A Forgotten Idea! Infinite Descent as a Problem-Solving Heuristic

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Abstract

As Shoenfeld and Polya stated, heuristics are one of the most important part of problem-solving. Therefore students should learn some heuristics like mathematical induction, pigeonhole principle, symmetry and so on.

Although the infinite descent heuristic is not so wildly known, it is very useful to solve various mathematical problems especially in number theory.

In this article we provide a framework to teach this heuristic to students. Also we present some examples to show how this method can be useful for solving problems.

1 Introduction

Heuristic is one of the main categories to solve a problem. It is a particular mental skill included strategies and techniques for problem-solving such as working backward, drawing figures, restate a problem in another way, symmetry, etc.(Polya 2014)(Schoenfeld 1985). The method of infinite descent and the method of mathematical induction, both came into use in the seventeenth century by Descartes, Cavalieri, Pascal, Wallis and Fermat. The method of infinite descent, which was invented by Fermat, is not so widely known, but Fermat applied this method to solve various problems such as the following propositions (Bussey 1918)

Proposition 1.1 (Bussey 1918) No number of the form $3k + 1$ can be of the form $x_2 + 3y_2$.

Proposition 1.2 (Bussey 1918) There is no right triangle whose sides are integers whose area is equal to the square of an integer.

The idea of this method is to show that if there is an integral solution to an equation then there is another integral solution which is smaller in some way. Repeating this process and comparing the sizes of the successive solutions leads to an infinitely decreasing sequence that is impossible in natural numbers. (Conrad 2012)

$$a_1 > a_2 > a_3 > \cdots$$

After seventeenth century, although the method of infinite descent did not receive enough attention, it is one of the most efficient methods of mathematics (Bussey 1918). For example, the proof of irrationality of $\sqrt{2}$ which is one of the classical problem of mathematics usually done in this way (Rudin 1964). But in the study of thirty math teachers it became clear that many of them (more than 80 percent) did not know it. Even those who knew the proof did not understand the process of infinite descent and consequently, they could not apply this method to other problems. This means they taught that the proof is limited to this problem and does not have the ability to generalize to other issues.

Since there are many problems that, their solutions are very difficult and time consuming without using infinite descent we provide an educational framework for teaching this method by presenting several examples of the application of this method in various problems. Although this method is used more in number theory, it can be used in geometry and algebra, too.

2 How it can be taught?

In this section, we present some examples to show the importance and beautifulness of this method. We tried to teach this method to several students via presenting some relevant examples and understand that between various examples which can be used for teaching this method, irrationality of $\sqrt{2}$ is the most understandable. Following we examine the process of the method in the proof.

Example 2.1 (Bilchev and Bilcheva n.d.) Proof that $\sqrt{2}$ is irrational.

Proof. Let $\sqrt{2}$ be a rational number, i.e. $\sqrt{2} = \frac{m}{n}$ where m and n are natural numbers. Since $m^2 = 2n^2$ we have $m = 2m_1$ where m_1 is a natural number. Substituting m_1 we get $2m_1^2 = n^2$, so we can conclude that n is even i.e. $n = 2n_1$ where n_1 is a natural number and $n_1 < n$ and $m_1 < m$. Continuing the same manner we will get decreasing sequences

$$m > m_1 > m_2 > \cdots$$

and

$$n > n_1 > n_2 > \cdots.$$

But it is impossible in natural number because of well-ordering principle of natural numbers and this is a contradiction. In such situation we can say that "it is a contradiction by infinite descent".

In fact, if we suppose that a natural number is a solution of a specific problem, and the existence of this solution leads to another smaller natural solution, this has a contradiction with well-ordering principle of natural numbers and we know it as "infinite descent".

If p does not be a perfect square number, the proof of irrationality of $\sqrt{p}$ is similar.

The application of this method in the algebraic problem is shown in the following example.

Example 2.2 (Bussey 1918) Show that $x^n - y^n$ is dividable by $x - y$ for all natural number n.

