\right)$$

where Φ is the distribution function of the standard normal distribution.
Thus, we obtain

$$\Phi\left(\frac{ln(x_{i_{(1)}} - \gamma_i) - \mu_i}{\sigma_i}\right) = \frac{1}{n+1}$$

Finally

$$\gamma_i + exp\left(\mu_i + \sigma_i \Phi^{-1}\left(\frac{1}{n+1}\right)\right) = x_{i_{(1)}} \; for \; i = 1, \cdots, k$$

Estimators can be determined from the following system:

$$\begin{cases} \widehat{\mu} = \frac{1}{n}\sum_{j=1}^{n} ln(x^j - \widehat{\gamma}) \\ \widehat{B} = \frac{1}{n}\sum_{j=1}^{n}(ln(x^j - \widehat{\gamma}) - \widehat{\mu})(ln(x^j - \widehat{\gamma}) - \widehat{\mu})' \\ \widehat{\gamma}_i + exp\left(\widehat{\mu}_i + \widehat{\sigma}_i \Phi^{-1}\left(\frac{1}{n+1}\right)\right) = x_{i_{(1)}} \; for \; i = 1 \cdots, k \end{cases}$$

The only difference between these two methods is that in the first method, we have $\Phi^{-1}\left(\frac{1}{n+1}\right)$ instead of $E\left(Z_{1,n}\right)$.

Remark 1. $\Phi^{-1}\left(\frac{1}{n+1}\right)$ *depends only on* n.

ii.3 The median method

Similarly, we will replace the equation (3) with an alternative equation.

Let Y a random variable that follow the normal distribution $N(\mu, \sigma^2)$ and a sample formed by n observations $y_1, \cdots, y_n$.
Since Y is a symmetric variable then we obtain the following apprixmation:

$$y_{(n)} - \tilde{y} = \tilde{y} - y_{(1)}$$

with

$$y_{(n)} = \max_{1 \le i \le n}\{y_i\}, \; y_{(1)} = \min_{1 \le i \le n}\{y_i\}$$

and $\tilde{y}$ is the median of Y determined by the sample $\{y_i, \; 1 \le i \le n\}$.
Therefore, if $Y = ln(X_i - \gamma_i)$, thus we obtain le following equation:

$$\frac{ln(x_{i_{(n)}} - \gamma_i) - ln(\tilde{x}_i - \gamma_i)}{ln(\tilde{x}_i - \gamma_i) - ln(x_{i_{(1)}} - \gamma_i)} = 1$$

with

$$x_{i_{(n)}} = \max_{1\leq j\leq n}\{x_{ij}\},\ x_{i_{(1)}} = \min_{1\leq j\leq n}\{x_{ij}\}$$

and $\tilde{x}_i$ is the median of X_i determined by the sample $\{x_{ij},\ 1 \leq j \leq n\}$.
Finally, we obtain

$$\gamma_i = \frac{x_{i_{(1)}}x_{i_{(n)}} - \tilde{x}_i^2}{x_{i_{(1)}} + x_{i_{(n)}} - 2\tilde{x}_i^2},\ for\ i = 1, \cdots k$$

Estimators are determined by:

$$\begin{cases} \widehat{\gamma}_i = \dfrac{x_{i_{(1)}}x_{i_{(n)}} - \tilde{x}_i^2}{x_{i_{(1)}} + x_{i_{(n)}} - 2\tilde{x}_i^2} \\ \widehat{\mu} = \frac{1}{n}\sum_{j=1}^{n} ln(x^j - \widehat{\gamma}) \\ \widehat{B} = \frac{1}{n}\sum_{j=1}^{n}(ln(x^j - \widehat{\gamma}) - \widehat{\mu})(ln(x^j - \widehat{\gamma}) - \widehat{\mu})' \end{cases}$$

ii.4 The method of the straight of Wicksell

Let $Z(x) = \dfrac{ln(x-\gamma_i) - \mu_i}{\sigma_i}$ for $1 \leq i \leq n$ and n fixed.
We suppose that $x = X_i$ is $\Lambda(\gamma_i, \mu_i, \sigma_i)$
which implies that $x = \gamma_i + exp(\sigma_i Z + \mu_i) \ = \ \gamma_i + \beta_i exp(\sigma_i Z)$
where $\beta_i = exp(\mu_i)$.

