International Journal of Scientific Engineering and Applied Science (IJSEAS) - Volume-1, Issue-7, October 2015
ISSN: 2395-3470
www.ijseas.com

Regular elements of semigroup $B_X(D)$ defined by $X-$ semilattice which is Union of Two Rhombes and a Chain

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Abstract : in this paper we take $Q=\{T_0,T_1,T_2,T_3,T_4,T_5,T_6,...,T_{m-1},T_m\}$ $(m\geq 6)$ subsemilattice of $X-$ semilattice of unions D where the elements T_i's are satisfying the following properties, $T_0\subset T_1\subset T_3\subset T_5\subset T_6\subset ...\subset T_{m-1}\subset T_m$, $T_0\subset T_2\subset T_3\subset T_5\subset T_6\subset ...\subset T_{m-1}\subset T_m$, $T_0\subset T_2\subset T_4\subset T_5\subset$ $\subset T_6\subset ...\subset T_{m-1}\subset T_m$, $T_1\setminus T_2\neq\varnothing$, $T_2\setminus T_1\neq\varnothing$, $T_1\setminus T_4\neq\varnothing$, $T_4\setminus T_1\neq\varnothing$, $T_3\setminus T_4\neq\varnothing$, $T_4\setminus T_3\neq\varnothing$, $T_1\cup T_2=T_3$, $T_4\cup T_1=T_4\cup T_3=T_5$. we will investige the properties of regular elements of the complete semigroup of binary relations $B_X(D)$ satisfying $V(D,\alpha)=Q$. And For the case where X is a finite set we derive formulas by means of which we can calculate the numbers of regular elements of the respective semigroup.

Introduction

1. Let X be an arbitrary nonempty set, D be a $X-$ semilattice of unions, i.e. a nonempty set of subsets of the set X that is closed with respect to the set-theoretic operations of unification of elements from D, f be an arbitrary mapping from X into D. To each such a mapping f there corresponds a binary relation α_f on the set X that satisfies the condition $\alpha_f=\bigcup_{x\in X}(\{x\}\times f(x))$. The set of all such α_f $(f:X\to D)$ is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by a $X-$ semilattice of unions D (see [2,3 2.1 p. 34]).

By $\varnothing$ we denote an empty binary relation or empty subset of the set X. The condition $(x,y)\in\alpha$ will be written in the form $x\alpha y$. Further let $x,y\in X$, $Y\subseteq X$, $\alpha\in B_X(D)$, $T\in D$, $\varnothing\neq D'\subseteq D$ and $t\in\breve{D}=\bigcup_{Y\in D}Y$. Then by symbols we denote the following sets:

$$y\alpha=\{x\in X\mid y\alpha x\},\ Y\alpha=\bigcup_{y\in Y}y\alpha,\ V(D,\alpha)=\{Y\alpha\mid Y\in D\},$$
$$X^*=\{T\mid\varnothing\neq T\subseteq X\},\ D'_t=\{Z'\in D'\mid t\in Z'\},\ D'_T=\{Z'\in D'\mid T\subseteq Z'\}.$$
$$\ddot{D}'_T=\{Z'\in D'\mid Z'\subseteq T\},\ l(D',T)=\cup(D'\setminus D'_T),\ Y^\alpha_T=\{x\in X\mid x\alpha=T\}.$$

Under the symbol $\wedge(D,D_t)$ we mean an exact lower bound of the set D_t in the semilattice D.

Definition 1.1. An element α taken from the semigroup $B_X(D)$ called a regular element of the semigroup $B_X(D)$ if in $B_X(D)$ there exists an element β such that $\alpha\circ\beta\circ\alpha=\alpha$ (see [1,2,3]).

Definition 1.2. We say that a complete $X-$ semilattice of unions D is an $XI-$ semilattice of unions if it satisfies the following two conditions:

a) $\wedge(D,D_t)\in D$ for any $t\in\breve{D}$;

b) $Z=\bigcup_{t\in Z}\wedge(D,D_t)$ for any nonempty element Z of D (see [2,3 definition 1.14.2]).

Definition 1.3. Let D be an arbitrary complete $X-$semilattice of unions, $\alpha \in B_X(D)$ and $Y_T^\alpha = \{x \in X \mid x\alpha = T\}$. If

$$V[\alpha] = \begin{cases} V(X^*,\alpha), \ if \ \varnothing \notin D, \\ V(X^*,\alpha), \ if \ \varnothing \in V(X^*,\alpha), \\ V(X^*,\alpha) \cup \{\varnothing\}, \ if \ \varnothing \notin V(X^*,\alpha) \text{ and } \varnothing \in D, \end{cases}$$

then it is obvious that any binary relation α of a semigroup $B_X(D)$ can always be written in the form $\alpha = \bigcup_{T \in V[\alpha]} (Y_T^\alpha \times T)$ the sequel, such a representation of a binary relation α will be called quasinormal.

Note that for a quasinormal representation of a binary relation α, not all sets Y_T^α $(T \in V[\alpha])$ can be different from an empty set. But for this representtation the following conditions are always fulfilled:

a) $Y_T^\alpha \cap Y_{T'}^\alpha = \varnothing$, for any $T, T' \in D$ and $T \neq T'$;

b) $X = \bigcup_{T \in V[\alpha]} Y_T^\alpha$

(see [2,3 definition 1.11.1]).

Definition 1.4. We say that a nonempty element T is a nonlimiting element of the set D' if $T \setminus l(D',T) \neq \varnothing$ and a nonempty element T is a limiting element of the set D' if $T \setminus l(D',T) = \varnothing$ (see [2,3 definition 1.13.1 and definition 1.13.2]).

