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THE EXTON'S QUADRUPLE HYPERGEOMETRIC FUNCTIONS OF MATRIX ARGUMENTS IN COMPLEX CASE

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ABSTRACT

In this paper we have defined the Exton's quadruple hypergeometric functionsof matrix arguments and established a number of results for these functions of matrix argument in complex case.

1. INTRODUCTION

Following Exton [1], we define first of all hypergeometric functions of four variables in matrix variate. All the matrices appearing in this paper are p x p real Hermitian positive definite and meanings of all the other symbols used are the same as in the work of Mathai [3,5].

FUNCTION OF MATRIX ARGUMENT IN THE COMPLEX CASE :

We consider real valued scalar function of a single matrix argument of the type $\tilde{Z} = \tilde{X} + i\tilde{Y}$ where $\tilde{X}$ and $\tilde{Y}$ are p x p matrices with real elements and $i=\sqrt{-1}$ as well as scaler functions of many matrices $\tilde{Z}_j$, j = 1, 2, …….K where each $\tilde{Z}_j$ is of the type $\tilde{Z}$ above in the real case. We confined our discussion to the situation where the argument matrix was real symmetric positive definite. This was done so that the fractional power of matrices and functions of such matrices could be uniquely defined. Corresponding properties are of we restrict to the class of Hermitian positive definite matrices.

Definition : Hermitian positive definite matrix due to Mathai [5], We will denote the conjugate of $\tilde{Z}$ by $\tilde{\tilde{Z}}$ if $\tilde{Z}$ hermitian, then $\tilde{Z} = \tilde{Z}*$, that is

$$\tilde{Z} = \tilde{Z}* \quad \Rightarrow \quad \tilde{X} + i\tilde{Y} = (\tilde{X} + i\tilde{Y})* = \tilde{X}' + i\tilde{Y}'$$

$$\Rightarrow \quad \tilde{X} = \tilde{X}' \text{ and } \tilde{Y} = \tilde{Y}'$$

Thus $\tilde{X}$ is the symmetric and $\tilde{Y}$ is skew symmetric. Further if $\tilde{Z}$ is hermitian positive definite, then all the eigen values of $\tilde{Z}$ are real and positive. Further, matrix variate gamma in the complex case is

$$\tilde{\Gamma}_p(\alpha) = \pi^{\frac{p(p-1)}{2}}\Gamma(\alpha)\Gamma(\alpha-1).......\Gamma(\alpha-\pi+1)$$

We will use the notation $\tilde{Z} > 0$ to indicate that $\tilde{Z}$ is hermitian positive definite. Constant matrices will be written without a tilde whether the elements are real or complex unless it has to be emphasized that the matrix involved has complex elements. Then in that case a constant matrix will also be written with a tilde.

2. DEFINITIONS

The following are the definitions of the Exton's quadruple hypergeometric functions with matrix arguments in complex case.

2.1 The Exton's K_1 function of matrix arguments,

$K_1 = K_1\left(a,a,a,a;b,b,b,c;d,e_1,e_2,d;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T}\right)$ is defined as that class of functions which has the following matrix transform:

$$M(K_1) = \int_{\tilde{X}>0}\int_{\tilde{Y}>0}\int_{\tilde{Z}>0}\int_{\tilde{T}>0}\left|\tilde{X}\right|^{\rho_1-p}\left|\tilde{Y}\right|^{\rho_2-p}\left|\tilde{Z}\right|^{\rho_3-p}\left|\tilde{T}\right|^{\rho_4-p}\times$$

$$K_1\left(a,a,a,a;b,b,b,c;d,e_1,e_2,d;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T}\right)d\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{T}$$

$$= \frac{\tilde{\Gamma}_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(a)}\frac{\tilde{\Gamma}_p(b-\rho_1-\rho_2-\rho_3)}{\tilde{\Gamma}_p(b)}\frac{\tilde{\Gamma}_p(c-\rho_4)}{\tilde{\Gamma}_p(c)}\times$$

$$\frac{\tilde{\Gamma}_p(d)}{\tilde{\Gamma}_p(d-\rho_1-\rho_4)}\frac{\tilde{\Gamma}_p(e_1)}{\tilde{\Gamma}_p(e_1-\rho_2)}\frac{\tilde{\Gamma}_p(e_2)}{\tilde{\Gamma}_p(e_2-\rho_3)}\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\rho_3)\tilde{\Gamma}_p(\rho_4)$$

for $\operatorname{Re}\left(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2-\rho_3, c-\rho_4, d-\rho_1-\rho_4, e_1-\rho_2, e_2-\rho_3, \rho_i\right) > p-1$

where, i=1,…,4. …(2.1)

