

## Convolution Inequality and $\cos \pi \rho$ theorem

**Seid M. Teum, Mobin Ahmad Syed and Neyaz Ahmad Sheikh**

*Department of Mathematics, Faculty Of Science, Jazan University, P.O. Box  
 2097, Jazan 45142, Saudi Arabia*

*Department of Mathematics, National Institute of Technology Srinagar(India) P.O  
 190006.*

*Email: seid\_mt@yahoo.com, profmobin@yahoo.com and neyaznit@yahoo.co.in*

**Abstract:** In this paper we study the asymptotic behaviour of a subharmonic functions of order less than half which are extremely to well known inequality usually referred to as the “ $\cos \pi \rho$  Theorem” due to Winman [1] and Valiron [2] and describe the regularity of related quantities. If  $U = \log|f|$ , where  $f$  is entire functions, the extremal problem for the  $\cos \pi \rho$  theorem has been studied by Drasin and D.F Shea [3]. The purpose of this paper is to obtain similar results for arbitrary subharmonic function extremal to the  $\cos \pi \rho$  theorem.

**Keywords/ Phrases:**  $\delta$ -subharmonic function, star-function, characteristic function, Polya peaks.

**AMS Subject Classification 2010:** 30A10, 30C14, 30D15

### 1. Introduction

Let  $U$  be a subharmonic function in the complex plane and

$$N(r, u) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta, \quad B(r, u) = \sup_{|z|=r} u(z) \quad \text{and} \quad A(r, u) = \inf_{|z|=r} u(z).$$

The Nevanlinna characteristic  $T(r, u)$  of  $u$  is defined by

$$T(r, u) = N(r, u)$$

and the order  $\rho$  of  $u$  by

$$\rho = \lim_{r \rightarrow \infty} \sup \frac{\log T(r, u)}{\log r}.$$

if  $\rho$  is finite,  $T(r, u)$  has sequence of Pòlya Peaks  $\{r_n\}$  of order  $\rho$ , that is,

$$T(r, u) \leq (1 + \epsilon_n) \left( \frac{r}{r_n} \right)^\rho T(r_n, u), \quad (\epsilon_n r_n \leq r \leq \frac{r_n}{\epsilon_n})$$

for some sequence  $\epsilon_n \rightarrow 0$  and  $\epsilon_n r_n \rightarrow \infty$  as  $n \rightarrow \infty$  (for the proof see Edrei [1], W. Hayman [2]).

Let  $\{r_n\}$  be a sequence of Pòlya peaks for  $T(r, u)$  of order  $\rho$ , and set

$$\beta = \frac{1}{2} m \{ \theta \in (-\pi, \pi] : U(re^{i\theta}) > 0 \}, \quad n = 1, 2, 3, \dots$$

where  $m$  is the Lebesgue measure on the real line. Put  $\beta_0 = \liminf \beta_n$ ,  $0 \leq \beta_n \leq \pi$ ,

then we have a subsequence  $\{r_{n_k}\}$  of  $\{r_n\}$  such that  $\beta_{n_k} \rightarrow \beta_n$  as  $k \rightarrow \infty$ . Since a subsequence of  $\{r_n\}$  is also a sequence is also a sequence of Pòlya peaks for  $T(r, u)$ , we assume that  $\beta_n \rightarrow \beta_0$  as  $n \rightarrow \infty$ .

We note that by A. Baernstein [3] (spread theorem)

$$\beta_0 = \pi \quad \text{for } 0 < \rho \leq \frac{1}{2} \quad (1)$$

Let  $u$  be a subharmonic function of order  $\rho$ ,  $0 < \rho < 1$ . Winman [4] and Valiron [5] proved that

$$\limsup_{r \rightarrow \infty} \frac{A(r, u)}{B(r, u)} \geq \cos \pi \rho, \quad (2)$$

is sharp, this result is usually referred as the “ $\cos \pi \rho$ -Theorem”.

If  $u = \log|f|$  is extremal, that is, if equality holds in (2) Drasin and D.F. Shea[6] have studied the asymptotic behaviour of the function  $f$ . In this paper we use convolution inequality due to M. Essèn, J. Rossi and D.F. Shea[7] via normal family of subharmonic functions extremal to (2) near Pòlya peaks. We will also see that the extremal functions are in some sense extremal to the well known sharp inequalities due to Paley[8], Ostrovski[9] and Edei[10]. The method employed here indicates how convolution inequalities are powerful tools in extremal problems. We state our main result.