Definition 1.5. The one-to-one mapping φ between the complete $X-$semilattices of unions $\phi(Q,Q)$ and D'' is called a complete isomorphism if the condition

$$\varphi(\cup D_1) = \bigcup_{T' \in D_1} \varphi(T')$$

is fulfilled for each nonempty subset D_1 of the semilattice D' (see [2,3 definition 6.3.2]).

Definition 1.6. Let α be some binary relation of the semigroup $B_X(D)$. We say that the complete isomorphism φ between the complete semilattices of unions Q and D' is a complete $\alpha-$isomorphism if

(a) $Q = V(D,\alpha)$;

(b) $\varphi(\varnothing) = \varnothing$ for $\varnothing \in V(D,\alpha)$ and $\varphi(T)\alpha = T$ for eny $T \in V(D,\alpha)$ (see [2,3 definition 6.3.3]).

Lemma 1.1. Let $Y = \{y_1, y_2, ..., y_k\}$ and $D_j = \{T_1, ..., T_j\}$ be some sets, where $k \geq 1$ and $j \geq 1$. Then the number $s(k,j)$ of all possible mappings of the set Y on any such subset of the set D_j' that $T_j \in D_j'$ can be calculated by the formula $s(k,j) = j^k - (j-1)^k$ (see [2,3 Corollary 1.18.1]).

lemma1.2. Let $D_j = \{T_1, T_2, ...T_j\}$, X and $Y-$ be three such sets, that $\varnothing \neq Y \subseteq X$. If f is such mapping of the set X, in the set D_j, for which $f(y) = T_j$ for some $y \in Y$, then the number s of all those mappings f of the set X in the set D_j is equal to $s = j^{|X \setminus Y|} \cdot \left(j^{|Y|} - (j-1)^{|Y|}\right)$ (see [2,3 Theorem 1.18.2]).

Theorem 1.1. Let $D = \{\breve{D}, Z_1, Z_2, ..., Z_{n-1}\}$ be some finite $X-$semilattice of unions and $C(D) = \{P_0, P_1, P_2, ..., P_{n-1}\}$ be the family of sets of pairwise nonintersecting subsets of the set X. If φ is a mapping of the semilattice D on the family of sets $C(D)$ which satisfies the condition $\varphi(\breve{D}) = P_0$ and $\varphi(Z_i) = P_i$ for any $i = 1, 2, ..., n-1$ and $\hat{D}_Z = D \setminus \{T \in D \mid Z \subseteq T\}$, then the following equalities are valid:

$$\breve{D} = P_0 \cup P_1 \cup P_2 \cup ... \cup P_{n-1}, \ Z_i = P_0 \cup \bigcup_{T \in \hat{D}_{Z_i}} \varphi(T). \qquad (*)$$

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice D are represented in the form (*), then among the parameters P_i $\left(i=0,1,2,\ldots,n-1\right)$ there exist such parameters that cannot be empty sets for D. Such sets P_i $\left(0<i\leq n-1\right)$ are called basis sources, whereas sets P_j $\left(0\leq j\leq n-1\right)$ which can be empty sets too are called completeness sources.

It is proved that under the mapping φ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping φ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one. (see [5])

Theorem 1.2. Let $\beta\in B_X\left(D\right)$. A binary relation β is a regular element of the semigroup $B_X\left(D\right)$ iff the complete $X-$ semilattice of unions $D'=V\left(D,\beta\right)$ satisfies the following two conditions:

a) $V\left(X^*,\beta\right)\subseteq D'$;

b) D' is a complete $XI-$ semilattice of unions (see [2,3 Theorem 6.3.1]).

Theorem 1.3. . Let D be a finite $X-$semilattice of unions and $\alpha\in B_X\left(D\right)$; $D\left(\alpha\right)$ be the set of those elements T of the semilattice $Q=V\left(D,\alpha\right)\setminus\{\varnothing\}$ which are nonlimiting elements of the set $\ddot{Q}_T$. Then a binary relation α having a quasinormal representation of the form $\alpha=\bigcup_{T\in V(D,\alpha)}\left(Y_T^\alpha\times T\right)$ is a regular element of the semigroup $B_X\left(D\right)$ iff $V\left(D,\alpha\right)$ is a $XI-$semilattice of unions and for some $\alpha-$isomorphism φ from $V\left(D,\alpha\right)$ to some $X-$ subsemilattice D' of the semilattice D the following conditions are fulfilled:

a) $\bigcup_{T'\in\ddot{D}(\alpha)_T} Y_{T'}^\alpha\supseteq\varphi\left(T\right)$ for any $T\in D\left(\alpha\right)$;

b) $Y_T^\alpha\cap\varphi\left(T\right)\neq\varnothing$ for any element T of the set $\ddot{D}\left(\alpha\right)_T$ (see [2,3 Theorem 6.3.3]).

Theorem 1.4 . let X be a finite set . if φ is a fixed element of the set $\Phi\left(Q,D'\right)$ and $\Omega\left(Q\right)=m_0$ then

$$\left|R\left(D'\right)\right|=m_0\cdot q\cdot\left|R_\varphi\left(Q,D'\right)\right|$$