2.2 $K_2 = K_2\left(a,a,a,a;b,b,b,c;d_1,d_2,d_3,d_4;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T}\right)$

$$M(K_2) = \int_{\tilde{X}>0}\int_{\tilde{Y}>0}\int_{\tilde{Z}>0}\int_{\tilde{T}>0}\left|\tilde{X}\right|^{\rho_1-p}\left|\tilde{Y}\right|^{\rho_2-p}\left|\tilde{Z}\right|^{\rho_3-p}\left|\tilde{T}\right|^{\rho_4-p}\times$$

$$K_2\left(a,a,a,a;b,b,b,c;d_1,d_2,d_3,d_4;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T}\right)d\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{T}$$

$$=\frac{\tilde{\Gamma}_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(a)}\frac{\tilde{\Gamma}_p(b-\rho_1-\rho_2-\rho_3)}{\tilde{\Gamma}_p(b)}\frac{\tilde{\Gamma}_p(c-\rho_4)}{\tilde{\Gamma}_p(c)}\times$$

$$\frac{\tilde{\Gamma}_p(d_1)}{\tilde{\Gamma}_p(d_1-\rho_1)}\frac{\tilde{\Gamma}_p(d_2)}{\tilde{\Gamma}_p(d_2-\rho_2)}\frac{\tilde{\Gamma}_p(d_3)}{\tilde{\Gamma}_p(d_3-\rho_3)}\frac{\tilde{\Gamma}_p(d_4)}{\tilde{\Gamma}_p(d_4-\rho_4)}\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\rho_3)\tilde{\Gamma}_p(\rho_4) \qquad \ldots(2.2)$$

for $\mathrm{Re}(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2-\rho_3, c-\rho_4, d_i-\rho_i, \rho_i) > p-1$ where, i=1,…,4.

2.3 $K_3 = K_3(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_2,c_1;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})$

$$M(K_3) = \int_{\tilde{X}>0}\int_{\tilde{Y}>0}\int_{\tilde{Z}>0}\int_{\tilde{T}>0} |\tilde{X}|^{\rho_1-p}|\tilde{Y}|^{\rho_2-p}|\tilde{Z}|^{\rho_3-p}|\tilde{T}|^{\rho_4-p}\times$$

$$K_3(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_2,c_1;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})d\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{T}$$

$$=\frac{\tilde{\Gamma}_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(a)}\frac{\tilde{\Gamma}_p(b_1-\rho_1-\rho_2)}{\tilde{\Gamma}_p(b_1)}\frac{\tilde{\Gamma}_p(b_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(b_2)}\times$$

$$\frac{\tilde{\Gamma}_p(c_1)}{\tilde{\Gamma}_p(c_1-\rho_1-\rho_4)}\frac{\tilde{\Gamma}_p(c_1)}{\tilde{\Gamma}_p(c_2-\rho_2-\rho_3)}\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\rho_3)\tilde{\Gamma}_p(\rho_4) \qquad \ldots(2.3)$$

for $\mathrm{Re}(a-\rho_1-\rho_2-\rho_3-\rho_4, b_1-\rho_1-\rho_2, b_2-\rho_3-\rho_4, c_1-\rho_1-\rho_4, c_2-\rho_2-\rho_3, \rho_i) > p-1$

where, i=1,…,4.

2.4 $K_4 = K_4(a,a,a,a;b_1,b_1,b_2,b_2;c,d_1,d_2,c;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})$

$$M(K_4) = \int_{\tilde{X}>0}\int_{\tilde{Y}>0}\int_{\tilde{Z}>0}\int_{\tilde{T}>0} |\tilde{X}|^{\rho_1-p}|\tilde{Y}|^{\rho_2-p}|\tilde{Z}|^{\rho_3-p}|\tilde{T}|^{\rho_4-p}\times$$

$$K_4(a,a,a,a;b_1,b_1,b_2,b_2;c,d_1,d_2,c;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})d\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{T}$$

$$=\frac{\tilde{\Gamma}_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(a)}\frac{\tilde{\Gamma}_p(b_1-\rho_1-\rho_2)}{\tilde{\Gamma}_p(b_1)}\frac{\tilde{\Gamma}_p(b_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(b_2)}\times$$