**Theorem 1.1** *Suppose  $u$  is a sub-harmonic function of order  $\rho$ ,  $0 \leq \rho < \frac{1}{2}$  and  $\{r_n\}$  a sequence of Pòlya peaks of order  $\rho$ . If  $u$  is extremal for (2) that is  $\limsup \frac{A(r_n, u)}{B(r_n, u)} = \cos(\pi \rho)$ , then*

$$(a) \lim_{n \rightarrow \infty} \frac{B(r r_n, u)}{T(r_n, u)} = \frac{\pi \rho r^\rho}{\sin \pi \rho}, \quad (0 < r < \infty)$$

$$(b) \lim_{n \rightarrow \infty} \frac{T(r r_n, u)}{B(r_n, u)} = r^\rho = \lim_{n \rightarrow \infty} \frac{N(r r_n, u)}{T(r_n, u)}, \text{ and}$$

(c) *there is a subsequence  $I$  of the positive integers such that  $u(r r_n e^{i\theta}) = (I + o(1))T(r_n, u) \frac{\pi \rho r^\rho}{\sin \pi \rho} \cos(\rho(\theta - \alpha))$  as  $n \rightarrow \infty$ ,  $n \in I$  for almost all  $\theta$ ,  $|\theta - \alpha| \leq \pi$  and for some  $\alpha \in (-\pi, \pi]$ .*

## 1 Some facts we need to prove the theorem 1

**a)** The star-function. Let  $u$  be a subharmonic function in the complex plane. Following A. Baerstein[11] we define the star-function of  $u$  by:

$u^*(re^{i\theta}) = \sup_{E} \frac{1}{2\pi} \int u(re^{i\phi}) d\phi$ , where the supremum is taken over all measurable set  $E$ , with  $mE = 2\theta$ , where  $m$  is the Lebesgue measure on  $\mathbb{R}$ . It is proved that

$$u^*(re^{i\theta}) = \frac{1}{\pi} \int_0^\theta u(re^{i\phi}) d\phi$$

where,  $\phi \rightarrow u(re^{i\theta})$  is the symmetric decreasing rearrangement of  $u(re^{i\phi})$  on  $[-\pi, \pi]$ . A. Baerstein[11] proved that  $u^*$  is subharmonic in the upper half plane  $\pi^+$  and continuous closure of  $\pi^+$  except possibly at the origin, and that the supremum in (4) is attained in some set  $E \subset [-\pi, \pi]$ . From the definition of  $T(r, u)$ , we have

$$T(r, u) = \max_\theta u^*(re^{i\theta}), \quad (0 \leq \theta \leq \pi), \quad N(r, u) = u^*(re^{i\theta}) \quad (3)$$

$$B(r, u) = \sup_{|z|=r} u(z) = \frac{\partial}{\partial \theta} u^*(re^{i\theta}) \Big|_{\theta=0} \quad (4)$$

$$A(r, u) = \inf_{|z|=r} u(z) = \pi \frac{\partial}{\partial \theta} u(re^{i\theta}) \Big|_{\theta=\pi} \quad (5)$$

**b)** We also need the following result due to Anderson, J.M and A. Baerstein [12]. Let  $u$  be subharmonic function in the plane of order  $0 \leq \rho < \infty$  and  $\{r_n\}$  a sequence of Pòlya peaks for  $T(r, u)$  of order  $\rho$ .

Set  $u_n = \frac{u(r_n z)}{T(r_n, u)}$ ,  $n = 1, 2, 3, \dots$ . Here we have  $T(r, u_n) = \frac{T(r r_n, u)}{T(r_n, u)}$ ,  $B(r, u) = \frac{B(r r_n, u)}{T(r_n, u)}$ , and