2.RESULTS

Let X be a finite set, D be a complete $X-$semilattice of unions, $m\geq 6$ and $Q=\{T_0,T_1,T_2,T_3,T_4,T_5,T_6,\ldots,T_{m-1},T_m\}$ $\left(m\geq 6\right)$ be a subsemilattice of unions of D satisfies the following conditions

$$\begin{aligned}&T_0\subset T_1\subset T_3\subset T_5\subset T_6\subset\ldots\subset T_{m-1}\subset T_m,\\&T_0\subset T_2\subset T_3\subset T_5\subset T_6\subset\ldots\subset T_{m-1}\subset T_m,\\&T_0\subset T_2\subset T_4\subset T_5\subset T_6\subset\ldots\subset T_{m-1}\subset T_m,\\&T_1\setminus T_2\neq\varnothing,\ T_2\setminus T_1\neq\varnothing,\ T_1\setminus T_4\neq\varnothing,\\&T_4\setminus T_1\neq\varnothing,\ T_3\setminus T_4\neq\varnothing,\ T_4\setminus T_3\neq\varnothing,\\&T_1\cup T_2=T_3,\ T_4\cup T_1=T_4\cup T_3=T_5.\end{aligned}\qquad(2.1)$$

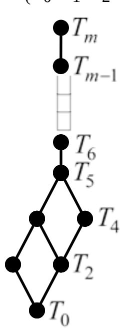


Fig.2.1

Note that the diagram of the given $X-$semilattice of Unions Q is shown fig.2.1

Let $P_0,P_1,\ldots,P_{m-1}$ and C be the pairwise nonintersecting Subset of the set X and

$$\varphi=\begin{pmatrix}T_0&T_1&T_2&T_3&T_4&T_5&T_6&\ldots&T_{m-1}&T_m\\P_0&P_1&P_2&P_3&P_4&P_5&P_6&\ldots&P_{m-1}&C\end{pmatrix}$$

is a mapping of the semilattice Q onto the family of sets $\{P_0, P_1, ..., P_{m-1}, C\}$ Then the formal equalities corresponding to the semilattice Q we have a form (see Theorem 1.1)

$$\begin{array}{l} T_m = C \cup P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup ... \cup P_{m-1}, \\ T_{m-1} = C \cup P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup ... \cup P_{m-2}, \\ ------------------------------ \\ T_6 = C \cup P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5, \\ T_5 = C \cup P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4, \\ T_4 = C \cup P_0 \cup P_1 \cup P_2 \cup P_3, \\ T_3 = C \cup P_0 \cup P_1 \cup P_2 \cup P_4, \\ T_2 = C \cup P_0 \cup P_1, \\ T_1 = C \cup P_0 \cup P_2 \cup P_4, \\ T_0 = C, \end{array} \qquad (2.2)$$

where $|C| \geq 0$, $|P_0| \geq 0$, $|P_2| \geq 0$ and $P_1, P_3, P_4, P_5, P_6, ...P_{m-1}, P_m \notin \{\varnothing\}$.

lemma 2.1. Let $Q = \{T_0, T_1, T_2, T_3, T_4, T_5, T_6, ..., T_{m-1}, T_m\}$ $(m \geq 6)$ be a subsemilattice of the semilattice D and Q subsemilattice satisfies (2.1) conditions, Then Q is always an XI – semilattice of unions.

Proof:

$$Q_t = \begin{cases} \{T_0, T_1, T_2, T_3, T_4, T_5, T_6, ..., T_{m-1}, T_m\} & if \quad t \in C, \\ \{T_1, T_2, T_3, T_4, T_5, T_6, ..., T_{m-1}, T_m\} & if \quad t \in P_0, \\ \{T_2, T_3, T_4, T_5, T_6, ..., T_{m-1}, T_m\} & if \quad t \in P_1, \\ \{T_1, T_3, T_4, T_5, T_6, ..., T_{m-1}, T_m\} & if \quad t \in P_2, \\ \{T_4, T_5, T_6, ..., T_{m-1}, T_m\} & if \quad t \in P_3, \\ \{T_1, T_3, T_5, T_6, ..., T_{m-1}, T_m\} & if \quad t \in P_4, \\ \{T_6, ..., T_{m-1}, T_m\} & if \quad t \in P_5, \\ \{T_7, ..., T_{m-1}, T_m\} & if \quad t \in P_6, \\ ------- & \\ \{T_{m-1}, T_m\} & if \quad t \in P_{m-2}, \\ \{T_m\} & if \quad t \in P_{m-1} \end{cases} \qquad \wedge(Q, Q_t) = \begin{cases} T_0, & if \quad t \in C, \\ T_0, & if \quad t \in P_0, \\ T_2, & if \quad t \in P_1, \\ T_0, & if \quad t \in P_2, \\ T_4, & if \quad t \in P_3, \\ T_1, & if \quad t \in P_4, \\ T_6, & if \quad t \in P_5, \\ T_7, & if \quad t \in P_6, \\ ------- & \\ T_{m-1}, & if \quad t \in P_{m-2}, \\ T_m, & if \quad t \in P_{m-1} \end{cases}$$

then We have obtained that $\wedge(Q, Q_t) \in D$ for any $t \in T_m$. Furthermore, if $Q^\wedge = \{\wedge(Q, Q_t) t \in T_m\}$, then $Q^\wedge = \{T_0, T_1, T_2, T_4, T_6, T_7, ..., T_m\}$ and it is easy to verify that any nonempty element of the semilattice Q is the union of some elements of the set $Q^\wedge$. Now, taking into account Definition 1.2, we obtain that Q is an XI – semilattice of unions. □

lemma. 2.2 if $Q = \{T_0, T_1, T_2, T_3, T_4, T_5, T_6, ..., T_{m-1}, T_m\}$ $(m \geq 6)$ is XI – semilattice of unions than $(T_4 \cap T_1, \ T_4 \setminus T_3, \ T_1 \setminus T_4, \ T_2 \setminus T_1, \ T_6 \setminus T_5, ..., \ T_m \setminus T_{m-1}, X \setminus T_m)$ is a partition of the set X.

