Some Interesting results in Entropy Concept

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Abstract

The concept of entropy is one of the basic concepts of science. In this paper we review some basic definitions in symbolic dynamics then we introduce induced algebra over polynomials and correspond each elements of algebra to a relevant matrix. At last we prove some results about the entropy of multiplication of matrices.

Keywords: *entropy, induced algebra, matrix representation, golden algebra.*

1. Introduction

In the last century entropy concept has played crucial role in the science. For example it has been used in physics (Zhang 2012), chemistry, thermodynamics (Kittel and Kroemer 1980), biology (Martyushev and Seleznev 2006), information theory (Marschak 1974), etc. Also this concept is used in different mathematical branches such as theory of fuzzy sets (DELUCA and TERMINI 1972), chaos theory, dynamic systems, etc. (Cambel 1993) There are various definitions for this concept which come from various points of view, for example, Shannon entropy was introduced by Claude E. Shannon in his 1948 paper "A Mathematical Theory of Communication" (Shannon 2001).

Entropy measures the complexity of mappings. The entropy of a shift is an important number, for it is invariant under conjugacy, can be computed for a wide class of shifts, and behaves well under standard operations like factor codes and products. For shifts, it also measures their "information capacity," or ability to transmit messages (Lind and Marcus 1995).

In this paper we establish a commutative and associative algebra, corresponds to a polynomial with real or complex coefficients. We present matrix representation for the elements of this algebra. The special case of such algebra is called golden algebra which is related to $x^2 - x - 1 = 0$ and golden ratio as a historical interesting issue (Markowsky 1992). We will show some properties of the entropy of these special matrices.

2. Preliminaries

This section is devoted to the very basic definitions in symbolic dynamics. The notations has been taken from (Lind and Marcus 1995), (Yuri and Pollicot 1998), and (Dastjerdi and Shaldehi 2012). The proofs of the relevant claims in this section can be found there.

Information is often represented as a sequence of discrete symbols drawn from a fixed finite set. There is a finite set $\mathcal{A}$ of symbols which we will call the alphabet. Elements of $\mathcal{A}$ are also called letters, and they will typically be denoted by $a, b, c, \ldots$ or sometimes by digits like $0, 1, 2, \ldots$ when this is more meaningful.

If $\mathcal{A}$ is a finite alphabet, then the full $\mathcal{A}$-shift is a collection of all bi-infinite sequences of symbols from $\mathcal{A}$. The full r-shift (or simply r-shift) is the full shift over the alphabet $\{0,1,2,\ldots,r-1\}$.

The $\mathcal{A}$-shift is denoted by

$$\mathcal{A}^{\mathbb{Z}} = \{x = (x_i)_{i\in\mathbb{Z}} : x_i \in \mathcal{A} \text{ for all } i \in \mathbb{Z}\}$$

The shift map σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ which its coordinates is $y_i = x_{i+1}$. Let X be a subset of a full shift, and $\mathcal{B}_n(x)$ denote the set of all n-blocks that occur in point in X. The language of X is the collection

$$\mathcal{B}(X) = \bigcup_{n=1}^{\infty} \mathcal{B}_n(X)$$

Let $\mathcal{F}$ be a collection of blocks over $\mathcal{A}$, which we will think of as being the forbidden blocks. For any such $\mathcal{F}$, define $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ which do not contain any block in $\mathcal{F}$. A shift space (or simply shift) is a subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks over $\mathcal{A}$.

Matrix A is *nonnegative* if each of its entries is nonnegative. The nonnegative matrix A is irreducible, if for each ordered pair of indices I; J there exist some $n \geq 0$ such that $A_{IJ}^{n} \geq 0$. The nonnegative matrix A is essential if none of its rows or columns is zero. Also the graph G is essential if and only if its adjacency matrix is essential. Let A be a nonnegative matrix with irreducible components A_1, $A_2, \dots$, A_k. The Perron eigenvalue λ_A of A is

$$\lambda_A = \max_{1\leq i\leq k} \lambda_{A_i}$$

Definition 2.1 (Lind and Marcus 1995) Let G be a graph with edge set $\xi$and adjacency matrix A. The edge shift χ_G or χ_A is the shift space over the alphabet A specified by

$$\chi_G = \chi_A = \{\xi = (\xi_i)_{i\in\mathbb{Z}} \in \varepsilon^{\mathbb{Z}}: t(\xi_i) = i(\xi_i + 1) \quad \forall i \in \mathbb{Z}\}$$

The shift map on χ_G or χ_A is called the edge shift map.

Lemma 2.1 (Lind and Marcus 1995) For an arbitrary nonnegative matrix A, its Perron eigenvalue λ_A is the largest eigenvalue of A.

