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Mathematical Modelling of Log Normal Atmospheric Turbulence Channels

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Abstract

In this study, a mathematical model was developed for systems involving complex numerical integration expressions. To this end, two-hop communication systems were analyzed, where the end-to-end performance depends on the harmonic mean of the hop signal-to-noise ratios (SNRs).For channels characterized by log-normal distribution, the harmonic mean of the two hop SNRs was derived. The resulting expression is in integral form and requires numerical evaluation. It was demonstrated that this integral expression can be approximated using relatively simple mathematical functions. Specifically, the curve fitting capabilities of the MATLAB platform were employed to approximate the harmonic mean expression by a Gaussian-like distribution. Based on the proposed approximation, the cumulative distribution function (CDF), moment generating function (MGF), statistical moments, outage probability, amount of fading, and transmission error probability were all derived in closed-form expressions. The proposed methodology is suitable for application in systems where performance evaluation involves lengthy and complex expressions

Keywords: Performance of Free Space Optic, Harmonic Mean, Performance Analyses, Probability Density Function.

1. Introduction

In recent years, the demand for improved opportunities, environments, and tools in the field of communication has significantly increased. Optical communication systems have become essential in various domains such as telephony, network data transmission, integrated optical mechanisms, cable television systems, transportation, military, and medical applications [1]. The potential for high-speed data transmission in optical networks renders them increasingly preferable over conventional communication systems [2].Optical communication systems offer several advantages compared to classical communication technologies. These include the abundance of silicon, which is the base material for optical waveguides, strong insulation capabilities, high transmission velocity, operational stability, immunity to electromagnetic interference, high reliability, cost efficiency, and especially, large capacity and low transmission loss with wide bandwidth [3].The usage of optical fibers — dielectric transmission media — continues to rise alongside traditional wired communication media such as copper and coaxial cables, owing to their significantly lower transmission losses compared to free-space wireless communication systems [1,2,3]. An important characteristic of optical fibers is their compatibility with existing communication infrastructure. Free Space Optics (FSO) systems, which enable data transmission for telecommunications and computer networks via light propagation through free space (e.g., air, vacuum), have gained attention in recent years due to several advantages over radio frequency-based systems. These advantages include high data rates; low cost, systematic deployment, portability, enhanced security, and exemption from licensing requirements. Fiber optic cables are commonly used in modern computer networks and high-speed communication applications. However, their deployment becomes impractical or prohibitively expensive in scenarios where physical connectivity is unfeasible. For instance, fiber optic installation in urban environments may incur costs ranging from $300,000 to $700,000 per kilometer, excluding additional infrastructure expenses. In contrast, FSO systems capable of delivering equivalent data rates can be deployed at a significantly lower cost, approximately $18,000 [4, 5].Despite their benefits, FSO systems also face challenges, primarily due to environmental conditions. Atmospheric phenomena such as rain, dust, snow, fog, or smog can disrupt the transmission path, potentially leading to communication outages. To address these limitations, ongoing research efforts aim to enhance FSO system robustness through the development of advanced hardware components and communication techniques.

2. Mathematical Modelling of Log Normal Atmospheric Turbulence Channels

The mathematical modeling of log-normal atmospheric turbulence channels involves the application of analytical tools and techniques to characterize the propagation behavior of optical wireless communication signals through the atmosphere, wherein the intensity fluctuations caused by turbulence are statistically represented by a log-normal distribution.

2.1 Probability Density Function of Log Normal Distribution

This section focuses on the performance analysis of free-space optical (FSO) channels by examining the outage probability and average capacity. Accordingly, closed-form expressions are derived for both outage probability and average capacity of optical links under atmospheric turbulence-induced fading, which is modeled using the log-normal distribution. These expressions account for turbulence strength, optical link length, and receiver aperture diameter—factors that significantly influence system performance. Initially, the probability density function (PDF) is obtained by employing the harmonic mean of two signals. Given two values X_1 and X_2, the harmonic mean is defined as[6, 9]

$$\mu_H\left(X_1, X_2\right) = \frac{2X_1X_2}{X_1 + X_2} \tag{2.1}$$

To facilitate the representation of the equation, three parameters, denoted as ($x\ z\ w$), are introduced. These parameters are defined as follows:

$$w = X_1 + X_2 \qquad z = 2X_1X_2 \qquad x = \frac{z}{w}. \tag{2.2}$$

or the calculation of the harmonic mean of two random variables, both variables are assumed to follow a log-normal distribution. The probability density function (PDF) of log-normal atmospheric turbulence is given by

$$p(x) = \frac{1}{2\mu\sigma\sqrt{2\pi}} \exp\left(-\frac{\left(\ln\left(\frac{\mu}{\bar{\mu}}\right) + \sigma^2\right)^2}{8\sigma^2}\right) \tag{2.3}$$

