Legendre Polynomials Approach for Bi-Level Linear Fractional Programming Problem

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Abstract

This paper presents a solution method of bi-level linear fractional programming problem (BLLFPP) using of Legendre polynomials. The Legendre polynomial is a series expansion that a representation of a function *f(x)* which is finite and single-valued in the interval [*a, b*]. Levels are classified as upper level and lower level, and they are weighted with respect to their classes before Legendre polynomials approach unified levels by using their weights. Thus, the problem is reduced to a single objective. Numerical example is provided to clarify the efficiency and feasibility of the suggested approach.

Key words: *Fractional programming; Series expansions; Management decision making; Linear programming.*

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1 Introduction

Fractional programming stems from the fact that programming models could better fit the real problems if we consider optimization of ratio between the physical and/or economic quantities. Literature survey reveals wide applications of fractional programming in different areas ranging from engineering to economics can be found in ([1], [2]).

The concept of bi-level programming problem (BLPP) was first introduced by Candler and Townsley in 1982 [3] as a tool to resolve the decision problems with two levels of structures in a hierarchical decision system. Actually, it is the special case of a multilevel programming problem (MLPP) for solving decentralized planning problems with multiple decision makers (DMs) in a hierarchical organization.

Bi-level programming has been applied in many real life problems such as agriculture, bio-fuel production, economic systems, finance, engineering, banking, management sciences, and transportation problem [4], [5], [6], [7] and [8].

In a BLPP, two DMs (upper and lower) are located at two different hierarchical levels, each independently controlling only one set of decision variables with different

and perhaps conflicting objectives. The execution of decisions is sequential, from the upper to lower levels; each independently optimizes its own objective. But the decision of the upper-level DM (the leader) often is affected by the reaction of the lower-level DM (the follower) because of his / her dissatisfaction with the solution. Consequently, decision deadlock arises frequently and the problem of distribution of proper decision power encountered in most of the practical situations.

The use of Taylor series as a solution method for solving bi-level linear fractional programming problem (BLLFPP) was introduced by Toksari in [9], where both objectives were transformed by using 1st order Taylor polynomial series. In that work [9], the Taylor series obtains polynomial objective functions which are equivalent to fractional objective functions. Thus, BLLFPP can be reduced into a single function. In other words, suitable transformations can be applied to formulate bi-level programming. In the compromised function, the first level are weighted more than the second level because the first level decision maker (DM) is called the center.

In the present paper, Legendre polynomials [10, 11] as another different approach for solving the same (BLLFPP) problem are suggested. On the other hand, the proposed approach facilitates computation to reduce the complexity in problem solving. The performance of the proposed approach is experimentally violated by numerical example considered by Toksari in [9]. Results demonstrate that the proposed approach here runs more effectively and accurate than the Taylor series approach in [9].

This paper is organized as follows: we start in Section 2 by formulating the model of bi-level linear fractional programming problem (BLLFP) along with the solution concept. In Section 3, Legendre polynomials approach for solving (BLLFPP) is suggested in finite steps. Numerical example is provided in Section 4 to illustrate the developed results. Finally, the paper is concluded in Section 5 with some open points for future research work in the field of bi-level linear fractional optimization.

2 Problem Formulation and the Solution Concept

Let both the leader and the follower have a motivation to cooperate with each other and try to maximize his / her own benefit, paying series attention to the preferences of the other. Then, the vectors of the decision variables x, y are under the control of the leader and the follower, respectively and $z_1(x, y), z_2(x, y)$ are their respective preference functions. Such a bi-level linear fractional programming problem can be formulated as follows:

$$\underset{(x)}{\text{maximize}} \quad z_1(x, y), \qquad (1)$$

$$\underset{(y)}{\text{maximize}} \quad z_2(x, y), \qquad (2)$$

$$\text{subject to}$$

$$(x, y) \in M, \qquad (3)$$

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where objective functions $z_k(x, y); (k = 1,2)$ are represented by a linear fractional function in the form:

$$z_k(x, y) = \frac{p_k(x, y)}{q_k(x, y)}, \qquad (4)$$

and the set M of constraints is given by:

$$M = \{(x, y) \in R^{N(1)} \times R^{N(2)} \, / \, Ax + By \leq b; x \geq 0, y \geq 0\}, \qquad (5)$$

where

$A \in R^{m \times N(1)}, B \in R^{m \times N(2)}, b \in R^m, x \in R^{N(1)}, y \in R^{N(2)}$ and $\text{rank}(A) = \text{rank}(B) = m.$

For a given $x \in R^{N(1)}$, the follower's feasible region, denoted by M_x is:

$$M_x = \{y \in R^{N(2)} \, / \, By \leq b - Ax, y \geq 0\} \qquad (6)$$

So that the follower's problems can be expressed as:

$$\underset{y \in M_x}{\text{maximize}} \quad z_2(x, y) \qquad (7)$$

Clearly, M and M_x are convex sets, see [12].