Let $N(Z) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{Z} \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t$ the distribution function of $N(0,1)$ and $F(x) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{Z(x)} \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t$ the distribution function of $\Lambda(\gamma_i, \mu_i, \sigma_i)$.
Then

$$F(x) = N(Z(x)) \tag{4}$$

If A_i is the mean of X_i then:

$$A_i = \gamma_i + \beta_i exp\left(\frac{{\sigma_i}^2}{2}\right)$$
$$Z(A_i) = \frac{\sigma_i}{2}$$
$$F(A_i) = N\left(\frac{\sigma_i}{2}\right)$$

Finally

$$\sigma_i = 2N^{-1}[F(A_i)] \tag{5}$$

Let $\{(x_{ij}, F(x_{ij})) : j = 1, \cdots, n\}$ a given random sample.
We calculate first the mean A_i, then by the interpolation method we calculate $F(A_i)$.
From (5) and (4) we determinate σ_i and $Z(x_{ij})$.
Finally, we determinate u_j such that $u_j = e^{\sigma_i Z(x_{ij})}$ for $j = 1, \cdots, n$ which is equivalent to $u_j = \dfrac{1}{\beta_i}x_{ij} - \dfrac{\gamma_i}{\beta_i}$.

If we make a linear approximation to the data by $\{(x_{ij}, u_j) : j = 1, \cdots, n\}$ $(u = cx + d)$, we can determine the parameters β_i and γ_i or μ_i and γ_i $\left(\beta_i = \frac{1}{c} \text{ and } \gamma_i = -\frac{d}{c}\right)$

If we make an approximation of $u = cx + d$ with quadratic mean, then we can get c and d minimizing :

$$f(c,d) = \sum_{j=1}^{n}(u - u_j)^2 = \sum_{j=1}^{n}(cx_{ij} + d - u_j)^2$$

The partial derivatives of $f(c,d)$ are:

$$\frac{\partial f(c,d)}{\partial d} = 2\sum_{j=1}^{n}(cx_{ij} + d - u_j)$$

$$\frac{\partial f(c,d)}{\partial c} = 2\sum_{j=1}^{n} x_{ij}(cx_{ij} + d - u_j)$$

If these partial derivatives are equal to zero we obtain the following system:

$$\begin{cases} \sum_{j=1}^{n}(cx_{ij} + d - u_j) = 0 \\ \sum_{j=1}^{n} x_{ij}(cx_{ij} + d - u_j) = 0 \end{cases}$$

which is equivalent to:

$$\begin{cases} \left(\sum_{j=1}^{n} x_{ij}\right) c + nd = \sum_{j=1}^{n} u_j \\ \left(\sum_{j=1}^{n} x_{ij}^2\right) c + \left(\sum_{j=1}^{n} x_{ij}\right) d = \sum_{j=1}^{n} x_{ij}u_j \end{cases}$$

thus

$$c = \frac{n\sum_{j=1}^{n} x_{ij}u_j - \sum_{j=1}^{n} x_{ij}\sum_{j=1}^{n} u_j}{n\sum_{j=1}^{n} x_{ij}^2 - \left(\sum_{j=1}^{n} x_{ij}\right)^2}$$

and

$$d = \frac{1}{n}\sum_{j=1}^{n} u_j - \frac{1}{n}\sum_{j=1}^{n} x_{ij}c$$

Finally, we have

$$\begin{cases} \mu_i = -lnc \\ \gamma_i = -\dfrac{d}{c} \\ B = \dfrac{1}{n}\sum_{j=1}^{n}\left(ln(x^j - \gamma) - \mu\right)\left(ln(x^j - \gamma) - \mu\right)' \end{cases}$$

III. Alternative methods of estimation in the case of lognormal process univariate with three parameters

In this section , we will estimate the parameters of lognormal process univariate with three parameters using the method of moments and the method of maximum likelihood modified.

i. The method of moments

Let be a discrete sample of the process $\{X_{t_0} = x_0, X_{t_1} = x_1, \cdots, X_{t_n} = x_n\}$ at the instants $\{t_0, t_1, \cdots, t_n\}$ with the initial condition $P[X_{t_0} = x_0] = 1$.
The density of X_t given $X_\tau = x$ is equal to:

$$f(\tau, x, t, y) = \frac{1}{[2\pi(t-\tau)\alpha]^{\frac{1}{2}}(y-\gamma)} exp\{-\frac{[ln(y-\gamma) - ln(x-\gamma) - \beta(t-\tau)]^2}{2(t-\tau)\alpha}\}$$

Therefore

$$\frac{ln(y-\gamma) - ln(x-\gamma) - \beta(t-\tau)}{\sqrt{t-\tau}} \ is\ N(0, \sqrt{\alpha})$$

In the case $t = t_i$ and $\tau = t_0$ we obtain:

$$Z_i = \frac{ln(x_i-\gamma) - ln(x_0-\gamma) - \beta(t_i-t_0)}{\sqrt{t_i-t_0}} \ is\ N(0, \sqrt{\alpha})$$