Here, σ denotes the standard deviation of the log-normal distribution, which is a function of the channel characteristics and is computed as [9, 10]

$$\sigma^2 = \exp\left[\frac{0.49\delta^2}{\left(1 + 0.18d^2 + 0.56\delta^{\frac{12}{5}}\right)^{\frac{7}{6}}} + \frac{0.51\delta^2}{\left(1 + 0.9d^2 + 0.62d^2\delta^{\left(\frac{12}{5}\right)}\right)^{\frac{5}{6}}}\right] - 1 \tag{2.4}$$

where

$$d = \sqrt{\frac{kD^2}{4L}}$$

and

$$k = \frac{2\pi}{\lambda}$$

is the optical wave number, L is the length of the optical link and D is the receiver's aperture diameter. The parameter δ is called Rytov variance and it is defined as

$$\delta^2 = 1.23\ C_n^2\ k^{\frac{7}{6}}\ L^{\frac{11}{6}}\ . \tag{2.5}$$

where C_n^2 is the altitude which is dependent on the turbulence strength varying from 10^{-17} to $10^{-13}\ m^{-2/3}$ according to atmospheric turbulence conditions [2,8]. The instantaneous electrical signal to noise ratio (SNR) is given as

$$\mu = \left(\frac{\eta I^2}{N_0}\right) = \frac{s^2}{N_0}$$

and the average electrical SNR is calculated as

$$\bar{\mu} = \eta \frac{E[I]}{N_0}.$$

Now we must use error function to define pdf equation of log normal model more simply. The exponential term in log-normal distribution in (2.3) can be expressed as in

$$p(x) = \exp\left(-\frac{\left(\ln\left(\frac{\mu}{\bar{\mu}}\right)+\sigma^2\right)^2}{8\sigma^2}\right) \rightarrow \exp\left[-\left(\frac{1}{8\sigma^2}\right)\left[\ln\left(\frac{\mu}{\bar{\mu}}\right)\right]^2 + \sigma^4 + 2\sigma^2 \ln\left(\frac{\mu}{\bar{\mu}}\right)\right]. \tag{2.6}$$

After this operation, probability density functions of log normal distribution take the form

$$p(x) = \frac{1}{2\sigma\sqrt{2\pi}} \exp\left[-\frac{\sigma^2}{8}\right] \bar{\mu}^{-\left(\frac{1}{4}\right)}\ \mu^{\frac{1}{4}}\ \ exp\left[-\left(\frac{1}{8\sigma^2}\right)\left[\ln\left(\frac{\mu}{\bar{\mu}}\right)\right]^2\right]. \tag{2.7}$$

If the constant value K defined as in

$$K = \frac{1}{2\sigma\sqrt{2\pi}} \exp\left[-\frac{\sigma^2}{8}\right] \bar{\mu}^{-\left(\frac{1}{4}\right)} \tag{2.8}$$

then the log-normal distribution in (2.7) can be written as in (2.9). At the end of these steps pdf equation transformed to an easier form which is expressed as equation(2.9)

$$p(x) = K \cdot \mu^{\frac{1}{4}} \cdot\ \ exp\left[\left[\ln\left(\frac{\mu}{\bar{\mu}}\right)\right]^2\right] \tag{2.9}$$

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2.2 Joint Probability Density Function

The joint pdf of the random variables Z, W i.e., $P_{z,w}(z,w)$ can be derived using the Jacobian transformation. The harmonic mean can be written of X_1 and X_2 as $X = Z/W$ where $X = 2X_1X_2$ and $W = X_1 + X_2$. Using [12, Sec. 6.2], the PDF of X can be written as,

$$p_x(x) = \int_{-\infty}^{\infty} |w| P_{z,w}(xw, w) dw \tag{2.10}$$

which can be evaluated with the help of [7, Eq. (3.383.4)] yielding,

$$P_{z,w} = \frac{1}{2\Delta}\left[P_{x_1,x_2}(X_{11}, X_{21}), P_{x_1,x_2}(X_{12}, X_{22})\right] \tag{2.11}$$

in which we have

$$X_{11}, X_{21} = w \pm \frac{\sqrt{w^2 - 2z}}{2} \;, X_{11}, X_{21} = w \pm \frac{\sqrt{w^2 - 2z}}{2} \;, \Delta = \sqrt{w^2 - 2z} \tag{2.12}$$