Proof. We know the proposition is true for $n = 1,2$. If there exists a natural number i.e. n_0 such that $x^{n_0} - y^{n_0}$ is not dividable by $x - y$ then by some calculations we have

$$\frac{x^{n_0} - y^{n_0}}{x - y} = x^{n_0-1} + \frac{x^{n_0-1} - y^{n_0-1}}{x - y} y$$

and it shows that $x^{n_0-1} - y^{n_0-1}$ is not dividable by $x - y$ and it is a contradiction by infinite descent.

In the next example, we will show the application of the method in a geometric problem.

Example 2.3 (Tat-Wing 2005) Let ΔABC be an acute triangle. Draw the perpendicular from vertex A to side BC such that the perpendicular cut BC at point H_1. From point H_1 draw a perpendicular to cut AC at the point H_2. Again from H_2 draw a perpendicular to meet AB at the point H_3. Then draw a perpendicular from H_3 to meet BC at H_4 and so on. Prove that all points H_i, $i \in N$ are distinct.
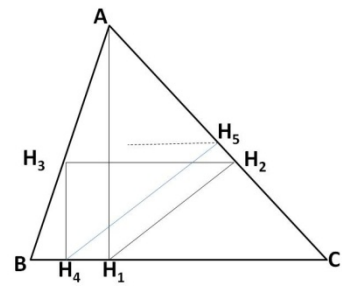

Fig2.1 All H_i are distinct.

Proof. At the first, we should note that the points A_i and A_{i+1} never coincide because ΔABC is an acute triangle. Suppose $A_i = A_j$ and $i < j$ so i equals to 1 otherwise A_{i-1} coincides with A_{j-1} that means $A_{i-1} = A_{j-1}$ and it is a contradiction by infinite descent. If A_1 coincides with A_j ($j \geq 3$) then the point A_{j-1} must be one of the vertices A, B, or C and it is impossible.

Example 2.4 (Bilchev and Bilcheva n.d.)(Tat-Wing 2005) $2n + 1$ natural number is given such that any of $2n$ numbers of them can be divided into two n-member groups such that the summation of members of each group is equal to another. Prove that all numbers are equal.

Solution. It is trivial that all of numbers must be even or odd and the summation of all $2n$ numbers is even. We can sort all of numbers as follow:

$$a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{2n-1} \leq a_{2n}$$

Subtracting a_0 from all numbers we get

$$0 \leq (a_1 - a_0) \leq (a_2 - a_0) \leq \cdots$$
$$\leq a_{2n} - a_0$$

that satisfy the condition of problem. Continuing this process, from the infinite descent we get a group which all members are zero. But it implies that all initial numbers are equal.

Example 2.5 (Engel 1998) Suppose that x and y are natural numbers such that $x^2 + y^2 + xy$ is dividable by 10. Proof that both x and y are dividable by 10.

Proof. Suppose not, so we can consider x and y as

$$x = 10t + k$$

and

$$y = 10m + n$$

such that k and n are not zero together. We have

$$x^2 + y^2 + xy = 100(t^2 + m^2 + tm) + 10(2tk + 2mn + tn + mk) + (k^2 + n^2 + kn)$$

Since $x^2 + y^2 + xy$ is dividable by 10 then $(k^2 + n^2 + kn)$ is dividable too and it is a contradiction by infinite descent.

2-1- Golden ratio and infinite descent!

One of the most famous mathematical constant is golden ratio which is represented with φ. Relation between the historical number and the historical method might be interesting. Since $\varphi = \frac{1+\sqrt{5}}{2}$ and $\sqrt{5}$ is an irrational number, it is trivial that φ is irrational. Here we present another proof based on infinite descent.