We might equate sampled moments Z_i to corresponding moments of the distribution Z_i to obtain the following equations:

$$\sum_{i=1}^{n} \frac{ln(x_i-\widehat{\gamma}) - ln(x_0-\widehat{\gamma}) - \widehat{\beta}(t_i-t_0)}{n\sqrt{t_i-t_0}} = 0$$
$$\sum_{i=1}^{n} \frac{[ln(x_i-\widehat{\gamma}) - ln(x_0-\widehat{\gamma}) - \widehat{\beta}(t_i-t_0)]^2}{n(t_i-t_0)} = \widehat{\alpha}$$
$$\sum_{i=1}^{n} \frac{[ln(x_i-\widehat{\gamma}) - ln(x_0-\widehat{\gamma}) - \widehat{\beta}(t_i-t_0)]^3}{n(t_i-t_0)^{\frac{3}{2}}} = 0$$

Finally, estimators are obtained from the following system:

$$\begin{cases} \widehat{\beta} = \frac{1}{\sum_{i=1}^{n}(t_i-t_0)^{\frac{1}{2}}} \sum_{i=1}^{n} \frac{ln(x_i-\widehat{\gamma}) - ln(x_0-\widehat{\gamma})}{(t_i-t_0)^{\frac{1}{2}}} \\ \widehat{\alpha} = \sum_{i=1}^{n} \frac{t_i-t_0}{n} \left(\frac{ln(x_i-\widehat{\gamma}) - ln(x_0-\widehat{\gamma})}{t_i-t_0} - \widehat{\beta} \right)^2 \\ \Lambda(\widehat{\gamma}) = 0 \end{cases}$$

with

$$\Lambda(\widehat{\gamma}) = \sum_{i=1}^{n} \frac{ln(x_i-\widehat{\gamma}) - ln(x_0-\widehat{\gamma}) - \frac{t_i-t_0}{\sum_{k=1}^{n}(t_k-t_0)^{\frac{1}{2}}} \sum_{j=1}^{n} \frac{ln(x_j-\widehat{\gamma}) - ln(x_0-\widehat{\gamma})}{(t_j-t_0)^{\frac{1}{2}}}}{(t_i-t_0)^{\frac{3}{2}}}$$

ii. The method of maximum likelihood modified

In this method , we will replace the third L.M.L estimating equation (Local Maximum Likelihood) [4] by a statistical function of rank 1 which contains more information on the threshold parameter γ.
As in the previous method , we will consider a discrete sample of the process $\{X_{t_0} = x_0, X_{t_1} =$

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$x_1, \cdots, X_{t_n} = x_n\}$ at the instants $\{t_0, t_1, \cdots, t_n\}$ with the initial condition $P[X_{t_0} = x_0] = 1$.
We obtain the maximum likelihood equations:

$$\widehat{\mu}(\gamma) = \frac{1}{t_n - t_0}[ln(x_n - \gamma) - ln(x_0 - \gamma)]$$
$$\widehat{\sigma}^2(\gamma) = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{t_j - t_{j-1}} \left(ln(x_j - \gamma) - ln(x_{j-1} - \gamma) - \widehat{\mu}(\gamma)(t_j - t_{j-1})\right)^2$$

More,
since

$$ln(X_t - \gamma)/X_{t_0} = x_0 \text{ } is \text{ } N(ln(x_0 - \gamma) + \mu(t - t_0),\ (t - t_0)^{\frac{1}{2}}\sigma)$$

then

$$Z_i = \frac{ln(X_{t_i} - \gamma) - ln(x_0 - \gamma) - \mu(t_i - t_0)}{(t_i - t_0)^{\frac{1}{2}}\sigma} \text{ } is \text{ } N(0, 1)$$

However, the last equation can be obtained from the following approximation:

$$E[Z_{(1)}] = Z_{(1)}$$

with
$Z_{(1)}$ the order statistic of rank 1 with n random independent variables $(Z_i)_{1 \le i \le n}$ distributed according to the normal law $N(0, 1)$
and $Z_{(1)} = \min_{1 \le k \le n}\{Z_k\}$ such that

$$Z_k = \frac{ln(x_k - \gamma) - ln(x_0 - \gamma) - \widehat{\mu}(\gamma)(t_k - t_0)}{(t_k - t_0)^{\frac{1}{2}}\widehat{\sigma}(\gamma)}$$

To determine the maximum likelihood modified estimators , we develop a table of x_k, γ_k, $\widehat{\mu}_k$, $\widehat{\sigma}_k$ and z_k for all $k = 1 \cdots n$.
Finally, estimators are γ_i, $\widehat{\mu}_i$ and $\widehat{\sigma}_i$ such that $Z_i = Z_{(1)}$.

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