$$\frac{\tilde{\Gamma}_p(c)}{\tilde{\Gamma}_p(c-\rho_1-\rho_4)}\frac{\tilde{\Gamma}_p(d_1)}{\tilde{\Gamma}_p(d_1-\rho_2)}\frac{\tilde{\Gamma}_p(d_2)}{\tilde{\Gamma}_p(d_2-\rho_3)}\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\rho_3)\tilde{\Gamma}_p(\rho_4) \qquad \ldots(2.4)$$

for $\mathrm{Re}(a-\rho_1-\rho_2-\rho_3-\rho_4, b_1-\rho_1-\rho_2, b_2-\rho_3-\rho_4, c-\rho_1-\rho_4, d_1-\rho_2, d_2-\rho_3, \rho_i) > p-1$

where, i=1,…,4.

2.5 $K_5 = K_5(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})$

$$M(K_5)=\int_{\tilde{X}>0}\int_{\tilde{Y}>0}\int_{\tilde{Z}>0}\int_{\tilde{T}>0}|\tilde{X}|^{\rho_1-p}|\tilde{Y}|^{\rho_2-p}|\tilde{Z}|^{\rho_3-p}|\tilde{T}|^{\rho_4-p}\times$$

$$K_5(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})d\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{T}$$

$$=\frac{\tilde{\Gamma}_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(a)}\frac{\tilde{\Gamma}_p(b_1-\rho_1-\rho_2)}{\tilde{\Gamma}_p(b_1)}\frac{\tilde{\Gamma}_p(b_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(b_2)}\times$$

$$\frac{\tilde{\Gamma}_p(c_1)}{\tilde{\Gamma}_p(c_1-\rho_1)}\frac{\tilde{\Gamma}_p(c_2)}{\tilde{\Gamma}_p(c_2-\rho_2)}\frac{\tilde{\Gamma}_p(c_3)}{\tilde{\Gamma}_p(c_3-\rho_3)}\frac{\tilde{\Gamma}_p(c_4)}{\tilde{\Gamma}_p(c_4-\rho_4)}\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\rho_3)\tilde{\Gamma}_p(\rho_4) \qquad \ldots(2.5)$$

for $\mathrm{Re}(a-\rho_1-\rho_2-\rho_3-\rho_4,b_1-\rho_1-\rho_2,b_2-\rho_3-\rho_4,c_1-\rho_1,c_2-\rho_2,c_3-\rho_3,c_4-\rho_4,\rho_i)>p-1$

where, i=1,…,4.

2.6 $K_6=K_6(a,a,a,a;b,b,c_1,c_2;e,d,d,d;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})$

$$M(K_6)=\int_{\tilde{X}>0}\int_{\tilde{Y}>0}\int_{\tilde{Z}>0}\int_{\tilde{T}>0}|\tilde{X}|^{\rho_1-p}|\tilde{Y}|^{\rho_2-p}|\tilde{Z}|^{\rho_3-p}|\tilde{T}|^{\rho_4-p}\times$$

$$K_6(a,a,a,a;b,b,c_1,c_2;e,d,d,d;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})d\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{T}$$

$$=\frac{\tilde{\Gamma}_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(a)}\frac{\tilde{\Gamma}_p(b-\rho_1-\rho_2)}{\tilde{\Gamma}_p(b)}\frac{\tilde{\Gamma}_p(c_1-\rho_3)}{\tilde{\Gamma}_p(c_1)}\times$$

$$\frac{\tilde{\Gamma}_p(c_2-\rho_4)}{\tilde{\Gamma}_p(c_2)}\frac{\tilde{\Gamma}_p(e)}{\tilde{\Gamma}_p(e-\rho_1)}\frac{\tilde{\Gamma}_p(d)}{\tilde{\Gamma}_p(d-\rho_2-\rho_3-\rho_4)}\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\rho_3)\tilde{\Gamma}_p(\rho_4) \qquad \ldots(2.6)$$

for $\mathrm{Re}(a-\rho_1-\rho_2-\rho_3-\rho_4,b-\rho_1-\rho_2,c_1-\rho_3,c_2-\rho_4,e-\rho_1,d-\rho_2-\rho_3-\rho_4,\rho_i)>p-1$

where, i=1,…,4.