Proof. the lemma immediately follows from the formal equalities (2.2)

Theorem 2.1. Let $Q = \{T_0, T_1, T_2, T_3, T_4, T_5, T_6, ..., T_{m-1}, T_m\}$ $(m \geq 6)$ be a subsemilattice of the semilattice D which satisfies (2.1) conditions (see Fig. 2.1). (see Fig. 1). A binary relation α of the semigroup $B_X(D)$ that has a quasinormal representation of the form $\alpha = \bigcup_{i=0}^{m} \left(Y_i^\alpha \times T_i\right)$, where $Q = V(D, \alpha)$, is a regular element of the semigroup $B_X(D)$ iff for some α – isomorohism φ of the semilattice Q on some X – subsemilattice $D' = \{\varphi(T_1), \varphi(T_2), ..., \varphi(T_m)\}$ of the semilattice D satisfies the conditions

$$\begin{array}{l} Y_0^\alpha \supseteq \varphi(T_0), \ Y_0^\alpha \cup Y_1^\alpha \supseteq \varphi(T_1), \ Y_0^\alpha \cup Y_2^\alpha \supseteq \varphi(T_2), \ Y_0^\alpha \cup Y_2^\alpha \cup Y_4^\alpha \supseteq \varphi(T_4), \\ Y_1^\alpha \cup Y_2^\alpha \cup ... \cup Y_p^\alpha \supseteq \varphi(T_p), \ Y_q^\alpha \cap \varphi(T_q) \neq \varnothing, \end{array} \qquad (2.3)$$

for any $p = 6, 7, ..., m-1$ and $q = 1, 2, 4, 6, 7, ..., m$.

Proof. To begin with, we recall that Q is an XI – semilattice of unions (see lemma 2.1). Now we are to find the nonlimiting element of the sets $\ddot{Q}_q^*$ of the semilattice $Q^* = Q \setminus \{\varnothing\}$. Indeed, let $T_q \in Q^*$, where $q = 0,1,2,...,m$. Then for $q = 0,1,2,...,m$ we obtain respectively

$$\begin{aligned}
&l\left(\ddot{Q}_{T_m}^*, T_m\right) = \cup\left(\{T_0, T_1, ..., T_m\} \setminus \{T_m\}\right) = \cup\{T_0, T_1, ..., T_{m-1}\} = T_{m-1},\\
&l\left(\ddot{Q}_{T_{m-1}}^*, T_{m-1}\right) = \cup\left(\{T_0, T_1, ..., T_{m-1}\} \setminus \{T_{m-1}\}\right) = \cup\{T_0, T_1, ..., T_{m-2}\} = T_{m-2},\\
&-----------------------------------\\
&l\left(\ddot{Q}_{T_6}^*, T_6\right) = \cup\left(\{T_0, T_1, T_2, T_3, T_4, T_5, T_6\} \setminus \{T_6\}\right) = \cup\{T_0, T_1, T_2, T_3, T_4, T_5\} = T_5,\\
&l\left(\ddot{Q}_{T_5}^*, T_5\right) = \cup\left(\{T_0, T_1, T_2, T_3, T_4, T_5\} \setminus \{T_5\}\right) = \cup\{T_0, T_1, T_2, T_3, T_4\} = T_5,\\
&l\left(\ddot{Q}_{T_4}^*, T_4\right) = \cup\left(\{T_0, T_2, T_4\} \setminus \{T_4\}\right) = \cup\{T_0, T_2\} = T_2,\\
&l\left(\ddot{Q}_{T_3}^*, T_3\right) = \cup\left(\{T_0, T_1, T_2, T_3\} \setminus \{T_3\}\right) = \cup\{T_0, T_1, T_2\} = T_3,\\
&l\left(\ddot{Q}_{T_2}^*, T_2\right) = \cup\left(\{T_0, T_2\} \setminus \{T_2\}\right) = \cup\{T_0\} = T_0,\\
&l\left(\ddot{Q}_{T_1}^*, T_1\right) = \cup\left(\{T_0, T_1\} \setminus \{T_1\}\right) = \cup\{T_0\} = T_0,\\
&l\left(\ddot{Q}_{T_0}^*, T_0\right) = \cup\left(\{T_0\} \setminus \{T_0\}\right) = \cup\{\varnothing\} = \varnothing,
\end{aligned}$$

Therefore

$$\begin{aligned}
&T_m \setminus l\left(\ddot{Q}_{T_m}^*, T_m\right) = T_m \setminus T_{m-1} \neq \varnothing,\ T_{m-1} \setminus l\left(\ddot{Q}_{T_{m-1}}^*, T_{m-1}\right) = T_{m-1} \setminus T_{m-2} \neq \varnothing,\\
&----------------------------------\\
&T_6 \setminus l\left(\ddot{Q}_{T_6}^*, T_6\right) = T_6 \setminus T_5 \neq \varnothing,\ T_5 \setminus l\left(\ddot{Q}_{T_5}^*, T_5\right) = T_5 \setminus T_5 = \varnothing,\\
&T_4 \setminus l\left(\ddot{Q}_{T_4}^*, T_4\right) = T_4 \setminus T_2 \neq \varnothing,\ T_3 \setminus l\left(\ddot{Q}_{T_3}^*, T_3\right) = T_3 \setminus T_3 = \varnothing,\\
&T_2 \setminus l\left(\ddot{Q}_{T_2}^*, T_2\right) = T_2 \setminus T_0 \neq \varnothing,\ T_1 \setminus l\left(\ddot{Q}_{T_1}^*, T_1\right) = T_1 \setminus T_0 \neq \varnothing,\\
&T_0 \setminus l\left(\ddot{Q}_{T_0}^*, T_0\right) = T_0 \setminus \varnothing \neq \varnothing,\ if\ T_0 \neq \varnothing,
\end{aligned}$$

