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Solution of Parabolic Partial Differential Equation of the form $u_t=u_{xx}+f(x,t)u$ using Lie Symmetry Analysis

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***Abstract*—The investigation of the exact solutions of nonlinear PDEs plays an important role in the study of nonlinear physical phenomena for instance in shallow water waves, fluid physics, general relativity and many others. Lie Symmetry analysis has played a significant role in the construction of exact solutions to nonlinear partial differential equations (PDEs). The modern approach for finding special solutions of systems of nonlinear PDEs was pioneered by Sophus Lie at the end of the nineteenth century. A variety of methods have been developed in the past few years by Ovsyannikov, Ibragimov and others. In this work, we present a Lie symmetry approach in solving the parabolic partial differential equation $u_t=u_{xx}+f(x,t)u$ with *f(x, t)* set as *xt* and *sinxt*. This will be achieved by developing infinitesimal transformations, generators, prolongations and the invariant transformations of the problem.**

Key words*— *Lie symmetries analysis, symmetry reduction, partial differential equations, Infinitesimal transformations, transformation generators, prolongations (extended transformations), invariant transformations

Second Order Parabolic Partial Differential Equation

We consider the second order parabolic partial differential equation:

$$u_t=u_{xx}+f(x,t)u \tag{1.1}$$

We then attempt to solve a special case of (1) by setting *f(x,t)* as *sinxt* and *xt* using Lie symmetry approach.

Case 1: f(x,t)= sinxt

When *f(x,t)* is set as *sinxt*, the (1) becomes

$$u_t=u_{xx}+u\sin xt \tag{1.2}$$

We let the generator X of (1.2) to be of the form

$$X=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u} \tag{1.3}$$

We determine all the coefficient functions ξ,τ,η so that the corresponding one-parameter Lie group of transformations $t^*=T(x,t,u;\varepsilon), x^*=X(x,t,u;\varepsilon), u^*=U(x,t,u;\varepsilon)$ for a symmetry group of (1.2).

For the symmetry condition to be satisfied by (1.2), then

$$X^{(2)}\left[u_t-u_{xx}-u\sin xt\right]=0 \tag{1.4}$$

where $X^{(2)}$ is the second prolongation of (1.3) given by

$$X^{(2)}=\xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial u}+\eta^x\frac{\partial}{\partial u_x}+\eta^t\frac{\partial}{\partial u_t}+\eta^{xx}\frac{\partial}{\partial u_{xx}}+\eta^{xt}\frac{\partial}{\partial u_{xt}}+\eta^{tt}\frac{\partial}{\partial u_{tt}} \tag{1.5}$$

Substituting (1.5) into (1.4) we obtain

$$\left[\xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial u}+\eta^x\frac{\partial}{\partial u_x}+\eta^t\frac{\partial}{\partial u_t}+\eta^{xx}\frac{\partial}{\partial u_{xx}}+\eta^{xt}\frac{\partial}{\partial u_{xt}}+\eta^{tt}\frac{\partial}{\partial u_{tt}}\right]\left[u_t-u_{xx}-u\sin xt\right]=0$$

The infinitesimal condition above reduces to

$$\eta^t=\eta^{xx}+\xi tu\cos xt+\tau xu\cos xt+\eta\sin xt \tag{1.6}$$

with η^t,η^{xx} explicitly defined in [1,2]. This must be satisfied whenever $u_t=u_{xx}+u\sin xt$. Substituting η^t,η^{xx} into (1.6), we obtain the equation

$$\eta_t-\xi_t u_x+(\eta_u-\tau_t)u_t-\xi_u u_x u_t-\tau_u u_t^2=\eta_{xx}+(2\eta_{xu}-\xi_{xx})-\tau_{xx}u_t+(\eta_{uu}-2\xi_{xu})u_x^2$$
$$-2\tau_{xu}u_x u_t-\xi_{uu}u_x^3-\tau_{uu}u_x^2u_t+\left(\eta_u-2\xi_x\right)u_{xx}-2\tau_x u_{xt}-3\xi_u u_x u_{xx}$$
$$-\tau_u u_t u_{xx}-2\tau_u u_x u_{xt}+\xi tu\cos xt+\tau xu\cos xt+\eta\sin xt \tag{1.7}$$

On replacing u_t by $u_{xx}+u\sin xt$ whenever it occurs and equating the coefficients of various polynomials in the first

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and second partial derivatives of u, we obtain the resulting equations for the Lie symmetry group of equation (1.2).