$u_n^*(Z) = \frac{u^*(r_n z)}{T(r_n, u)}$ . Anderson J.M and A. Baerstein [12] have proved that there is a subharmonic function  $v$  and a subsequence  $I = \{n_k\}$  of positive integers such that the following statements hold as  $n \rightarrow \infty$  in  $I$ :

$$(i) \lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} |u_n(re^{i\theta}) - v(re^{i\theta})| d\theta = \lim_{n \rightarrow \infty} N(r, |u_n - v|) = 0, \quad 0 < r < \infty, \quad (6)$$

$$(ii) \lim_{n \rightarrow \infty} T(r, u_n) = T(r, v) \leq r^\rho, \quad (7)$$

$$(iii) \lim_{n \rightarrow \infty} N(r, r^{(n)}) = N(r, v). \quad (8)$$

Since  $T(I, u_n) = 1$ , it follows from (ii) that  $T(I, v) = 1$ . We refer to the subharmonic function  $v$  as the limit function of  $u$ . We remark that if  $u$  is subharmonic of order  $\rho$ ,  $0 \leq \rho < \frac{1}{2}$ , then by (1),  $\beta_n \rightarrow \pi$  as  $n \rightarrow \infty$  and using (6) one can show (for the proof see Seid[13]) that

$$u_n^*(e^{i\pi}) \rightarrow v^*(e^{i\pi}) = 1 \text{ as } n \rightarrow \infty. \quad (9)$$

We restate Theorem 1.1 in terms of the limit function  $v$  of  $u$ .

**Theorem 2.1** *Let  $u$  be a sub-harmonic function of order  $\rho$ ,  $0 \leq \rho < \frac{1}{2}$  and  $\{r_n\}$  a sequence of Pòlya peaks of order  $\rho$ . if  $u$  is extremal to (2) and  $v$  is a limit function of  $u$ , then*

$$(a) B(r, v) = \frac{\pi \rho r^\rho}{\sin \pi \rho},$$

$$(b) A(r, v) = \pi \rho r^\rho \frac{\cos \pi \rho}{\sin \pi \rho},$$

$$(c) N(r, v) = r^\rho = T(r, v), (0 < r < \infty),$$

$$(d) \text{ if } n(t) \text{ is the Reisz mass in the ball } B(0, t) \text{ then } n(t) = \rho r^\rho,$$

$$(e) v(re^{i\theta}) = \frac{\pi \rho r^\rho}{\sin \pi \rho} \cos \rho(\theta - \alpha) |\theta - \pi| \leq \pi, \text{ for some } \alpha \in [-\pi, \pi].$$

Thus, if  $u$  is extremal to (2), then the above theorem implies the limit function  $v$  of  $u$  satisfies:

$$(i) \frac{B(r, v)}{T(r, v)} = \frac{\pi \rho}{\sin \pi \rho}, \text{ which shows that } v \text{ is extremal to inequality.}$$

$$\lim_{r \rightarrow \infty} \inf \frac{B(r, u)}{T(r, u)} \leq \frac{\pi \rho}{\sin \pi \rho} \text{ due to Paley [8].}$$

$$(ii) \frac{A(r, u)}{T(r, u)} \geq \pi \lambda \frac{\cos \pi \rho}{\sin \pi \rho}, \text{ that is } v \text{ is extremal to the inequality } \lim_{r \rightarrow \infty} \sup \frac{A(r, u)}{T(r, u)} \geq \pi \rho \frac{\cos \rho}{\sin \rho} \text{ due to Ostrovskii [9] and Edrei [10],}$$

$$(iii) \frac{A(r, v)}{B(r, v)} = \cos \pi \rho \dots \text{ that is extremal to the } \cos \pi \rho \text{ inequality}$$

$$\lim_{r \rightarrow \infty} \sup \frac{A(r, u)}{B(r, u)} \geq \cos \pi \rho. \text{ due to Valiron and Wiman.}$$

All the above equalities shows that in some sense if  $u$  is extremal to Valiron and Wiman inequality, then it is extremal to Paley inequality, and to the Ostrovskii and Edrei inequality.

### Proof of Theorem 2.1:

We shall prove Theorem 2.1, and Theorem 1.1 follows from (6), (7) and (8).

**Lemma 1.** Let  $u$  be a subharmonic function of order  $0 < \rho < \frac{1}{2}$ , and  $\{r_n\}$  is a sequence of polya peaks for  $T(r, u)$  of order  $\rho$ . If  $v$  is a limit function of  $u$ , then

$$\lim_{n \rightarrow \infty} B(r, u_n) = \lim_{n \rightarrow \infty} \frac{B(r r_n, u)}{T(r_n, u)} = B(r, v) \leq \frac{\pi \rho}{\sin \pi \rho} \quad (10)$$

**Proof.** Fix  $r > 0$  and put  $B(r, u_n) = u_n(re^{i\alpha_n})$ ,  $\alpha_n \in (-\pi, \pi]$ ,  $n = 1, 2, 3, \dots$

