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Compressed Sensing: Progresses and Challenges

[1] Ishani Mishra, [2] Divya Sharma, [3] Dr Sanjay Jain

[1] Assistant Professor , Department of ECE, New Horizon College Of Engineering, Bangalore
[2] Assistant Professor , Department of ECE, New Horizon College Of Engineering, Bangalore
[3] HOD , Department Of ECE, New Horizon College Of Engineering, Bangalore

Abstract: **CS is an emerging theory which permits radically new sensing devices that simultaneously acquire and compress certain signals using very efficient randomized sensing protocols. The implications of the CS theory are very far-reaching and will likely impact analog-to-digital conversion, data compression, medical imaging, sensor networks, digital communication, statistical model selection, and more. Yet some important theoretical questions remain open, and seemingly obvious applications keep escaping the grip of compressive sensing. In this paper1 we discuss some of the recent progresses in compressed sensing and point out key challenges and opportunities as the area of compressed sensing and sparse representations keeps evolving. We also attempt to assess the long term impact of compressive sensing.**

Keywords: Sparsity ,RIP ,Incoherence

1. Inroduction

Compressed Sensing is a novel research area which was introduced in 2006, and since then has already become a key concept in various areas of applied Mathematics, Computer science, and Electrical engineering. It surprisingly predicts that high-dimensional signals, which allow a sparse representation by a suitable basis or, more generally, a frame, can be recovered from what was previously considered highly incomplete linear measurements by using efficient algorithms.

Compressed sensing or compressive sampling (CS) is a simple and efficient signal acquisition technique that collects a few measurements about the signal of interest and later uses optimization techniques for reconstructing the original signal from what appears to be an incomplete set of measurements. It surprisingly predicts that high-dimensional signals, which allow a sparse representation by a suitable basis or, more generally, a frame, can be recovered from what was previously considered highly incomplete linear measurements by using efficient algorithms.

2.Fundamentals of Compressed Sensing:

Sensing of the time domain signal y(t) is defined as the process of collecting some measurements about y(t) by correlating y(t) with some sensing waveforms { $\Phi_j(t)$}, i.e.,

$$x_j = \langle y, \Phi_j \rangle, \quad j = 1, 2, \ldots, m. \qquad (2.1)$$

Based on the sensing waveforms, the entries of the vector x have different interpretations. For example, if the sensing waveforms are sinusoids, then x is a vector of Fourier coefficients, and if the sensing waveforms are Dirac delta functions, then x is a vector of sampled values of y(t).To simplify the presentation of the CS technique we will restrict our attention to discrete signals y Є R^n. Accordingly, equation 1.1[1] can be rewritten in matrix form as

$$x = \Phi y \qquad (2.2)$$

where the jth row of the sensing matrix $\Phi \in R^{m*n}$ is the discrete representationof the jth sensing function $\Phi_j(t)$, and $y \in R^n$ is the discrete representation of y(t).

Based on this model, compressed sensing is defined as the sensing process for which the number m of available measurements is much smaller than the dimension n of the signal y. The problem associated with compressed sensing is that we have to solve an under–determined system of equations to recover the original signal y from the measurement vector x. However, since the number of equations is less than the number of unknowns, it is known that this system has infinitely many solutions , and thus it is necessary to impose constraints on the candidate solution to identify which of these candidate solutions is the desired one.

2.1Sparsity

Before explaining the importance of the sparsity constraint in solving under-determined systems of equations, we present the following definitions:[2]

i) Sparsity = $|\{s[i] = 0\}|$ = number of zero entries in s, where $|\cdot|$ denotes cardinality of a set.
ii) Diversity = $|\{s[i] \neq 0\}|$ = number of nonzero entries in s.

iii) k–sparse vector: a k–sparse vector is defined as the vector that has at most k non-zero entries.

An underdetermined system of linear equations has infinite candidate solutions of the form $y = y_0 + N$ where y_0 is any vector that satisfies $x = \Phi y_0$ and $N := N(\Phi)$ is the null space of Φ. If the candidate solution vector is known to be k–sparse, and under some conditions on the sensing matrix Φ the solution vector can be uniquely determined using an optimization technique. Fortunately, this also applies to non sparse vectors that can be sparsely represented in a suitably selected basis Ψ i.e.,[3]

$$y = \Psi s \quad (2.3)$$

where the coefficient vector s is sparse.
Clearly y and s are equivalent representations of the signal, with y in the time or space domain and s in the $\boldsymbol{\Psi}$ domain. In some applications, it may be natural to choose as an orthonormal basis, while in others the signal y may only be sparsely represented when $\boldsymbol{\Psi}$ is a redundant dictionary; i.e., it has more columns than rows. Combining equation (1.2) and (1.3) and taking into consideration the case of noisy measurements,the sensing process can be written as

$$x = \Phi\Psi s + v = As + v, \quad (2.4)$$

where $A = \Phi\Psi \in R^{m*n}$ and $v \in R^n$ is a noise vector. Assuming that the coefficient vector s is k–sparse, then s, and hence $y = \Psi s$, can only be estimated from x if the matrices Φ,Ψ and A satisfy the properties described in the next subsection.[4],[5]

2.2 Incoherence and Restricted Isometric Properties

The sparsity of the solution vector, or its representation in some basis, is a necessary but not sufficient condition for finding a unique solution to an underdetermined system of linear equations. In addition to the sparsity principle, CS relies on another principle which is the incoherence between the sensing matrix Φ and the sparsity basis Ψ. The incoherence principle is also related to an equivalent property, which is associated with A, called restricted isometric property (RIP).