The set $\{x \in R^{N(1)} \, / \, x \geq 0\}$ is called the policy space of (BLLFPP) (1)-(3) and the set

$$M_p = \{x \in R^{N(1)} \, / \, x \geq 0, M_x \neq \phi\} \qquad (8)$$

is called the feasible policy space of (BLLFPP) (1)-(3).

The concept of optimal solution of problem (BLLFPP) (1)-(3) is given in the following definition.

Definition1.

If (x^*, y^*) is a feasible solution of (BLLFPP) (1)-(3) and no other feasible solution $(x, y) \in M$ exists, such that $z_k(x^*, y^*) \leq z_k(x, y)$ for at least one k; then (x^*, y^*) is the optimal solution of the problem (BLLFPP) (1)-(3).

3 Legendre Polynomials Approach for Bi-level Linear Fractional Programming

In the BLLFPP (1)-(3), objective functions are transformed by using Legendre polynomials at first, and then a satisfactory value(s) for the variable(s) of the model is obtained by solving the model; which has a single objective function. Here, Legendre polynomials obtain polynomial objective functions which are equivalent to the fractional objective function. Then, the BLLFPP (1)-(3) can be reduced into a single

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objective. In the compromised objective function, weight of the first level is more than weight of the second level because the first level decision maker (FLDM) is called the center.

The proposed approach for Bi-level Linear Fractional Programming BLLFPP (1)-(3) can be explained in the following finite steps:

Step 1.

Determine the feasible policy space $M_p = \{x \in R^{N(1)} / x \geq 0, M_x \neq \phi\}$ of problem (BLLFPP) (1)-(3) i.e. determine the intervals $x \in [a_1, b_1]$, $y \in [a_2, b_2]$ which satisfy the feasibility of solution.

Step 2.

Transform the intervals $[a_1, b_1]$ and $[a_2, b_2]$ of the *x*-axis and *y*-axis, respectively into the closed interval [-1, 1] of the *t*-axis, where:

$$t_1 = \frac{2}{b_1 - a_1}[x - \frac{b_1 + a_1}{2}], a_1 \prec b_1, \qquad t_2 = \frac{2}{b_2 - a_2}[y - \frac{b_2 + a_2}{2}], a_2 \prec b_2$$

Step 3.

Transform the objective functions $z_k(x, y); (k = 1,2)$ by using Legendre polynomials defined in [10, 11] in the form:

$$\hat{z}_k(x, y) \cong \sum_{j=0}^{n} \sum_{i=0}^{m} a_{ij} P_i(x) P_j(y); k = 1,2 \qquad (9)$$

where the coefficients a_{ij} are computed by the formula:

$$a_{ij} = \frac{1}{h_i h_j} \int_{b_2}^{a_2} \int_{b_1}^{a_1} z_k(x, y) P_i(x) P_j(y) \, dx \, dy; k = 1,2 \qquad (10)$$

such that

$$h_i = \frac{b_1 - a_1}{2i + 1}; \quad h_j = \frac{b_2 - a_2}{2j + 1}$$

and

$$P_i(x) = \frac{1}{i!(b_1 - a_1)^i} D_x^i (x - a_1)^i (x - b_1)^i, \qquad (11)$$

$$P_j(y) = \frac{1}{j!(b_2 - a_2)^j} D_x^j (y - a_2)^j (y - b_2)^j, \qquad (12)$$

The symbols *D* means differentiation and ! means factorial.

Step 4.

Find satisfactory (x^*, y^*) by solving the reduced problem to a single objective. In the compromised objective function, weight of the first level is more than weight of the second level because the first level decision maker (FLDM) is called the center.

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The BLLFPP (1)-(3) now is converted into a new mathematical model which is written as follows:

$$\text{maximize} \quad w_1 \hat{z}_1(x, y) + w_2 \hat{z}_2(x, y), \qquad (13)$$

$$\text{subject to}$$

$$M = \{(x, y) \in R^{N(1)} \times R^{N(2)} \ / \ Ax + By \le b; x \ge 0, y \ge 0\}, \qquad (14)$$

where $w_1 \ge 0, w_2 \ge 0$ and $\sum_{k=1}^{2} w_k = 1$ such that $w_1 \succ w_2$.

Thus, a new model (13)-(14) , which is equivalent to BLLFPP (1)-(3) is obtained and can be solved using any linear programming software package based on the Simplex method, for example TORA optimization system which is a Windows-based software designed for use with many of the techniques presented in the book of H.A. Taha [13].