i.e. $T_q \setminus l\left(\ddot{Q}_{T_q}, T_q\right) \neq \varnothing$, where $q = 1,2,4,6,7,...,m$. Thus we have obtained that T_3, T_5 are the limiting elements of the sets $\ddot{Q}_{T_3}^*$, $\ddot{Q}_{T_5}^*$ and the T_q are the nonlimiting elements of the set $\ddot{Q}_{T_q}^*$, where $q = 1,2,4,6,7,...,m$. (see definition 1.4)
Now, in view of Theorem 1.3 a binary relation α of the semigroup $B_X(D)$ is a regular element of this semigroup iff there exists an α – isomorphism φ of the semilattice Q on some X – subsemilattice $D' = \{\varphi(T_0), ..., \varphi(T_m)\}$ of the semilattice Q such that

$$\begin{aligned}
&Y_0^\alpha \supseteq \varphi(T_0),\ Y_0^\alpha \cup Y_1^\alpha \supseteq \varphi(T_1),\ Y_0^\alpha \cup Y_2^\alpha \supseteq \varphi(T_2),\ Y_0^\alpha \cup Y_2^\alpha \cup Y_4^\alpha \supseteq \varphi(T_4),\\
&Y_1^\alpha \cup Y_2^\alpha \cup ... \cup Y_p^\alpha \supseteq \varphi(T_p),\ Y_q^\alpha \cap \varphi(T_q) \neq \varnothing,
\end{aligned}$$

for any $p = 6,7,...,m$ and $q = 1,2,4,6,7,...,m$.

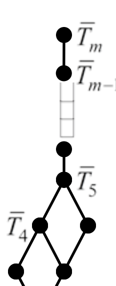
It is clearly understood that the inclusion $Y_1^\alpha \cup ... \cup Y_m^\alpha = X \supseteq \varphi(T_m)$ is always valid. Therefore

$$\begin{aligned}
&Y_0^\alpha \supseteq \varphi(T_0),\ Y_0^\alpha \cup Y_1^\alpha \supseteq \varphi(T_1),\ Y_0^\alpha \cup Y_2^\alpha \supseteq \varphi(T_2),\ Y_0^\alpha \cup Y_2^\alpha \cup Y_4^\alpha \supseteq \varphi(T_4),\\
&Y_1^\alpha \cup Y_2^\alpha \cup ... \cup Y_p^\alpha \supseteq \varphi(T_p),\ Y_q^\alpha \cap \varphi(T_q) \neq \varnothing,
\end{aligned}$$

for any $p = 6,7,...,m-1$ and $q = 1,2,4,6,7,...,m$.

Theorem is proved.

Theorem 2.2. Let $Q = \{T_0, T_1, T_2, T_3, ..., T_m\}$ $(m \geq 6)$ be a subsemilattice of the semilattice D which satisfies (2.1) conditions (see Fig. 2.1)



If the XI – semilattices Q and $D' = \{\overline{T}_0, \overline{T}_1, \overline{T}_2, ..., \overline{T}_m\}$ are α – isomorphic and $|\Omega(Q)| = m_0$, then the following equality is valid:

$$\left|R(D')\right| = m_0 \cdot \left(3^{\left|\overline{T}_4\setminus\overline{T}_3\right|} - 2^{\left|\overline{T}_4\setminus\overline{T}_3\right|}\right)\cdot\left(2^{\left|\overline{T}_1\setminus\overline{T}_4\right|}-1\right)\cdot\left(2^{\left|\overline{T}_2\setminus\overline{T}_1\right|}-1\right)\cdot\left(7^{\left|\overline{T}_6\setminus\overline{T}_5\right|}-6^{\left|\overline{T}_6\setminus\overline{T}_5\right|}\right)\cdot$$
$$\cdot\left(8^{\left|\overline{T}_7\setminus\overline{T}_6\right|}-7^{\left|\overline{T}_7\setminus\overline{T}_6\right|}\right)\cdots\left(m^{\left|\overline{T}_{m-1}\setminus\overline{T}_{m-2}\right|}-(m-1)^{\left|\overline{T}_{m-1}\setminus\overline{T}_{m-2}\right|}\right)\cdot(m+1)^{\left|X\setminus\overline{T}_m\right|}.$$

Proof. In the first place, we note that the semilattice Q has only one automorphisms (i.e. $\left|\Phi(Q,Q)\right|=1$). Let $\alpha\in\overline{R}(Q,D')$ and a quasinormal representation of a regular binary relation α have the form

$$\alpha=\bigcup_{i=1}^{m}\left(Y_i^\alpha\times T_i\right)$$

Then according to Theorem 2.1 the condition $\alpha\in\overline{R}(Q,D')$ is fulfilled if

$$\begin{aligned}&Y_0^\alpha\supseteq\overline{T}_0,\ Y_0^\alpha\cup Y_1^\alpha\supseteq\overline{T}_1,\ Y_0^\alpha\cup Y_2^\alpha\supseteq\overline{T}_2,\\&Y_0^\alpha\cup Y_2^\alpha\cup Y_4^\alpha\supseteq\overline{T}_4,\ Y_1^\alpha\cup Y_2^\alpha\cup\ldots\cup Y_p^\alpha\supseteq\overline{T}_p,\\&Y_q^\alpha\cap\overline{T}_q\neq\varnothing.\end{aligned}\qquad(2.4)$$

for any $p=6,7,\ldots,m-1$ and $q=1,2,4,6,7,\ldots,m$.