Consider the matrix $\begin{pmatrix} a & b \\ bb_0 & a + ba_0 \end{pmatrix}$ then Perron eigenvalue is

$$\lambda_A = a + b\left(\frac{a_0 + \sqrt{a_0^2 + 4b_0}}{2}\right)$$

Definition 2.2 (Lind and Marcus 1995) Let X be a shift space. The entropy of X is defined by

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{B}_n(X)|.$$

Theorem 2.1 (Lind and Marcus 1995) Let A be the related matrix to the graph G then $h(\chi_G) = \log \lambda_A$. In addition

$$h(\chi_G) = \log(\ a + b\left(\frac{a_0 + \sqrt{a_0^2 + 4b_0}}{2}\right))$$

3. Main Result

Now, we introduce an algebra of matrices which in special case is called golden algebra. We assume that all matrices are nonnegative. Let $\mathbb{K}$ is the ring of real numbers and $a_0, b_0 \in \mathbb{K}$ be fixed elements. Assume that $\lambda^2 - a_0\lambda - b_0 = 0$, we define an algebra over $\mathbb{K}$ as follows:

$$F = \{x = a + b\lambda | a, b \in \mathbb{K}, \lambda^2 = a_0\lambda + b_0\}$$

with the operations

$$x + x_1 = (a + a_1) + (b + b_1)\lambda$$

and

$$tx = ta + tb\lambda \, , t \in \mathbb{K}$$

and

$$xx_1 = (a + b\lambda)(a_1 + b_1\lambda)$$
$$= (aa_1 + bb_1b_0) + (aa_1 + ba_1 + bb_1a_0)\lambda$$

where $x = a + b\lambda$ and $x_1 = a_1 + b_1\lambda$. It is trivial that F with the above operations is an algebra and we can represent each element of F with a matrix

$$A = \begin{pmatrix} a & b \\ bb_0 & a + ba_0 \end{pmatrix}.$$

Indeed this algebra is isomorphic with $\frac{R[x]}{<f(x)>}$ by the following map

$$x = a + b\lambda \rightarrow \begin{pmatrix} a & b \\ bb_0 & a + ba_0 \end{pmatrix}$$

where $f(x) = x^2 - a_0x - b_0$ and $R[x]$ is the algebra of polynomials (Nazari, Delbaznasab et al.).

Using matrix representation, it can be seen that the multiplication operation of F is ordinary products of matrices. We know that

$$\det(A) = a^2 + aba_0 - b^2b_0$$

and

$$trac(A) = 2a + ba_0.$$

Notice that $\det(A)$ can be reducible as

$$(a - b\lambda_1)(a + b\lambda_2)$$

where λ_1 and λ_2 are the roots of equation

$$\lambda^2 - a_0\lambda - b_0 = 0.$$

Characteristics polynomial of A is

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$$t^2-(2a+ba_0)t+a^2+aba_0-b^2b_0=0$$

clearly the roots of characteristics equation are

$$t=a+b\left(\frac{a_0\pm\sqrt{a_0^2+4b_0}}{2}\right)$$

Since the roots of

$$\lambda^2-a_0\lambda-b_0=0.$$

are

$$\lambda=\frac{a_0\pm\sqrt{a_0^2+4b_0}}{2}$$

then we have

$$t=a+b\lambda$$

and

$$\lambda^2=a_0\lambda+b_0.$$

On the other words, the roots of characteristic polynomial belong to F.

Consider the special conditions $a_0=b_0=1$. We can obtain

$$\lambda=\frac{1\pm\sqrt{5}}{2}$$

that $\frac{1+\sqrt{5}}{2}$ is golden number (Livio 2008). Its matrix representation is

$$\begin{pmatrix} a & b \\ b & a+b \end{pmatrix}.$$

Now we prove the interesting result about entropy on products of matrices.

Theorem 3.1 Let

$$A=\begin{pmatrix} a & b \\ bb_0 & a+ba_0 \end{pmatrix}$$

and

$$A_1=\begin{pmatrix} a_1 & b_1 \\ b_1b_0 & a_1+b_1a_0 \end{pmatrix}$$

then

$$h(\chi_{AA_1})=h(\chi_A)+h(\chi_{A_1}).$$

Proof. We have

$$AA_1=$$

$$\begin{pmatrix} aa_1+bb_1b_0 & ab_1+ba_1+bb_1a_0 \\ ab_1b_0+ba_1b_0+bb_1a_0b_0 & bb_1b_0+aa_1+ab_1a_0+ba_1a_0+bb_1a_0^2 \end{pmatrix}$$

then

$$h(\chi_{AA_1})=$$

$$\log(aa_1+bb_1b_0+(ab_1+ba_1+bb_1a_0)\lambda)$$

which

$$\lambda=\frac{a_0+\sqrt{a_0^2+4b_0}}{2}$$

therefore

$$h(\chi_{AA_1})=\log[(a+b\lambda)(a_1+b_1\lambda)]=$$

$$h(\chi_A)+h(\chi_{A_1})$$

Noticing that examination of this property for polynomials of degree 3 and 4 is similar; we show the property for a polynomial of degree 4.