Assume  $\alpha_n \rightarrow \alpha_0$  as  $n \rightarrow \infty$ . The for  $s > r$ , we have

$$B(r, u_n) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} u_n(se^{i\theta}) P_r(\theta - \alpha_n) d\theta$$

where,  $P_r$  is the Poisson kernel. By (6) and the dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \sup B(r, u_n) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} v(se^{i\theta}) P_{\frac{r}{s}}(\theta - \alpha) d\theta \leq B(s, v)$$

Since this holds for any  $s > r$  and  $B(s, v)$  is a continuous function of  $s$ , letting  $s$  approach  $r$  and by (16), we get

$$\lim_{n \rightarrow \infty} B(r, u_n) \leq B(r, v), \quad (0 < r < \infty) \quad (11)$$

Now fix  $r > 0$  arbitrary. Put  $B(r, v) = v(re^{i\theta})$  for  $\alpha \in (-\pi, \pi]$ . Then for any  $s > 0$ , we have

$$B(r, \theta) \leq \frac{1}{2\pi} \int_0^{2\pi} v(se^{i\theta}) P_{\frac{r}{s}}(\theta - \alpha) d\theta.$$

Again using (6), we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} v(se^{i\theta}) P_{\frac{r}{s}}(\theta - \alpha) d\theta < \frac{1}{2\pi} \int_0^{2\pi} u_n(se^{i\theta}) P_{\frac{r}{s}}(\theta - \alpha) d\theta + o(1) \leq B(s, u_n) + o(1) \quad \text{as } n \rightarrow \infty$$

Consequently,

$$B(r, v) \leq \lim_{n \rightarrow \infty} \inf B(r, u_n) \quad (12)$$

The result follows from (11) and (12). To see the remaining inequality, let  $v^*$  be the star-function of  $v$ . then by phragmén Lindelöf principle, we have

$$v^*(re^{i\theta}) \leq r^\rho \frac{\sin \rho \theta}{\sin \pi \rho}, \quad 0 \leq \theta \leq \pi \quad (13)$$

Thus  $\left. \frac{\partial}{\partial \theta} v^*(re^{i\theta}) \right|_{\theta=0} \leq \frac{r^\rho \rho}{\sin \pi \rho}$  and this implies by (4) that

$$B(r, v) \leq \frac{\pi \rho r^\rho}{\sin \pi \rho}$$

We need the following convolution inequality due to Essén, Rossi and Shea [7]

Let  $u = u_1 - u_2$  be a  $\delta$  subharmonic function in the plane,  $0 \leq a < b \leq \pi$ ,

$\gamma = \frac{b-a}{\pi}$ ,  $z = re^{i\theta}$ , ( $0 < \theta < \pi$ .) and  $0, r < \frac{1}{2}R$  Then

$$u^*(z^\gamma e^{i\alpha}) \leq \int_0^r \bar{u}(t^\gamma e^{ib}) K_1\left(\frac{r}{t}, \theta\right) \frac{dt}{t} + \int_0^R u^*(t^\gamma e^{i\theta}) K_2\left(\frac{r}{t}, \theta\right) \frac{dt}{t} + 8 \left(\frac{r}{R}\right)^{\frac{1}{2}} T(2R^\gamma, u) \quad \left(0 < r < \frac{R}{2}\right)$$

For our purpose we consider the case  $u$  is subharmonic,  $a = 0, b = \pi$ , so that  $\gamma = 1$  and the inequality reduces to:

$$u^*(z) \leq \int_0^r A((t, u)) K_1\left(\frac{r}{t}, \theta\right) \frac{dt}{t} + 8 \left(\frac{r}{R}\right)^{\frac{1}{2}} T(2R, u), \quad \left(0 < r < \frac{R}{2}\right) \quad (14)$$

where  $K_1(r, \theta)$  is a non-negative function and satisfies

$$\int_0^\infty K_1(t, \theta) \frac{dt}{t^{\rho+1}} = \frac{\sin \rho \theta}{\pi \rho \cos \rho \pi}, \quad (15)$$

for  $0 < \theta \leq \pi$  and  $|\rho| < \frac{1}{2}$ .