The restricted isometry property is a notion introduced in [6] and has proved to be very useful in studying the general robustness of CS. RIP provides a very useful tool for determining sufficient conditions that guarantee exact reconstruction of a sparse solution vector for different reconstructing (decoding) algorithms.

Consider the following definition.

For each integer k = 1, 2, . . ., the isometry constant δk of a matrix A is defined as the smallest number [7] such that

$$(1 - \delta k)\|s\|_{\ell 2}^{2} \leq \|As\|_{\ell 2}^{2} \leq (1 + \delta k)\|s\|_{\ell 2}^{2} \quad (1.5)$$

holds all k–sparse vectors s.

It will be loosely said that a matrix A obeys the RIP of order k if δk is not too close to 1. When the RIP holds, the Euclidean length of k–sparse signals is approximately preserved by A, which in turn implies that k–sparse vectors cannot be in the nullspace of A. Clearly this is very important as otherwise there would be no hope of reconstructing these k–sparse vectors. The RIP can also be interpreted as all subsets of k columns taken from A being nearly orthogonal (the columns of A cannot be exactly orthogonal since we have more columns than rows).[8],[9]

3 Progresses and Challenges

Compressive sensing took the signal processing community by storm. Compressive sampling and sensing allow for efficient signal acquisition and storage by capitalizing on the fact that many real-world signals inherently have far fewer degrees of freedom than the signal size might indicate. In some cases, for example, the signals of interest can be expressed as sparse linear combinations of elements from some dictionary, and the sparsity of the representation, in turn, permits efficient algorithms for signal processing. In other cases, the conciseness of the signal model may impose a low-dimensional geometric (often manifold-like) structure to the signal class as a subset of the high-dimensional ambient signal space. Some application areas that capitalize on concise and low-dimensional geometric models include image compression; parameter estimation and image registration; imaging and signal reconstruction.

3.1 Structured Sensing Matrices

Much of the theory concerning explicit performance bounds for compressive sensing revolves around Gaussian and other random matrices. These results have immense value as they show us, in principle, the possibilities of compressive sensing.However, in reality we usually do not have the luxury to choose A as we please. Instead the sensing matrix is often dictated by the physical properties of the sensing process (e.g.,the laws of wave propagation) as well as by constraints related to its practical implementability. Furthermore, sensing matrices with a specific structure can give rise to fast algorithms for matrix-vector multiplication, which will significantly speed up recovery algorithms. Thus the typical sensing matrix in practice is not Gaussian or Bernoulli, but one with a very specific structure, This includes deterministic sensing matrices as well as matrices whose entries are random variables which are coupled across rows and columns in a peculiar way. This can make it highly nontrivial to apply standard proof techniques from the compressive sensing literature.[10]-[12]

Over the last few years researchers have developed a fairly good understanding of how to derive compressive sensing theory for a variety of structured sensing matrices that arise in applications, Despite this admirable progress, the derived bounds obtained so far are not as strong as those for Gaussian-type random matrices. One either needs to collect more measurements or enforce more restrictive bounds on the signal sparsitycompared to Gaussian matrices, or one has to sacrifice*universality*. Here, universality means that a fixed (random) sensing matrix guarantees recovery of *all* sufficiently sparse signals. In comparison, to obtain competitive theoretical bounds using structured sensing matrices we may have to assume that the locations and/or the signs of the non-zero entries of the signal are randomly chosen [13]–[16]. As a consequence the performance guarantees obtained are not universal, as they only hold for *most* signals.

After careful construction of algebraic Structure of matrix for each individual case, often provide the best theoretical performance bounds for structured matrices – and yet, as mentioned before, these bounds still fall short of those for Gaussian matrices. Can we overcome these limitations of the existing theory by developing a collection of tools that allows us to build a compressive sensing theory for structured matrices that is (almost) on par with that for random matrices?

If we have the frredom to design the sensing matrix, then the only condition we impose is that we want deterministic (explicit) constructions with the goal to establish performance bounds that are comparable to those of random matrices, for instance by establishing appropriate RIP bounds. Most bounds to date on the RIP for deterministic matrix constructions are based on the coherence, which in turn causes the number of required samples to scale quadratically with the signal sparsity.

This poses the question, whether we can come up with deterministic matrices which satisfy the RIP in the optimal range of parameters. It may well be that the so constructed matrices will have little use in practice. But if we succeed in this enterprise, I expect the mathematical techniques developed for this purpose to have impact far beyond compressivesensing.