4 Numerical Example

The following example studied earlier by Toksari [9] is considered to demonstrate the effectiveness and accuracy of the suggested Legendre polynomials approach.

$$\max z_1(x, y) = \frac{x + 2y}{x + y + 1}$$

$$\max z_2(x, y) = \frac{2x + y}{2x + 3y + 1}$$

$$\text{Subject to}$$

$$-x + 2y \le 3$$
$$2x - y \le 3$$
$$x + y \ge 3$$
$$x, y \ge 0$$

$$x \in [1,3], y \in [1,3]$$

If a function $z(x, y)$ is defined as $x \in [a_1, b_1], y \in [a_2, b_2]$ it is some times necessary in the applications to expand the function in Legendre series "Orthogonal "so we have to transform the intervals $[a_1, b_1], [a_2, b_2]$ to $[-1,1]$.

So let $\quad x = \frac{2}{b_1 - a_1}\left[X - \frac{b_1 + a_1}{2} \right] \quad, \quad y = \frac{2}{b_2 - a_2}\left[Y - \frac{b_2 + a_2}{2} \right]$

with $\quad a_1 < b_1 \ , \ a_2 < b_2$

It is therefore;

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$$z(x, y) \approx \hat{z}(x, y) = \sum_{j=0}^{n} \sum_{i=0}^{m} a_{ij} p_i(x) p_j(y)$$

Such that

$$a_{ij} = \frac{1}{h_i h_j} \int_{b_2}^{a_2} \int_{b_1}^{a_1} z(x, y) p_i(x) p_j(y) dx dy \;\;,$$

$$h_i = \frac{b_1 - a_1}{2i + 1} \;,\; h_j = \frac{b_2 - a_2}{2j + 1} \;\;,$$

$$p_i(x) = \left[\frac{1}{i!(b_1 - a_1)^i} \right] \left[D_x^i \left[(x - a_1)^i (x - b_1)^i \right] \right] \;,$$

$$\cdot \, p_j(y) = \left[\frac{1}{j!(b_2 - a_2)^j} \right] \left[D_y^j \left[(y - a_2)^j (y - b_2)^j \right] \right]$$

The bi-level linear fractional programming problem (BLLFPP) under consideration could be converted to a single objective linear fractional programming problem (SOLFPP) under w_1 (weight of first level) is equal 0.51 and w_2 (weight of second level) is equal 0.49 since; the first level weight must be more than the weight of the second level as follows:

$$\max z(x, y) = 0.51 (z_1(x, y)) + 0.49 (z_2(x, y))$$

$$\max z(x, y) = 0.51 \left(\frac{x + 2y}{x + y + 1} \right) + 0.49 \left(\frac{2x + y}{2x + 3y + 1} \right)$$

Subject to

$$-x + 2y \leq 3$$
$$2x - y \leq 3$$
$$x + y \geq 3$$
$$x, y \geq 0$$

where $\quad 1 \leq x, y \leq 3$

and we suppose that

$$a_1 = a_2 = 1 \;,\; b_1 = b_2 = 3.$$

Let $m = 1, n = 1$

Then

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$$p_0(x) = 1 \; , \quad p_1(x) = x - 2$$
$$p_0(y) = 1 \; , \quad p_1(y) = y - 2$$

Also

$$h_i = \frac{2}{2i+1} \; , \; h_j = \frac{2}{2j+1} \quad , \; \frac{1}{h_i h_j} = \frac{(2i+1)(2j+1)}{4}$$

Then

$$\frac{1}{h_0 h_0} = \frac{1}{4} \quad , \quad \frac{1}{h_0 h_1} = \frac{3}{4} \quad , \quad \frac{1}{h_1 h_0} = \frac{3}{4} \quad , \quad \frac{1}{h_1 h_1} = \frac{9}{4}$$

Since

$$a_{ij} = \frac{1}{h_i h_j} \int_{b_2}^{a_2} \int_{b_1}^{a_1} z(x,y) p_i(x) p_j(y) dx dy$$

Then , we calculate the coefficients as:

$$a_{00} = \frac{1}{4} \int_3^1 \int_3^1 z(x,y) dx dy = 0.8749922799$$

$$a_{01} = \frac{3}{4} \int_3^1 \int_3^1 (z(x,y))(y-2) dx dy = 0.055743254$$

$$a_{10} = \frac{3}{4} \int_3^1 \int_3^1 (z(x,y))(x-2) dx dy = 0.022617872$$

$$a_{11} = \frac{9}{4} \int_3^1 \int_3^1 (z(x,y))(x-2)(y-2) dx dy = -0.019846537$$

Then the objective functions are transformed by using Legendre polynomial as follows:

$$\hat{z}(x,y) = 0.8749922799 + 0.022617872(x-2) + 0.055743254(y-2)$$
$$- 0.019846537(x-2)(y-2)$$