### Proof of Theorem 2.1.

Suppose,  $\lim_{r \rightarrow \infty} \sup \frac{A(r, u)}{B(r, u)} = \cos \pi \rho = C(\rho)$ , ( $0 < \rho < \frac{1}{2}$ ). Then, we have

$$A(r, u) < (C(\rho) + o(1))B(r, u), \quad \text{as } r \rightarrow \infty.$$

Let  $\{r_n\}$  be a sequence of Pòlya peaks for  $T(r, u)$  of order  $\rho$ . We apply (14) with  $R = R_n = \frac{1}{2\epsilon_n} r_n$ ,  $\rho_n = \epsilon_n r_n$  and  $z = re^{i\theta}$  and letting  $n \rightarrow \infty$  to get

$$u^*(z) \leq (C(\rho) + o(1)) \int_{\rho_n}^R B(t, u) K_1\left(\frac{r_n}{t}, \pi\right) \frac{dt}{t} + \int_0^{\rho_n} A(t, u) K_1\left(\frac{r_n}{t}, \pi\right) \frac{dt}{t} + 8 \left(\frac{r_n}{R}\right)^{\frac{1}{2}} T(2R, u)$$

Setting  $t = sr_n$ , we get

$$u^*(z) \leq (C(\rho) + o(1)) \int_{\frac{1}{\epsilon_n}}^{\epsilon_n} B(sr_n, u) K_1\left(\frac{r}{s}, \pi\right) \frac{ds}{s} + \int_{\frac{1}{\epsilon_n}}^{\infty} A(sr_n, u) K_1\left(\frac{1}{s}, \pi\right) \frac{ds}{s} + 8 \left(\frac{r_n}{R}\right)^{\frac{1}{2}} T(2R, u) \quad (16)$$

as  $n \rightarrow \infty$ .

Since  $A(r, u) < (C(\rho) + o(1))B(sr_n, u)$  as  $r \rightarrow \infty$ ; and  $8 \left(\frac{r_n}{R}\right)^{\frac{1}{2}} T(2R, u) \leq 82^{\frac{1}{2}} \epsilon^{\frac{1}{2}} (1 + \epsilon_n) \left(d \frac{1}{\epsilon_n}\right)^{\rho} T(r, u)$ ,. Dividing throughout the above inequality (16) by  $T(r, u)$ , using (9), Lemma I and letting  $n \rightarrow \infty$ , we obtain by Lebesgue's dominating convergence theorem

$$1 \leq C(\rho) \int_0^\infty B(s, v) K_1\left(\frac{1}{s}, \pi\right) \frac{ds}{s} \leq C(\rho) \int_0^\infty s^\rho \frac{\pi \rho}{\sin \pi \rho} K_1\left(\frac{1}{s}, \pi\right) \frac{ds}{s} = 1.$$

Thus equality holds throughout. As  $B(s, v) \leq s^\rho \frac{\pi \rho}{\sin \pi \rho}$  (by Lemma 1), we have

$B(s, v) = s^\rho \frac{\pi \rho}{\sin \pi \rho}$  holds for almost all  $s \geq 0$ . Since  $B(s, v)$  is a continuous function of  $s$ , we conclude

$$B(s, v) = s^\rho \frac{\pi \rho}{\sin \pi \rho} \quad (0 \leq s < \infty)$$

Now we apply the convolution inequality due to Petrenko [13] (1969) with  $\gamma = \frac{\alpha}{\pi}$ , ( $0 < \alpha < \pi$ ) and (13) to the limit function  $v$  to obtain

$$B(r, v) = r^\rho \frac{\pi \rho}{\sin \pi \rho} \leq \int_0^\infty v^*(te^{i\alpha}) K\left(\frac{r}{t}, \gamma\right) \frac{dt}{t} \leq \frac{\sin \rho \alpha}{\sin \pi \rho} \int_0^\infty t^\lambda k\left(\frac{r}{t}, \gamma\right) \frac{dt}{t} = r^\lambda \frac{\pi \rho}{\sin \pi \rho}.$$

Thus the equality holds throughout. Using (13), the continuity of  $v^*$ , and basic fact in Lebesgue integral we conclude that

$$v^*(te^{i\alpha}) = t^\rho \frac{\sin \rho \alpha}{\sin \pi \rho}.$$

Hence from (12) and the maximum principle for subharmonic function, we have

$$v^*(re^{i\theta}) = r^\rho \frac{\sin \rho \alpha}{\sin \pi \rho}, \quad (0 \leq \theta \leq \pi, 0 < r < \infty) \quad (17)$$