Structured sparsity is only one of many kinds of prior information we may have about the signal or image. Besides obvious constraints such as non-negativity of the signal coefficients,there is also application-specific prior information,such as the likelihood of certain molecule configurations or a minimum distance between sparse coefficients due to some repelling force. In particular in the low SNR regime the proper utilization of available prior information can have abig impact on the quality of the recovered signal. The aim is to develop frameworks that can incorporate various kinds of prior information both at the theoretical and the algorithmic level of compressive sensing.

3.2 Development of Appropriate Signal Recovery Algorithm

Compressive sampling offers a new paradigm for acquiring signals that are compressible with respect to an orthonormal basis. The major algorithmic challenge in compressive sampling is to approximate a compressible signal from noisy samples. paper Some new iterative recovery algorithms should be developed based on optimization approaches. Moreover, these algorithms should offer rigorous bounds on computational cost and storage. It is likely to be extremely efficient for practical problems because it requires only matrix–vector multiplies with the sampling matrix. For compressible signals, the running time is just O(N $\log_2$ N),where N is the length of the signal.

Compressive sampling refers to the idea that, for certain types of signals, a small number of non adaptive samples carries sufficient information to approximate the signal well. Research in this area has two major components:[17]

Sampling: How many samples are necessary to reconstruct signals to a specified precision?What type of samples? How can these sampling schemes be implemented in practice?

Reconstruction: Given the compressive samples, what algorithms can efficiently construct a signal approximation?

The literature already contains a well-developed theory of sampling, which we summarize below.Although algorithmic work has been progressing, the state of knowledge is less than complete. We assert that a practical signal reconstruction algorithm should have all of the following properties.

i) It should accept samples from a variety of sampling schemes.

ii) It should succeed using a minimal number of samples.

iii) It should be robust when samples are contaminated with noise.

iv) It should provide optimal error guarantees for every target signal.

v) It should offer provably efficient resource usage.

3.3 Hardware Design

The concept of compressive sensing has inspired the development of new data acquisition hardware. By now we have seen compressive sensing “in action” in a variety of applications, such as MRI, astronomy, and analog-to-digital conversion. Yet, the construction of compressive sensingbased hardware is still a great challenge. But the process of developing compressive sensing hardware is not the job of the domain scientist alone. The knowledge gained during this process feeds back into the production cycle of compressive sensing, as theoreticians (have to) learn how to adapt their theory to more realistic scenarios, and in turn may then be able to provide the practitioner with better insight into performance bounds and improved design guidelines. Noise is a major limiting factor. Calibration remains a big problem. An efficient feedback loop between the different scientists working on theory, algorithms, and hardware designwill be key to ensure further breakthroughs in this area.

4. Nonlinear compressed sensing

So far we have assumed that the observations we arecollecting can be modeled as linear functionals of the form <x, ak>, k = 1, . . . ,m, where a_k* is a sensing vector representing a row of A. However in many applications we can only take nonlinear measurements. An important example is the case where we observe signal intensities, i.e., the measurements are of the form $<x, a_k>^2|$, the phase information is missing. The problem is then to reconstruct x from intensity measurements only. A classical example is the problem of recovering a signal or image from the intensity measurements of its Fourier transform. Problems of this kind, known as phase retrieval arise in numerous applications, including X-ray crystallography, diffraction imaging, astronomy, and quantum tomography.

Concepts from compressive sensing and matrix completion have recently inspired a new approach to phase retrieval called PhaseLift . It has been shown that if the vectors ak are sampled independently and uniformly at random on the unit sphere, then the signal x can be recovered exactly (up to a global phase factor) from quadraticmeasurements by solving a trace-norm minimization problem provided that m is on the order of n log n measurements4. PhaseLift does not assume that the signal is sparse. It is natural to ask if we can extend the compressive sensing theory to the recovery of sparse signals from intensity measurements. Some initial results can be found in [18], but it is clear that this development is still in its infancy and much more remains to be done. For instance, it would be very useful for a variety of applications to know how many measurements are required to recover an s-sparse signal $x \in C^n$ from Fourier-type intensity measurements.

5. Future scope of Compressed Sensing:

There is a growing gap between the amount of data we generate and the amount of data we are able to store, communicate, and process. In the year 2011 we produced already twice as many data as could be stored . And the gap keeps widening. As long as this development continues there is an urgent need for novel data acquisition concepts like compressive sensing**.**

There is an obvious intellectual achievement, in which compressive sensing and sparse representations play a key role :Advanced probability theory and (in particular) random matrix theory, convex optimization, and applied harmonic analysis will become and already have become standard ingredients of the toolbox of many engineers. At the same time, mathematicians will have gained a much deeper understanding of how to confront real-world applications. Furthermore, compressive sensing has advanced the development of $\ell 1$-minimization algorithms, and more generally of non smooth optimization. These algorithms find wide-spread use in many disciplines, including physics, biology, and economics.

6. Conclusion

To revolutionize technology we will need to develop hardware and algorithms via an integrated, trans disciplinary approach. Hence, in the future when we design sensors, processors, and other devices, we may no longer speak only about hardware and software, where each of these two components is developed essentially separately. Instead, we may have to add a third category, which we could call hybrid ware or mathematical sensing, where the physical device and the mathematical algorithm are completely intertwined and codesigned right from the beginning.

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