Thus, the final form of the (BLLFPP) is obtained as follows:

Find (x,y) so as to:

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$$\max \; \hat{z}(x, y)$$

Subject to

$$-x + 2y \le 3$$
$$2x - y \le 3$$
$$x + y \ge 3$$
$$x, y \ge 0$$

The problem is solved and the solution of the above problem is as follows:

$$x = 3 \;\; \text{and} \;\; y = 3$$

So

$$\hat{z}(3,3) = 0.933506870 \;\; \text{and} \;\; z(3,3) = 0.931339$$

with the error will be 0.00216758

Repeat the steps again

Let $m = 2, n = 2$

Then

$$p_0(x) = 1 \;, \;\; p_1(x) = x - 2 \;, \;\; p_1(x) = \frac{3}{2}x^2 - 6x + \frac{11}{2}$$
$$p_0(y) = 1 \;, \;\; p_1(y) = y - 2 \;, \;\; p_1(y) = \frac{3}{2}y^2 - 6y + \frac{11}{2}$$

Also

$$h_i = \frac{2}{2i+1} \;, \;\; h_j = \frac{2}{2j+1} \;\;\;, \;\; \frac{1}{h_i h_j} = \frac{(2i+1)(2j+1)}{4}$$

Then

$$\frac{1}{h_0 h_0} = \frac{1}{4} \;, \;\; \frac{1}{h_0 h_1} = \frac{3}{4} \;, \;\; \frac{1}{h_0 h_2} = \frac{5}{4} \;, \;\; \frac{1}{h_1 h_0} = \frac{3}{4} \;, \;\; \frac{1}{h_1 h_1} = \frac{9}{4} \;, \;\; \frac{1}{h_1 h_2} = \frac{15}{4}$$
$$\frac{1}{h_2 h_0} = \frac{5}{4} \;, \;\; \frac{1}{h_2 h_1} = \frac{15}{4} \;, \;\; \frac{1}{h_2 h_2} = \frac{25}{4}$$

Since

$$a_{ij} = \frac{1}{h_i h_j} \int_{b_2}^{a_2} \int_{b_1}^{a_1} z(x, y) p_i(x) p_j(y) \, dx \, dy$$

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Then, the Legendre coefficients can be calculated as:

$$a_{00} = 0.874992 \quad , a_{01} = 0.0557433 \ , a_{02} = -0.00636636$$
$$a_{10} = 0.0226179 \quad , a_{11} = -0.0198465 \ , a_{12} = 0.0044732$$
$$a_{20} = -0.00283789 \ , a_{21} = 0.00367849 \ , a_{22} = -0.00124123$$

Then, the objective functions are transformed by using Legendre polynomial as follows:

$\hat{z}(x,y) = 0.874992 + 0.0226179 \ (-2 + x) - 0.00283789 \ (5.5 - 6\ x + 1.5\ x^2)$

$+ 0.0557433 \ (-2 + y) - 0.0198465 \ (-2 + x) \ (-2 + y) + 0.00367849 \ (5.5 - 6x + 1.5\ x^2) \ (-2 + y)$

$-0.00636636 \ (5.5 - 6\ y + 1.5\ y^2) + 0.0044732 \ (-2. + x) \ (5.5 - 6\ y + 1.5\ y^2)$

$-0.00124123 \ (5.5 - 6\ x + 1.5\ x^2) \ (5.5 - 6\ y + 1.5\ y^2)$

Thus, the final form of the (BLLFPP) is rewritten in the following manner:

Find (x, y) so as to:

$$\max \hat{z}(x, y)$$
$$\text{Subject to}$$
$$-x + 2y \le 3$$
$$2x - y \le 3$$
$$x + y \ge 3$$
$$x, y \ge 0$$

The problem is solved and the solution of the above problem is found: $x = 3$ and $y = 3$.

Therefore, we obtain:

$$\hat{z}(3,3) = 0.931213 \text{ and } z(3,3) = 0.931339$$
$$\text{with the error will be } 0.000126205$$

5 Conclusions

Legendre Polynomials Approach for Bi-level Linear Fractional Programming (BLLFPP) proposed in this study can be extended to solve quadratic bi-level programming problems without involving any extra computational load. The proposed approach can be extended to multilevel decentralized planning problems for distribution of decision powers to decision makers in a large hierarchical decision organization. Generally, an extension of the approach to fuzzy and stochastic

nonlinear bi-level and multilevel linear fractional programming problems is one of our current research problems.

However, it is hoped that the approach presented here can contribute to future study in the field of practical hierarchical decision-making problems. Finally, results demonstrate that the proposed approach here runs more effectively and accurate than the Taylor series approach suggested by Toksari in [9].

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