Monomial terms	Equation
$u_x u_{xt}$	$0=-2\tau_u$ (i)
u_{xt}	$0=-2\tau_x$ (ii)
u_{xx}^2	$-\tau_u=-\tau_u$ (iii)
$u_x^2 u_{xx}$	$0=-\tau_{uu}$ (iv)
$u_x u_{xx}$	$-\xi_u=-2\tau_{xu}-3\xi_u$ (v)
u_{xx}	$\eta_u-\tau_t-2\tau_u u\sin xt=-\tau_{xx}+\eta_u-2\xi_x-\tau_u u\sin xt$ (vi)
u_x^3	$0=-\xi_{uu}$ (vii)
u_x^2	$0=\eta_{uu}-2\xi_{xu}-\tau_{uu}u\sin xt$ (viii)
u_x	$-\xi_t-\xi_u u\sin xt=2\eta_{xu}-\xi_{xx}-2\tau_{xu}u\sin xt$ (ix)
1	$\eta_t+(\eta_u-\tau_t)u\sin xt-\tau_u u^2\sin^2 xt=\eta_{xx}+\xi tu\cos xt$ $+\tau xu\cos xt+\eta\sin xt$ (x)

The general solution to the above determining equations (i) to (x) becomes

$$\xi=0,\tau=0,\eta=c$$

Therefore, the symmetry algebra of equation (1.2) is

$$X=u\frac{\partial}{\partial u} \qquad (1.7)$$

The Lie group admitted by (1.7) is thus given by

$G:V(x,t,u;\varepsilon)\rightarrow V(x,t,e^{\varepsilon}u)$ which is a trivial solution.

Case 2: $f(x,t)=xt$

If *f(x,t)* is set as *xt*, then (1.1) becomes

$$u_t=u_{xx}+xtu \qquad (2.1)$$

We let the generator X of (2.1) to be of the form (1.3). We then determine the coefficient functions ξ,τ,η so that the corresponding one-parameter Lie group of transformations $t^*=T(t,x,u;\varepsilon),x^*=X(t,x,u;\varepsilon),u=U(t,x,u;\varepsilon)$ for a symmetry group of (1.2).

For the symmetry condition to be satisfied by (2.1), then

$$X^{(2)}\left[u_t-u_{xx}-xtu\right]=0 \qquad (2.2)$$

where $X^{(2)}$ is the second prolongation of (1.3) given by (1.5).

Substituting (1.5) into (2.2), we obtain

$$\left[\xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial u}+\eta^x\frac{\partial}{\partial u_x}+\eta^t\frac{\partial}{\partial u_t}+\eta^{xx}\frac{\partial}{\partial u_{xx}}+\eta^{xt}\frac{\partial}{\partial u_{xt}}+\eta^{tt}\frac{\partial}{\partial u_{tt}}\right]\left[u_t-u_{xx}-xtu\right]=0$$

This reduces to

$$\eta^t=\eta^{xx}+\xi tu+\tau xu+\eta xt \qquad (2.3)$$

Substituting η^t,η^{xx} into (2.3), we obtain the equation

$$\eta_t-\xi_t u_x+(\eta_u-\tau_t)u_t-\xi_u u_x u_t-\tau_u u_t^2=\eta_{xx}+(2\eta_{xu}-\xi_{xx})-\tau_{xx}u_t$$
$$+(\eta_{uu}-2\xi_{xu})u_x^2-2\tau_{xu}u_x u_t-\xi_{uu}u_x^3+\tau_{uu}u_x^2 u_t(\eta_u-2\xi_x)u_{xx}-2\tau_x u_{xt}$$
$$-3\xi_u u_x u_{xx}-\tau_u u_t u_{xx}-2\tau_u u_x u_{xt}+\tau xu+\eta xt$$

On replacing u_t by $u_{xx}+xtu$ whenever it occurs and equating the coefficients of various polynomials in the first and second partial derivatives of u, we obtain the resulting equations for the Lie symmetry group of equation (2.1)

Monomial terms	**Equation**
$u_x u_{xt}$	$0=-2\tau_u$ **(a)**
u_{xt}	$0=-2\tau_x$ **(b)**
u_{xx}^2	$-\tau_u=-\tau_u$ **(c)**
$u_x^2 u_{xx}$	$0=-\tau_{uu}$ **(d)**
$u_x u_{xx}$	$-\xi_u=-2\tau_{xu}-3\xi_u$ **(e)**
u_{xx}	$\eta_u-\tau_t-2xtu\tau_t=-\tau_{xx}+\eta_u-2\xi_x-\tau_u xtu$ **(f)**
u_x^3	$0=-\xi_{uu}$ **(g)**
u_x^2	$0=\eta_{uu}-2\xi_{xu}-\tau_{uu}xtu$ **(h)**
u_x	$-\xi_t-\xi_u xtu=2\eta_{xu}-\xi_{xx}-2\tau_{xu}xtu$ **(i)**
1	$\eta_t+(\eta_u-\tau_t)xtu-\tau_u x^2t^2u^2=\eta_{xx}+\xi tu+\tau xu+\eta xt$ **(j)**