The golden equation which is related to golden ratio is

$$\frac{x}{1} = \frac{x+1}{x}$$

If φ is a rational number i.e. $\frac{a}{b}$ then substituting in golden equation we get

$$a^2 - b^2 - ab = 0$$

or

$$a^2 = ab + b^2$$

It can be easily seen that both a and b must be even. Consider

$$a = 2a_1$$

and

$$b = 2b_1$$

where $a_1 < a$ and $b_1 < b$. Substituting we have

$$a_1^2 = a_1 b_1 + b_1^2$$

which is a same equation and it is a contradiction by infinite descent (Engel 1998).

3 Is it necessary?

Faced with the problem-solving methods, there is a question we need to ask ourselves: is it necessary to learn the method? And is the method really useful for solving problems?

In this section, we present some evidences which indicate that knowing this heuristic can be effective in solving problems simpler. For example, the proof of Fermat last theorem for n=3, 4 is based on infinite descent (Grant and Perella 1999)(Barbara 2007).

To show that infinite descent can solve problems simpler, let us to solve example 2.5 in different way.

Proof 2.5. It is trivial that both x and y are even. So it is sufficient to prove that both of them are dividable by 5. Suppose not, consider that

$$x = 5t + k$$

and

$$y = 5m + n$$

such that k and n are not zero together. We have

$$x^2 + y^2 + xy = 25(t^2 + m^2 + tm) + 5(2tk + 2mn + tn + mk) + (k^2 + n^2 + kn)$$

Since $x^2 + y^2 + xy$ is dividable by 5 then $(k^2 + n^2 + kn)$ must be dividable by 5. But the following table shows that for all possible cases of k and n, $k^2 + n^2 + kn$ is not dividable by 5.

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k	N	$k^2 + n^2 + kn$
0	1	1
0	2	4
0	3	9
0	4	16
1	1	3
1	2	7
1	3	13
1	4	21
2	2	12
2	3	19
2	4	28
3	3	27
3	4	37
4	4	28

Table3-1. For all possible cases of k and n
$k^2 + n^2 + kn$ is not dividable by 5.

The next problem shows that how this method can be useful for solving some complicated problems.

Example 2.6 (Tat-Wing 2005) Find all prime numbers p such that satisfy the equation

$$p^n = x^3 + y^3$$

(x, y, and n are natural numbers.)

Proof. It is easily can be seen that $2^1 = 1^3 + 1^3$ and $3^2 = 1^3 + 2^3$. If the equation has another solution then $p \geq 5$ and at least one of x or y must be greater than 1. We will prove that if a solution satisfy the equation with power n, then there is a solution with power $m < n$.

We have
$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$
that
$$(x + y) \geq 3$$
and
$$x^2 - xy + y^2 = (x - y)^2 + xy \geq 2$$
Since
$$p^n = (x + y)(x^2 - xy + y^2)$$
both $(x + y)$ and $(x^2 - xy + y^2)$ must be dividable by p otherwise one of them is equal to 1 and another is equal to p^n. But it is impossible because

    a. If $(x^2 - xy + y^2) = p^n$ then $(x + y) = 1$ and it is a contradiction.

    b. If $(x + y) = p^n$ given that one of x or y is greater than 1, then $(x^2 - xy + y^2)$ is greater than or equal to 3.

Since
$$3xy = (x + y)^2 - (x^2 - xy + y^2)$$
and both xy and $x + y$ are dividable by p then $p|x$ and $p|y$.

We have
$$p|x, y \Rightarrow p^3|x^3, y^3 \Rightarrow$$
$$\left(\frac{x}{p}\right)^3 + \left(\frac{y}{p}\right)^3 = \frac{p^n}{p^3} = p^{n-3}$$

and it is a contradiction by infinite descent.

4 Conclusion

The method of infinite descent is an important heuristic in problem-solving. In this article we provide some examples such that in their solutions, this method was evident and presented examples for teaching this heuristic. Also examined problems show that using this method can simplify and shorten the process of obtaining solution. Therefore, to improve teachers and student's problem-solving skills, it is recommended that this method be taught.

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