Substituting the parameters in (2.11) and the log-normal distribution expression in (2.9) into obtained (2.14).

$$P_{z,w}(z,w) = \frac{1}{2\sqrt{w^2 - 2z}}\left[P(x_{11})P(x_{21}) + P(x_{12})P(x_{22})\right] \tag{2.13}$$

$$P_{z,w}(z,w) = \frac{2}{2\sqrt{w^2 - 2z}}\left[\begin{matrix} K\left(\frac{(w+\Delta)}{2}\right)^{\frac{1}{4}} exp\left(\frac{1}{8\sigma^2}\left[ln\left(\frac{(w+\Delta)}{2}\frac{1}{x}\right)\right]^2\right) \\ K\left(\frac{(w-\Delta)}{2}\right)^{\frac{1}{4}} exp\left(\frac{1}{8\sigma^2}\left[ln\left(\frac{(w-\Delta)}{2}\frac{1}{x}\right)\right]^2\right) \end{matrix}\right]. \tag{2.14}$$

The probability density function of X=Z/W can be calculated using

$$P_X(x) = \int_{2x}^{\infty} w P_{z,w}(xw, w) dw$$

leading to

$$p(x) = K\int_{2x}^{\infty} w^{-\left(\frac{3}{4}\right)}(w - 2x)^{-\left(\frac{1}{2}\right)} exp\left(-\frac{1}{8\sigma^2}\left[\left[\frac{w+\Delta}{2a}\right]^2\left[\frac{w-\Delta}{2a}\right]^2\right]\right) dw. \tag{2.15}$$

The resulting integral does not admit a closed-form solution. Therefore, numerical computation was performed using MATLAB to evaluate the integral, and the corresponding function P(X) was plotted with respect to the signal-to-noise ratio (SNR), denoted as X. The graph of P(X) is presented in Fig. 1.

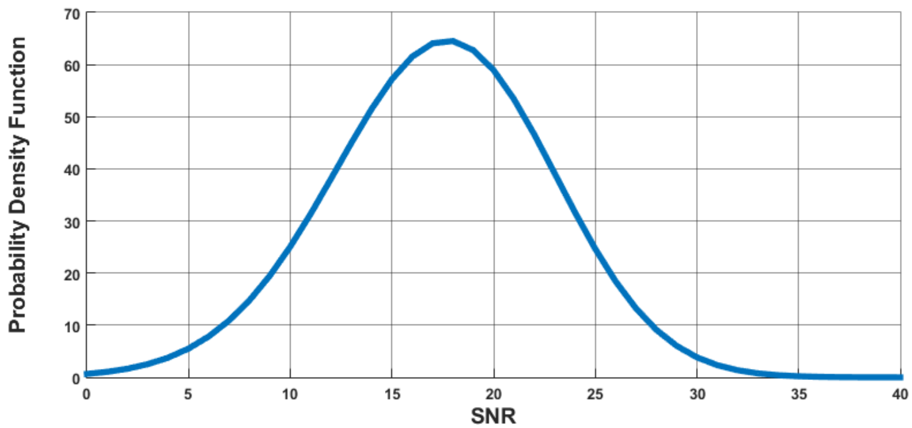

Figure 1 Probability Density Function to Signal Noise Ratio

3. Detection of Best Fitting Curves in MATLAB

The curve fitting toolbox in MATLAB was employed to identify an analytical expression that closely approximates the probability density function described in P(X). In order to represent the integral in a closed-form expression, various fitting models available in MATLAB—such as Gaussian, Fourier, and Polynomial—were considered. Among these, the Gaussian model with a specified degree was selected due to its superior fitting accuracy and the analytical simplicity it offers for integration. The performance evaluation and coefficient values of the Gaussian model are presented in Fig. 2. For comparison, the results of the Polynomial and Fourier models are illustrated in Fig. 3 and Fig. 4, respectively. Furthermore, Fig. 2 through Fig. 4 provide a comparative assessment of the models in terms of the number of terms, general form of the equations, goodness-of-fit metrics, and the corresponding coefficient values.