The general solution to the above determining equations (a) to (j) becomes

$$\tau=\frac{1}{2}a_1t^2+a_2t+a_3 \qquad (2.4)$$

$$\xi=\frac{1}{2}a_1xt+\frac{1}{2}a_2x-\frac{1}{3}a_1t^4-\frac{5}{6}a_2t^3-a_3t^2+a_4t+a_5 \qquad (2.5)$$

$$\eta=-\frac{1}{8}a_1x^2u+\frac{2}{3}a_1t^3xu+\frac{5}{4}a_2t^3xu+a_3txu-\frac{1}{2}a_4xu-\frac{1}{4}a_5tu$$
$$-\frac{1}{18}a_1t^6u-\frac{1}{6}a_2t^5u-\frac{1}{4}a_3t^4u+\frac{1}{3}a_4t^3u+\frac{1}{2}a_5t^2u+a_6u+\beta(x,t) \tag{2.6}$$

The Lie algebra of the point symmetries are therefore given by

$$X_1=\left(\frac{1}{2}xt-\frac{1}{3}t^4\right)\frac{\partial}{\partial x}+\frac{1}{2}t^2\frac{\partial}{\partial t}+\left(-\frac{1}{8}x^2u+\frac{2}{3}t^3xu-\frac{1}{4}tu-\frac{1}{18}t^6u\right)\frac{\partial}{\partial u}$$

$$X_2=\left(\frac{1}{2}x-\frac{5}{6}t^3\right)\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}+\left(\frac{5}{4}t^3xu-\frac{1}{6}t^5u\right)\frac{\partial}{\partial u}$$

$$X_3=-t^2\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\left(txu-\frac{1}{4}t^4u\right)\frac{\partial}{\partial u}$$

$$X_4=t\frac{\partial}{\partial x}+\left(-\frac{1}{2}xu+\frac{1}{3}t^3u\right)\frac{\partial}{\partial u}$$

$$X_5=\frac{\partial}{\partial x}+\frac{1}{2}t^2u\frac{\partial}{\partial u}$$

$$X_6=u\frac{\partial}{\partial u}$$

Lie groups admitted by equation (2.1)

The one parameter groups G_i admitted by equation (2.1) are determined by solving corresponding Lie equations below

$$X_4:\frac{dx^*}{d\varepsilon}=t^*,\quad \frac{dt^*}{d\varepsilon}=0,\quad \frac{du^*}{d\varepsilon}=-\frac{1}{2}x^*u^*+\frac{1}{3}t^{*3}u^*$$

$$X_5:\frac{dx^*}{d\varepsilon}=1,\quad \frac{dt^*}{d\varepsilon}=0,\quad \frac{du^*}{d\varepsilon}=\frac{1}{2}t^{*2}u^*$$

with initial conditions $t^*_{\varepsilon=0}=t$, $x^*_{\varepsilon=0}=x$, $u^*_{\varepsilon=0}=u$

This leads to

$$X_4:G_4:V(x,t,u;\varepsilon)\to V_4\left(x+\varepsilon t,t,e^{-\frac{1}{2}t\varepsilon^2-\frac{1}{2}\varepsilon x+\frac{1}{3}\varepsilon t^3}u\right)$$

$$X_4:G_5:V(x,t,u;\varepsilon)\to V_5\left(x+\varepsilon,t,e^{\frac{1}{2}\varepsilon t^2}u\right)$$

Invariant solutions of the equation (2.1)

If a group of transformations maps a solution into itself, we arrive at the **group invariant** solution. Given infinitesimal symmetry of equation (1) the invariant solution under the one-parameter group generated by a generator *V* are obtained as follows: We calculate two independent invariants $J1=k(x,t)$ and $J2=\mu(x,t,u)$ by solving the equation

$$X(J)=\xi(x,t,u)\frac{\partial J}{\partial x}+\tau(x,t,u)\frac{\partial J}{\partial t}+\eta(x,t,u)\frac{\partial J}{\partial u} \tag{2.7}$$

or its system of characteristics

$$\frac{dx}{\xi(x,t,u)}=\frac{dt}{\tau(x,t,u)}=\frac{du}{\eta(x,t,u)} \tag{2.8}$$

We then designate one of the invariants as a function of the other e.g

$$\mu=\phi(k) \tag{2.9}$$

Finally we substitute expression for μ ,in equation (2.9) and obtain ordinary differential equation for the unknown function $\phi(k)$ of one variable. This procedure reduces the number of independent variables by one.