Now, assume that f_α is a mapping of the set X in D such that $f_\alpha(t)=t\alpha$ for any $t\in X$. $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$, $f_{4\alpha}$, $f_{p\alpha}$ $(p=6,7,\ldots,m)$, $f_{m+1\,\alpha}$ are respectively the restrictions of the mapping f_α on the sets $\overline{T}_4\cap\overline{T}_1$, $\overline{T}_4\setminus\overline{T}_3$, $\overline{T}_1\setminus\overline{T}_4$, $\overline{T}_2\setminus\overline{T}_1$, $\overline{T}_6\setminus\overline{T}_5,\ldots,$ $\overline{T}_m\setminus\overline{T}_{m-1}$ and $X\setminus\overline{T}_m$. We have, by assumption, that these sets do not intersect pairwise and the set-theoretic union of these sets is equal to X.

Let us establish the properties of the mappings $t\in X$. $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$, $f_{4\alpha}$, $f_{p\alpha}$ $(p=6,7,\ldots,m)$, $f_{m+1\,\alpha}$.

1) $t\in\overline{T}_4\cap\overline{T}_1$. Hence by virtue of the inclusions (2.4) we have

$$t\in\left(Y_0^\alpha\cup Y_2^\alpha\cup Y_4^\alpha\right)\cap\left(Y_0^\alpha\cup Y_1^\alpha\right)=Y_0^\alpha,$$

i.e., $t\alpha=T_0$ by the definition of the set Y_0^α. Thus $f_{1\alpha}(t)=T_0$ for any $t\in\overline{T}_4\cap\overline{T}_1$.

2) $t\in\overline{T}_4\setminus\overline{T}_3$,. In that case, by virtue of inclusion (2.4) we have $t\in\overline{T}_4\setminus\overline{T}_3\subseteq\overline{T}_4\subseteq Y_0^\alpha\cup Y_2^\alpha\cup Y_4^\alpha$. Therefore $t\alpha\in\{T_0,T_2,T_4\}$ by the definition of the sets $Y_0^\alpha,Y_2^\alpha,Y_4^\alpha$. Thus $f_{2\alpha}(t)\in\{T_0,T_2,T_4\}$ for any $t\in\overline{T}_4\setminus\overline{T}_3$,.

On the other hand, the inequality $Y_4^\alpha\cap\overline{T}_4\neq\varnothing$ is true. Therefore $t_4\in Y_4^\alpha$ for some element $t_4\in\overline{T}_4$. Hence it follows that $t_4\alpha=T_4$. Furthermore, if $t_4\in\overline{T}_3$, then $t_4\alpha\in\{T_0,T_1,T_2,T_3\}$. However the latter condition contradicts the equality $t_4\alpha=T_4$. The contradiction obtained shows that $t_4\in\overline{T}_4\setminus\overline{T}_3$,. Thus $f_{2\alpha}(t_4)=T_4$ for some $t_4\in\overline{T}_4\setminus\overline{T}_3$.

3) $t\in\overline{T}_1\setminus\overline{T}_4$. In that case, by virtue of inclusion (2.4) we have $t\in\overline{T}_1\setminus\overline{T}_4\subseteq\overline{T}_1\subseteq Y_0^\alpha\cup Y_1^\alpha$. Therefore $t\alpha\in\{T_0,T_1\}$ by the definition of the sets Y_0^α,Y_1^α. Thus $f_{3\alpha}(t)\in\{T_0,T_1\}$ for any $t\in\overline{T}_1\setminus\overline{T}_4$.

On the other hand, the inequality $Y_1^\alpha\cap\overline{T}_1\neq\varnothing$ is true. Therefore $t_1\in Y_1^\alpha$ for some element $t_1\in\overline{T}_1$. Hence it follows that $t_1\alpha=T_1$. Furthermore, if $t_1\in\overline{T}_4$, then $t_1\alpha\in\{T_0,T_2,T_4\}$. However the latter condition contradicts the equality $t_1\alpha=T_1$. The contradiction obtained shows that $t_1\in\overline{T}_1\setminus\overline{T}_4$. Thus $f_{3\alpha}(t_1)=T_1$ for some $t_1\in\overline{T}_1\setminus\overline{T}_4$.

4) $t\in\overline{T}_2\setminus\overline{T}_1$. In that case, by virtue of inclusion (2.4) we have $t\in\overline{T}_2\setminus\overline{T}_1\subseteq\overline{T}_2\subseteq Y_0^\alpha\cup Y_2^\alpha$. Therefore $t\alpha\in\{T_0,T_2\}$ by the definition of the sets Y_0^α,Y_2^α. Thus $f_{4\alpha}(t)\in\{T_0,T_2\}$ for any $t\in\overline{T}_2\setminus\overline{T}_1$.

On the other hand, the inequality $Y_2^\alpha\cap\overline{T}_2\neq\varnothing$ is true. Therefore $t_2\in Y_2^\alpha$ for some element $t_2\in\overline{T}_2$. Hence it follows that $t_2\alpha=T_2$. Furthermore, if $t_2\in\overline{T}_1$, then $t_2\alpha\in\{T_0,T_1\}$. However the latter condition contradicts the equality $t_2\alpha=T_2$. The contradiction obtained shows that $t_2\in\overline{T}_2\setminus\overline{T}_1$. Thus $f_{4\alpha}(t_2)=T_2$ for some $t_2\in\overline{T}_2\setminus\overline{T}_1$.

5) $t \in \overline{T}_s \setminus \overline{T}_{s-1}$ $(s = 6, 7, ..., m)$. In that case, by virtue of inclusion (2.4) we have

$$t \in \overline{T}_s \setminus \overline{T}_{s-1} \subseteq \overline{T}_s \subseteq Y_0^\alpha \cup Y_1^\alpha \cup Y_2^\alpha \cup ... \cup Y_s^\alpha .$$

Therefore $t\alpha \in \{T_0, T_1, ..., T_s\}$ by the definition of the sets $Y_0^\alpha, Y_1^\alpha, ..., Y_s^\alpha$. Thus $f_{s\alpha}(t) \in \{T_0, T_1, ..., T_s\}$ for any $t \in \overline{T}_s \setminus \overline{T}_{s-1}$.