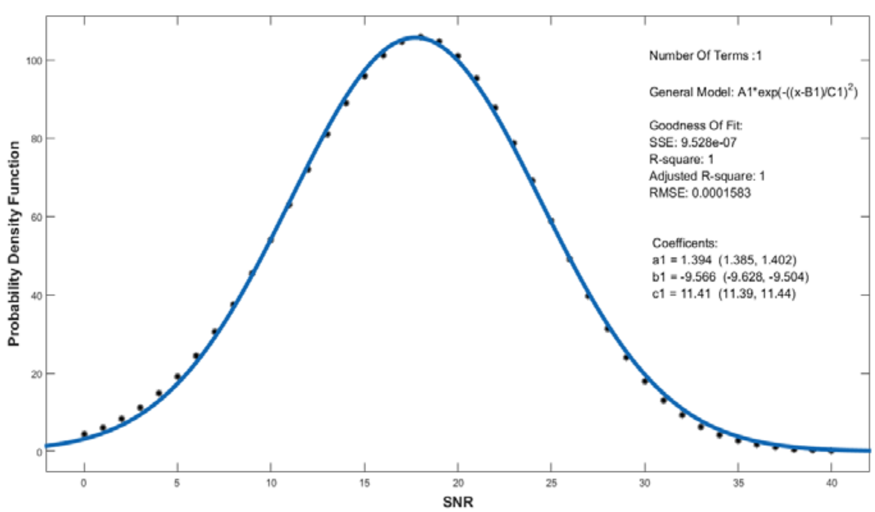

Figure 2 Gaussian Equation Model

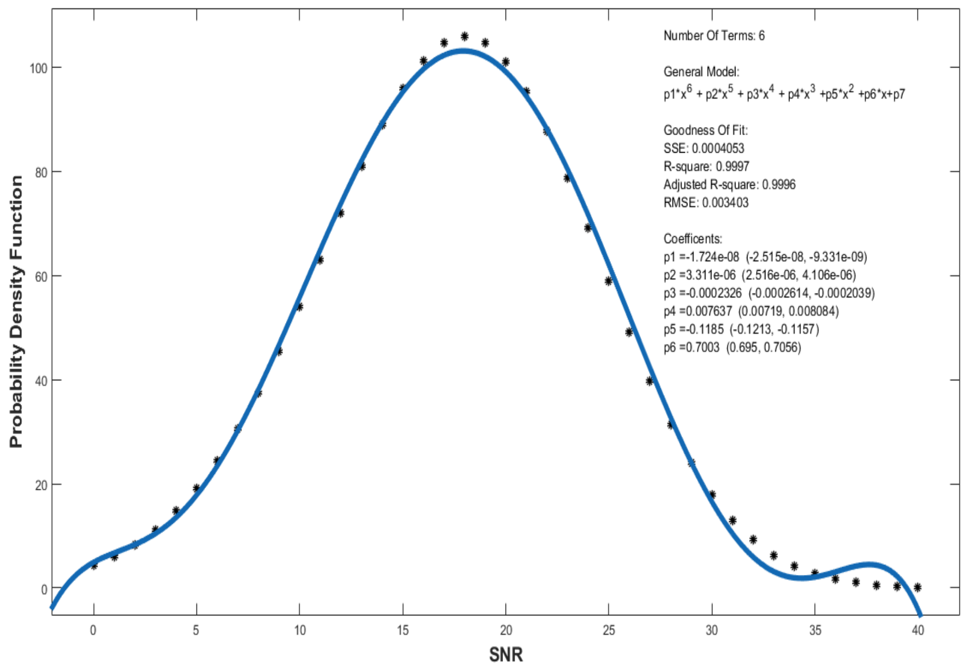

Figure 3 Polynomial Equation

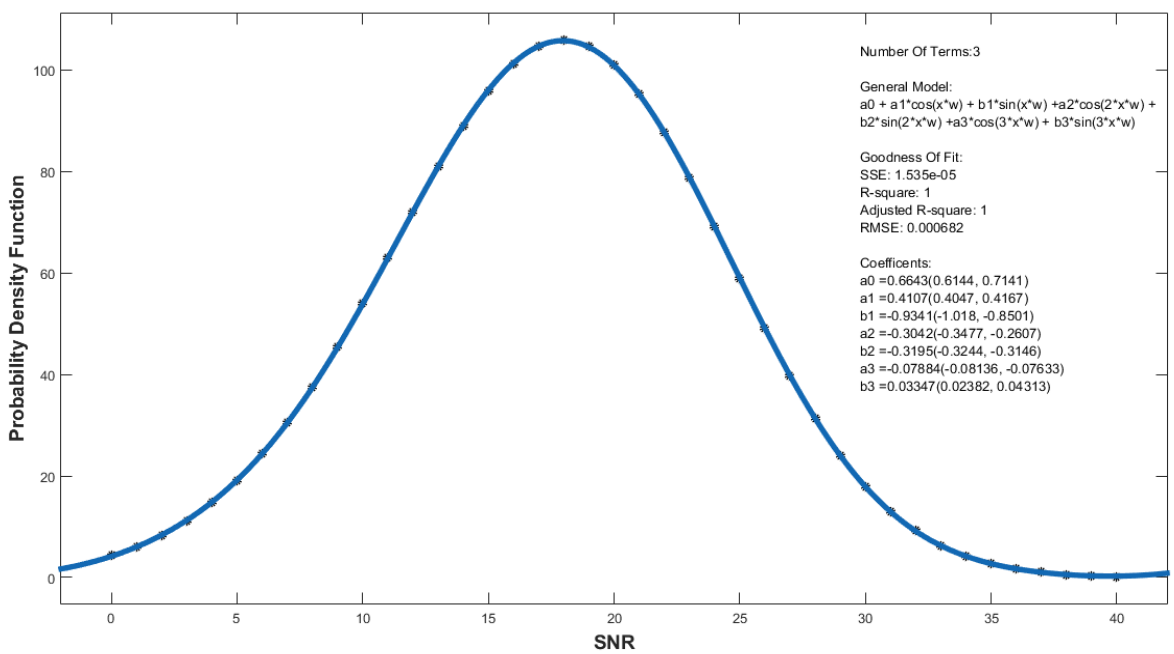

Figure 4 Fourier Equation Model

Following the selection of the Gaussian model with an appropriate degree, the complex integral expression originally defined in P(X) was replaced with the corresponding Gaussian approximation obtained via MATLAB's curve fitting utility. Thus, the function P(X) is accurately approximated by the Gaussian expression as follows:

$$p(x) = A * exp\left(-\left(\frac{(X-B)}{C}\right)^2\right). \tag{2.16}$$

where A, B, and C are the model parameters determined through curve fitting..

4.Conclusion

This study presented the development of a closed-form mathematical model for systems whose performance metrics are originally expressed in the form of complex numerical integrals. In particular, two-hop communication systems were considered, where the end-to-end performance depends on the harmonic mean of the hop signal-to-noise ratios (SNRs). The analysis focused on two-hop systems operating over log-normally distributed optical communication channels, and an expression for the harmonic mean in such scenarios was derived. To obtain an analytically tractable form, the harmonic mean expression was approximated using a Gaussian-like model derived through MATLAB's curve fitting utility. Based on this approximation, closed-form expressions were subsequently developed for the cumulative distribution function (CDF), moment generating function (MGF), and transmission error probability associated with two-hop communication systems affected by log-normal turbulence. Free-space optical (FSO) communication offers several advantages over traditional optical, radio frequency, and microwave systems, including lower deployment cost and reduced setup time. With minor modifications, conventional optical components can be employed within FSO systems. While FSO technology holds significant potential due to its cost-effectiveness and applicability, its performance is affected by atmospheric attenuation, leading to transmission power loss. Therefore, careful consideration and reassessment of atmospheric conditions are essential when designing FSO systems. In this work, closed-form expressions were derived for evaluating the average capacity and outage probability of a representative two-hop FSO system under atmospheric turbulence modeled by a log-normal distribution. The study examined the system's performance and reliability as functions of key physical parameters, including the optical link length, receiver aperture diameter, and turbulence severity along the transmission path. Given knowledge of the transmission distance and atmospheric turbulence strength, the proposed model enables straightforward and accurate performance prediction for optical communication channels under realistic conditions

Appendix
N/A

Acknowledgments
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