Suppose that $w^4=1$ is a polynomial in complex numbers. If

$$x=a_0+a_1w+a_2w^2+a_3w^3$$

and

$$y=b_0+b_1w+b_2w^2+b_3w^3$$

then we define the following multiplication:

$$xy=(a_0b_0+a_1b_3+a_3b_1+a_2b_2)+$$
$$(a_0b_1+a_1b_0+a_2b_3+a_3b_2)w+$$
$$(a_0b_2+a_1b_1+a_2b_0+a_3b_3)w^2+$$
$$(a_0b_3+a_1b_2+a_2b_1+a_3b_0)w^3$$

It can be shown that this operation is associative and distributive; obviously these operations can create algebra.

Matrix representation for each elements of this algebra is

$$A=\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_3 & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_1 & a_2 & a_3 & a_0 \end{pmatrix}.$$

The set of all matrices A create a four-dimensional algebra, actually each element

$$x=a_0+a_1w+a_2w^2+a_3w^3$$

can be considered as (a_0,a_1,a_2,a_3). Also we have

$$\det(A)=a_0^4-a_1^4+a_2^4-a_3^4$$
$$-4a_0^2a_1a_3+4a_1^2a_0a_2-4a_2^2a_1a_3+4a_3^2a_0a_2$$
$$-2a_0^2a_2^2+2a_1^2a_3^2$$

The roots of $w^4=1$ are $w_0=1$, $w_1=-1$, $w_2=i$, and $w_3=-i$. Corresponding to these roots we define four functions from $\mathbb{R}^4$ to $\mathbb{C}$.

$$\phi_0(x)=\phi_0(a_0,a_1,a_2,a_3)=a_0+a_1+a_2+a_3$$

corresponds to $w_0=1$.

$$\phi_1(x)=a_0-a_1+a_2-a_3$$

corresponds to $w_1=-1$.

$$\phi_2(x) = a_0 - a_2 + (a_1 - a_3)i$$
corresponds to $w_2 = i$.
$$\phi_3(x) = a_0 - a_2 + (a_3 - a_1)i$$
corresponds to $w_3 = -i$.

For any $x = a_0 + a_1 w + a_2 w^2 + a_3 w^3$, the eigenvalues of the corresponding matrix A are $\phi_0(x), \phi_1(x), \phi_2(x)$, and $\phi_3(x)$.

With some calculations it can be shown that each function has multiplication property i.e.:
$$\phi_j(xy) = \phi_j(x)\phi_j(y)$$
for $j = 0,1,2,3$. Also we can see that
$$\det(A) = \prod_{j=0}^{n} \phi_j(x)$$
Now we prove theorem 3.1 for the case n=4.

Theorem 3.2 If A and B are square matrices with nonnegative arrays such that $\det(A)\det(B) \neq 0$, then
$$h(AB) = h(A) + h(B).$$
Proof. First we now that if A and B are nonnegative matrices then AB is nonnegative square matrix too, in addition if $\det(A)\det(B) \neq 0$ then $\det(AB) \neq 0$. If x and y correspond to A and B respectively, then xy corresponds to AB. Assume that the eigenvalues of A are $\phi_j(x)$ for $j = 0,1,2,3$. Similarly, eigenvalues of B are $\phi_j(y)$ and eigenvalues of AB are $\phi_j(xy)$ for $j = 0,1,2,3$. We define
$$\Phi(x) = \max_{0\le j\le 3} \phi_j(x)$$
and
$$\Phi(y) = \max_{0\le j\le 3} \phi_j(y)$$
and
$$\Phi(xy) = \max_{0\le j\le 3} \phi_j(xy)$$
Since $\Phi(x)$ and $\Phi(y)$ have multiplication property then
$$\begin{aligned} h(AB) &= \log\big(\Phi(xy)\big) = \log(\Phi(x)\Phi(y)) \\ &= \log\big(\Phi(x)\big) + \log\big(\Phi(y)\big) \\ &= h(A) + h(B) \end{aligned}$$

4. Conclusions

In this article, we proved a multiplication property of eigenvalues for some special matrices. Indeed it can be seen as an issue in linear algebra which consequently leads to some results in entropy. We conjecture that this proposition can be prove in polynomials with higher degree.

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