2.7 $K_7=K_7(a,a,a,a;b,b,c_1,c_2;d_1,d_2,d_1,d_2;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})$

$$M(K_7)=\int_{\tilde{X}>0}\int_{\tilde{Y}>0}\int_{\tilde{Z}>0}\int_{\tilde{T}>0}|\tilde{X}|^{\rho_1-p}|\tilde{Y}|^{\rho_2-p}|\tilde{Z}|^{\rho_3-p}|\tilde{T}|^{\rho_4-p}\times$$

$$K_7(a,a,a,a;b,b,c_1,c_2;d_1,d_2,d_1,d_2;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})d\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{T}$$

$$=\frac{\tilde{\Gamma}_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(a)}\frac{\tilde{\Gamma}_p(b-\rho_1-\rho_2)}{\tilde{\Gamma}_p(b)}\frac{\tilde{\Gamma}_p(c_1-\rho_3)}{\tilde{\Gamma}_p(c_1)}\times$$

$$\frac{\tilde{\Gamma}_p(c_2-\rho_4)}{\tilde{\Gamma}_p(c_2)}\frac{\tilde{\Gamma}_p(d_1)}{\tilde{\Gamma}_p(d_1-\rho_1-\rho_3)}\frac{\tilde{\Gamma}_p(d_2)}{\tilde{\Gamma}_p(d_2-\rho_2-\rho_4)}\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\rho_3)\tilde{\Gamma}_p(\rho_4) \qquad \ldots(2.7)$$

for $\mathrm{Re}(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2, c_1-\rho_3, c_2-\rho_4, d_1-\rho_1-\rho_3, d-\rho_2-\rho_4, \rho_i) > p-1$

where, i=1,…,4.

2.8 $K_8 = K_8(a,a,a,a;b,b,c_1,c_2;d,e_1,d,e_2;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})$

$$M(K_8) = \int\limits_{\tilde{X}>0}\int\limits_{\tilde{Y}>0}\int\limits_{\tilde{Z}>0}\int\limits_{\tilde{T}>0}|\tilde{X}|^{\rho_1-p}|\tilde{Y}|^{\rho_2-p}|\tilde{Z}|^{\rho_3-p}|\tilde{T}|^{\rho_4-p}\times$$

$$K_8(a,a,a,a;b,b,c_1,c_2;d,e_1,d,e_2;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})d\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{T}$$

$$=\frac{\tilde{\Gamma}_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(a)}\frac{\tilde{\Gamma}_p(b-\rho_1-\rho_2)}{\tilde{\Gamma}_p(b)}\frac{\tilde{\Gamma}_p(c_1-\rho_3)}{\tilde{\Gamma}_p(c_1)}\times$$

$$\frac{\tilde{\Gamma}_p(c_2-\rho_4)}{\tilde{\Gamma}_p(c_2)}\frac{\tilde{\Gamma}_p(d)}{\tilde{\Gamma}_p(d-\rho_1-\rho_3)}\frac{\tilde{\Gamma}_p(e_1)}{\tilde{\Gamma}_p(e_1-\rho_2)}\frac{\tilde{\Gamma}_p(e_2)}{\tilde{\Gamma}_p(e_2-\rho_4)}\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\rho_3)\tilde{\Gamma}_p(\rho_4)$$

for $\mathrm{Re}(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2, c_1-\rho_3, c_2-\rho_4, d-\rho_1-\rho_3, e_1-\rho_2, e_2-\rho_4, \rho_i) > p-1$

where, i=1,…,4. …(2.8)

2.9 $K_9 = K_9(a,a,a,a;b,b,c_1,c_2;e_1,e_2,d,d;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})$

$$M(K_9) = \int\limits_{\tilde{X}>0}\int\limits_{\tilde{Y}>0}\int\limits_{\tilde{Z}>0}\int\limits_{\tilde{T}>0}|\tilde{X}|^{\rho_1-p}|\tilde{Y}|^{\rho_2-p}|\tilde{Z}|^{\rho_3-p}|\tilde{T}|^{\rho_4-p}\times$$

$$K_9(a,a,a,a;b,b,c_1,c_2;e_1,e_2,d,d;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})d\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{T}$$

$$=\frac{\tilde{\Gamma}_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(a)}\frac{\tilde{\Gamma}_p(b-\rho_1-\rho_2)}{\tilde{\Gamma}_p(b)}\frac{\tilde{\Gamma}_p(c_1-\rho_3)}{\tilde{\Gamma}_p(c_1)}\times$$