Below is a list of Generators (X_i) and their corresponding Invariant Solutions (u)

$$X_3:u=e^{\frac{1}{2}t^2x-\frac{1}{20}t^5}\left[c_1\cos\left(\frac{1}{\sqrt{2}}xt^2+\frac{1}{3\sqrt{2}}t^5\right)+c_2\sin\left(\frac{1}{\sqrt{2}}xt^2+\frac{1}{3\sqrt{2}}t^5\right)\right] \tag{3.0}$$

$$X_4:u=c_3e^{\frac{2}{9}t^3+\frac{1}{2}t^2x-\frac{x^2}{4t}} \tag{3.1}$$

$$X_5:u=c_4e^{\frac{1}{20}t^5+\frac{1}{2}t^2x} \tag{3.2}$$

Symmetry solutions of equation (2.1)

Consider the equation (2.1).

According to Olver P.J [9], a symmetry group of (2.1) is a local group of transformation G with the property that whenever $u=\phi(x)$ is a solution of (2.1) and whenever $g.u$ is defined for $g\in G$, then $u^*=g.u$ is also a solution of (2.1).

(a) Given the generator $X_4=t\frac{\partial}{\partial x}+\left(-\frac{1}{2}xu-\frac{1}{3}t^3u\right)\frac{\partial}{\partial u}$ of equation (2.1) for the symmetry group
$$G_4\left(x^*,t^*,u^*\right):\left(x+\varepsilon t,t,e^{-\frac{1}{2}t\varepsilon^2-\frac{1}{2}\varepsilon x+\frac{1}{3}\varepsilon t^2}u\right).$$

This yields the groups $x^* = x + \varepsilon t$, $t^* = t$, $u^* = e^{-\frac{1}{2}t\varepsilon^2 - \frac{1}{2}\varepsilon x + \frac{1}{3}\varepsilon t^2} u$ and applying the inverse mapping, we obtain the symmetry solution

$$u^* = e^{-\frac{1}{2}\varepsilon x^* + \frac{1}{3}\varepsilon t^{*2}} u(x,t) \qquad (3.3)$$

where $u(x,t)$ is a known invariant solution of (2.1).

Inserting each of the invariant solutions (3.0), (3.1) and (3.2) into (3.3), we obtain the following symmetry solutions of (2.1)

(i) $$u^* = e^{-\frac{1}{2}\varepsilon x^* + \frac{1}{3}\varepsilon t^{*2} + \frac{1}{2}t^2 x - \frac{1}{20}t^5}\left[c_1 \cos\left(\frac{1}{\sqrt{2}}xt^2 + \frac{1}{3\sqrt{2}}t^5\right) + c_2 \sin\left(\frac{1}{\sqrt{2}}xt^2 + \frac{1}{3\sqrt{2}}t^5\right)\right]$$

(3.4)

(ii) $$u^* = C_1 e^{-\frac{1}{2}\varepsilon x^* + \frac{1}{3}\varepsilon t^{*2} + \frac{2}{9}t^3 + \frac{1}{2}t^2 x - \frac{x^2}{4t}}$$

(3.5)

(iii) $$u^* = C_2 e^{-\frac{1}{2}\varepsilon x^* + \frac{1}{3}\varepsilon t^{*2} + \frac{1}{20}t^5 + \frac{1}{2}t^2 x}$$

(3.6)

(b) Applying the inverse mapping to the symmetry groups of the generator

$X_5 : G_5\left(x^*, t^*, u^*\right) : \left(x + \varepsilon, t, e^{\frac{1}{2}\varepsilon t^2} u\right)$, we obtain the solution

$$u^* = e^{\frac{1}{2}\varepsilon t^{*2}} u(x,t) \qquad (3.7)$$

Inserting each of the invariant solutions (3.0), (3.1) and (3.2) into (3.7), we obtain the following symmetry solutions

(i) $$u^* = e^{\frac{1}{2}\varepsilon t^{*2} + \frac{1}{2}t^2 x - \frac{1}{20}t^5}\left[c_1 \cos\left(\frac{1}{\sqrt{2}}xt^2 + \frac{1}{3\sqrt{2}}t^5\right) + c_2 \sin\left(\frac{1}{\sqrt{2}}xt^2 + \frac{1}{3\sqrt{2}\cdot}t^5\right)\right]$$

(3.8)

(ii) $$u^* = C_1 e^{\frac{1}{2}\varepsilon t^{*2} + \frac{2}{9}t^3 + \frac{1}{2}t^2 x - \frac{x^2}{4t}}$$

(3.9)

(iii) $$u^* = C_2 e^{\frac{1}{2}\varepsilon t^{*2} + \frac{1}{20}t^5 + \frac{1}{2}t^2 x}$$

(4.0)

Conclusion

In this paper, we have looked at methods of group invariant solutions, based on the theory of continuous group of transformations, better known as 'Lie groups', acting on the space of independent and dependent variables of the system. More specifically, we have developed and used the infinitesimal transformations, symmetry generators, prolongations, groups and invariant solutions of the parabolic partial differential equation $u_t = u_{xx} + f(x,t)$ to find the symmetry solutions (3.4 – 4.0)

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