On the other hand, the inequality $Y_s^\alpha \cap \overline{T}_s \neq \varnothing$ is true. Therefore $t_s \in Y_s^\alpha$ for some element $t_s \in \overline{T}_s$. Hence it follows that $t_s\alpha = T_s$. Furthermore, if $t_s \in \overline{T}_{s-1}$, then $t_s\alpha \in \{T_0, T_1, ..., T_{s-1}\}$. However the latter condition contradicts the equality $t_s\alpha = T_s$. The contradiction obtained shows that $t_s \in \overline{T}_s \setminus \overline{T}_{s-1}$. Thus $f_{s\alpha}(t_s) = T_s$ for some $t_s \in \overline{T}_s \setminus \overline{T}_{s-1}$.

6) $t \in X \setminus \overline{T}_m$. Then by virtue of the condition $X = \bigcup_{i=0}^{m} Y_i^\alpha$ we have $t \in \bigcup_{i=0}^{m} Y_i^\alpha$. Hence we obtain $t\alpha \in \{T_0, T_1, T_2, ..., T_m\}$.

Thus $f_{m+1\,\alpha}(t) \in \{T_0, T_1, T_2, ..., T_m\}$ for any $t \in X \setminus \overline{T}_m$.

Therefore for a binary relation $\alpha \in \overline{R}(Q, D')$ there exists an ordered system $(f_{1\alpha}, f_{2\alpha}, ..., f_{m+1\alpha})$

Now let

$$f_1 : \overline{T}_4 \cap \overline{T}_1 \to \{T_0\},\ f_2 : \overline{T}_4 \setminus \overline{T}_3 \to \{T_0, T_2, T_4\},\ f_3 : \overline{T}_1 \setminus \overline{T}_4 \to \{T_0, T_1\},\ f_4 : \overline{T}_2 \setminus \overline{T}_1 \to \{T_0, T_2\}$$

$$f_s : \overline{T}_s \setminus \overline{T}_{s-1} \to \{T_0, T_1, ..., T_s\},\ s = 6, 7, ..., m,\quad f_{m+1} : X \setminus \overline{T}_m \to \{T_0, T_1, ..., T_m\}$$

be the mappings satisfying the following conditions:

7) $f_1(t) = T_0$ for any $t \in \overline{T}_4 \cap \overline{T}_1$;

8) $f_2(t) \in \{T_0, T_2, T_4\}$ for any $t \in \overline{T}_4 \setminus \overline{T}_3$ and $f_2(t_4) = T_4$ for some $t_4 \in \overline{T}_4 \setminus \overline{T}_3$;

9) $f_3(t) \in \{T_0, T_1\}$ for any $t \in \overline{T}_1 \setminus \overline{T}_4$ and $f_3(t_1) = T_1$ for some $t_1 \in \overline{T}_1 \setminus \overline{T}_4$;

10) $f_4(t) \in \{T_0, T_2\}$ for any $t \in \overline{T}_2 \setminus \overline{T}_1$ and $f_4(t_2) = T_2$ for some $t_2 \in \overline{T}_2 \setminus \overline{T}_1$;

11) $f_s(t) \in \{T_0, T_1, T_2, ..., T_s\}$ for any $t \in \overline{T}_s \setminus \overline{T}_{s-1}$, and $f_s(t_s) = T_s$ for some $t_s \in \overline{T}_s \setminus \overline{T}_{s-1}$, where $s = 6, 7, ..., m$;

12) $f_{m+1}(t) \in \{T_0, T_1, T_2, ..., T_m\}$ for any $t \in X \setminus \overline{T}_m$.

Now we write the mapping $f : X \to D$ as follows:

$$f(t) = \begin{cases} f_1(t), & if\ \ t \in \overline{T}_4 \cap \overline{T}_1, \\ f_2(t), & if\ \ t \in \overline{T}_4 \setminus \overline{T}_3, \\ f_3(t), & if\ \ t \in \overline{T}_1 \setminus \overline{T}_4, \\ f_4(t), & if\ \ t \in \overline{T}_2 \setminus \overline{T}_1, \\ f_s(t), & if\ \ t \in \overline{T}_s \setminus \overline{T}_{s-1},\ p = 6, 7, ..., m, \\ f_{m+1}(t), & if\ \ t \in X \setminus \overline{T}_m. \end{cases}$$

To the mapping f we put into correspondence the relation $\beta = \bigcup_{t \in X} (\{t\} \times f(t))$.

Now let $Y_i^\beta = \{t \in X \mid t\beta = T_i\}$, where $i = 0, 1, 2, ..., m$. With this notation, the binary relation β is represented as $\beta = \bigcup_{i=0}^{m} (Y_i^\beta \times T_i)$. Moreover, from the defnition of the binary relation β we immediately obtain

$$\begin{array}{l} Y_0^\beta \supseteq \overline{T}_0,\ Y_0^\beta \cup Y_1^\beta \supseteq \overline{T}_1,\ Y_0^\beta \cup Y_2^\beta \supseteq \overline{T}_2, \\ Y_0^\beta \cup Y_2^\beta \cup Y_4^\beta \supseteq \overline{T}_4,\ Y_1^\beta \cup Y_2^\beta \cup ... \cup Y_p^\beta \supseteq \overline{T}_p, \\ Y_q^\beta \cap \overline{T}_q \neq \varnothing. \end{array}$$

for any $p = 6, 7, ..., m-1$ and $q = 1, 2, 4, 6, 7, ..., m$ since $f_2(t_4) = T_4$ for some $t_4 \in \overline{T}_4 \setminus \overline{T}_3$, $f_3(t_1) = T_1$ for some $t_1 \in \overline{T}_1 \setminus \overline{T}_4$, $f_4(t_2) = T_2$ for some $t_2 \in \overline{T}_2 \setminus \overline{T}_1$, $f_s(t_s) = T_s$ for some $t_s \in \overline{T}_s \setminus \overline{T}_{s-1}$, where $s = 6, 7, ..., m$.