Thus from (3), (4) and (5), we have

$$B(r, v) = \frac{\pi \rho r^\rho}{\sin \pi \rho}, A(r, v) = \pi \rho r^\rho \frac{\cos \pi \rho}{\sin \pi \rho} \quad \text{and} \quad N(r, v) = r^\rho = T(r, v), \quad (0 < r < \infty).$$

Let  $\bar{v}$  be the symmetric decreasing arrangement of  $v$ , so that

$$v^*(re^{i\theta}) = \frac{1}{\pi} \int_0^\theta P(re^{i\alpha}) d\alpha, \quad (0 \leq \theta \leq \pi)$$

Thus using (17), we have

$$\bar{v}(re^{i\theta}) = \lambda \pi r^\rho \frac{\cos \lambda \theta}{\sin \pi \lambda}, \quad (|\theta| \leq \pi) \quad (18)$$

and  $\bar{v}$  is harmonic in  $\{Z : -\pi < \arg(Z) < \pi\}$ . A well known result of Esseen and Shea [15] (1978/79) shows that

$$v(ze^{i\alpha}) = \bar{v}(Z), \quad (|\arg(Z)| \leq \pi). \quad \text{for some } \alpha \in (-\pi, \pi].$$

Thus setting  $Z = re^{i(\theta-\alpha)}$ , where  $|\theta - \alpha| \leq \pi$  and using (16) we get

$$v(re^{i\theta}) = \frac{\pi \lambda r^\rho}{\sin \pi \lambda} \cos(\theta - \alpha), \quad (|\theta - \alpha| \leq \pi).$$

This proves (f). Assertion (e) follows from (d) and Jensen's formula.

A standard result in the theory of Lebesgue integral and (6) shows that there is a subsequence  $I$  of positive integers such that

$$u_n(re^{i\theta}) = (o(1) + 1)v(re^{i\theta})$$

as  $n \rightarrow \infty$  ( $n \in I$ ) and for almost all  $\theta$ , ( $|\theta - \alpha| \leq \pi$ )

This completes the proof of Theorem 2.1.

## References

- [1] A.Edrei,Sums of deficiencies of Meromorphic functions, J. Analyse Math. 14(1965), 70-107.
- [2] Hayman, W. K, Subharmonic functions, Vol.2(1989), Academic Press, New York.
- [3] A. Baernstein, Proof of Edreis conjecture, Proc. London Math. Soc.(3)26(1973), 418-434.
- [4] Wiman (1915) Ubereine Eigenschaft der ganzan Functionen von der Höhe Null. Math. Ann, 76(1915), 197-211.
- [5] G.Valiron ,Sur les fonctions entières d'ordre fini et ordre infini, et en particulier les fonctions à correspondance régulière, Ann. Fac. Sci. Univ. Toulouèse (3)5 (1913), 117 -257.
- [6] D.Drasin and D.F.Shea, Asymptotic properties of entire functions extremal for the  $\cos \pi \rho$  theorem, Bull. Amer. Soc. 75(1969) 119-122.
- [7] M.Essèn, Jhon Rossi and D.F.Shea, a convolution Inequality with applications to function theory II , Journal of Analyse Mathématique, Vol. 61(1993,).
- [8] Paley A.Note on integral functions. Cambridge Phil.Soc. 28(1932),(262-265).
- [9] I.V.Ostrovskii ,Deficiencies of Meromorphic functions of order less than one, Dokl. Akad. Nauk, SSSR 150,(1963), 32-35.
- [10] A.Edrei : A local form of the Phragmén Lindelöf indicator ,Mathematika,17 (1970) 149 -172.
- [11] ] A. Baernstein: Integral means, univalent functions and circular symmetrization .Acta Math. 133,(1974), 133-169.
- [12] J. M. Anderson and Baernstein A : The size of the set of which a meromorphic function is large Proc. London Math. Soc. 36-5(1978), 518-539.
- [13] S. M. Tejum : Application of normal family to the spread inequality and the Paley type inequality , Sinet : Ethiop. J. Sci. 34(2) (2011) 107 112.
- [14] Petrenko. V. P.: The growth of meromorphic functions of finite order ,Izv. Akad. Nauk. USSR ser. Mat. 33 (1969), 414 454.



- [15] ] M. Essen and D.F.Shea: On some questions of uniqueness in the theory of symmetrization, Ann. Acad. Sci. Fenni. Series A. I. Mathematics, Vol.4, (1978/79), 311-340.