Hence by virtue of Theorem 2.1 we conclude that the binary relation β is a regular element of the semigroup $B_X(D)$ that belongs to the set $\overline{R}(Q,D')$.

By the lemma 1.1 and lemma 1.3 The numbers of all mappings of the form $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$, $f_{4\alpha}$, $f_{p\alpha}$ $(p=6,7,...,m)$, $f_{m+1\,\alpha}$ $\left(\alpha \in \overline{R}(Q,D')\right)$ are equal respectively to

$$1,\ 3^{\left|\overline{T}_4\setminus\overline{T}_3\right|}-2^{\left|\overline{T}_4\setminus\overline{T}_3\right|},\ 2^{\left|\overline{T}_1\setminus\overline{T}_4\right|}-1,\ 2^{\left|\overline{T}_2\setminus\overline{T}_1\right|}-1\ldots,(s+1)^{\left|\overline{T}_s\setminus\overline{T}_{s-1}\right|}-s^{\left|\overline{T}_s\setminus\overline{T}_{s-1}\right|},\ (m+1)^{\left|X\setminus\overline{T}_m\right|}.$$

Therefore the equality

$$\left|\overline{R}(Q,D')\right|=\left(3^{\left|\overline{T}_4\setminus\overline{T}_3\right|}-2^{\left|\overline{T}_4\setminus\overline{T}_3\right|}\right)\cdot\left(2^{\left|\overline{T}_1\setminus\overline{T}_4\right|}-1\right)\cdot\left(2^{\left|\overline{T}_2\setminus\overline{T}_1\right|}-1\right)\cdot\left((s+1)^{\left|\overline{T}_s\setminus\overline{T}_{s-1}\right|}-s^{\left|\overline{T}_s\setminus\overline{T}_{s-1}\right|}\right)\cdot(m+1)^{\left|X\setminus\overline{T}_m\right|}$$

is valid, where $s=6,7,...,m$.

Now, using the equalities $\left|\Omega(Q)\right|=m_0$, $\left|\Phi(Q,D')\right|=1$ and theorem1.4, we Obtain

$$\left|R(D')\right|=m_0\cdot\left(2^{\left|\overline{T}_1\setminus\overline{T}_4\right|}-1\right)\cdot\left(2^{\left|\overline{T}_2\setminus\overline{T}_1\right|}-1\right)\cdot\left(3^{\left|\overline{T}_4\setminus\overline{T}_3\right|}-2^{\left|\overline{T}_4\setminus\overline{T}_3\right|}\right)\cdot\left(7^{\left|\overline{T}_6\setminus\overline{T}_5\right|}-6^{\left|\overline{T}_6\setminus\overline{T}_5\right|}\right)\cdot$$
$$\cdot\left(8^{\left|\overline{T}_7\setminus\overline{T}_6\right|}-7^{\left|\overline{T}_7\setminus\overline{T}_6\right|}\right)\cdots\left(m^{\left|\overline{T}_{m-1}\setminus\overline{T}_{m-2}\right|}-(m-1)^{\left|\overline{T}_{m-1}\setminus\overline{T}_{m-2}\right|}\right)\cdot(m+1)^{\left|X\setminus\overline{T}_m\right|}.$$

Theorem is proved.

Corollary 2.1. *Let* $Q=\{T_0,T_1,T_2,...,T_6\}$ *be a subsemilattice of the semilattice* D *and*

$$T_0\subset T_1\subset T_3\subset T_5\subset T_6,\ T_0\subset T_2\subset T_3\subset T_5\subset T_6,$$
$$T_0\subset T_2\subset T_4\subset T_5\subset T_6, T_1\setminus T_2\neq\varnothing,\ T_2\setminus T_1\neq\varnothing,$$
$$T_1\setminus T_4\neq\varnothing,\ T_4\setminus T_1\neq\varnothing,\ T_3\setminus T_4\neq\varnothing,\ T_4\setminus T_3\neq\varnothing,$$
$$T_1\cup T_2=T_3,\ T_4\cup T_1=T_4\cup T_3=T_5.$$

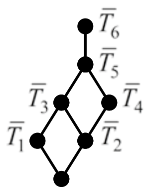


Fig. 4

(see Fig. 13.6.4). If the $XI-$*semilattices* Q *and* $D'=\{\overline{T}_1,\overline{T}_2,...,\overline{T}_6\}$ *are* $\alpha-$*isomorphic and* $\left|\Omega(Q)\right|=m_0$, *the equality*

$$\left|R(D')\right|=m_0\cdot\left(2^{\left|\overline{T}_1\setminus\overline{T}_4\right|}-1\right)\cdot\left(2^{\left|\overline{T}_2\setminus\overline{T}_1\right|}-1\right)\cdot\left(3^{\left|\overline{T}_4\setminus\overline{T}_3\right|}-2^{\left|\overline{T}_4\setminus\overline{T}_3\right|}\right)\cdot\left(7^{\left|\overline{T}_6\setminus\overline{T}_5\right|}-6^{\left|\overline{T}_6\setminus\overline{T}_5\right|}\right)\cdot 7^{\left|X\setminus\overline{T}_m\right|}.$$

is valid.

Proof. The corollary immediately follows from Theorem 2.2

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