$$\frac{\tilde{\Gamma}_p(c_2-\rho_4)}{\tilde{\Gamma}_p(c_2)}\frac{\tilde{\Gamma}_p(e_1)}{\tilde{\Gamma}_p(e_1-\rho_1)}\frac{\tilde{\Gamma}_p(e_2)}{\tilde{\Gamma}_p(e_2-\rho_2)}\frac{\tilde{\Gamma}_p(d)}{\tilde{\Gamma}_p(d-\rho_3-\rho_4)}\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\rho_3)\tilde{\Gamma}_p(\rho_4)$$

for $\mathrm{Re}(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2, c_1-\rho_3, c_2-\rho_4, e_1-\rho_1, e_2-\rho_2, d-\rho_3-\rho_4, \rho_i) > p-1$

where, i=1,…,4. …(2.9)

2.10 $K_{10} = K_{10}(a,a,a,a;b,b,c_1,c_2;d_1,d_2,d_3,d_4;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})$

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$$M(K_{10}) = \int_{\tilde{X}>0}\int_{\tilde{Y}>0}\int_{\tilde{Z}>0}\int_{\tilde{T}>0} |\tilde{X}|^{\rho_1-p}|\tilde{Y}|^{\rho_2-p}|\tilde{Z}|^{\rho_3-p}|\tilde{T}|^{\rho_4-p} \times$$

$$K_{10}(a,a,a,a;b,b,c_1,c_2;d_1,d_2,d_3,d_4;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})d\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{T}$$

$$= \frac{\tilde{\Gamma}_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(a)}\frac{\tilde{\Gamma}_p(b-\rho_1-\rho_2)}{\tilde{\Gamma}_p(b)}\frac{\tilde{\Gamma}_p(c_1-\rho_3)}{\tilde{\Gamma}_p(c_1)} \times$$

$$\frac{\tilde{\Gamma}_p(c_2-\rho_4)}{\tilde{\Gamma}_p(c_2)}\frac{\tilde{\Gamma}_p(d_1)}{\tilde{\Gamma}_p(d_1-\rho_1)}\frac{\tilde{\Gamma}_p(d_2)}{\tilde{\Gamma}_p(d_2-\rho_2)}\frac{\tilde{\Gamma}_p(d_3)}{\tilde{\Gamma}_p(d_3-\rho_3)}\frac{\tilde{\Gamma}_p(d_4)}{\tilde{\Gamma}_p(d_4-\rho_4)}\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\rho_3)\tilde{\Gamma}_p(\rho_4)$$

for $\mathrm{Re}(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2, c_1-\rho_3, c_2-\rho_4, d_i-\rho_i, \rho_i) > p-1$ …(2.10)

where, i=1,…,4.

Theorem 3.1

$$K_5(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})$$

$$= \frac{1}{\tilde{\Gamma}_p(a)}\int_{\tilde{S}>0} e^{-tr(\tilde{S})}|\tilde{S}|^{a-p}\psi_2(b_1;c_1,c_2;-\tilde{S}^{1/2}\tilde{X}\tilde{S}^{1/2},-\tilde{S}^{1/2}\tilde{Y}\tilde{S}^{1/2})\times$$

$$\psi_2(b_2;c_3,c_4;-\tilde{S}^{1/2}\tilde{Z}\tilde{S}^{1/2},-\tilde{S}^{1/2}\tilde{T}\tilde{S}^{1/2})d\tilde{S}$$

for $\mathrm{Re}(a) > p-1$ …(3.1)

Proof: Taking the M-transform of the right side of eq.(3.1) with respect to the variables $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{T}$ and the parameters $\rho_1, \rho_2, \rho_3, \rho_4$ respectively, we get,

$$\int_{\tilde{X}>0}\int_{\tilde{Y}>0}\int_{\tilde{Z}>0}\int_{\tilde{T}>0} |\tilde{X}|^{\rho_1-p}|\tilde{Y}|^{\rho_2-p}|\tilde{Z}|^{\rho_3-p}|\tilde{T}|^{\rho_4-p}\psi_2(b_1;c_1,c_2;-\tilde{S}^{1/2}\tilde{X}\tilde{S}^{1/2},-\tilde{S}^{1/2}\tilde{Y}\tilde{S}^{1/2})\times$$

$$\psi_2(b_2;c_3,c_4;-\tilde{S}^{1/2}\tilde{Z}\tilde{S}^{1/2},-\tilde{S}^{1/2}\tilde{T}\tilde{S}^{1/2})d\tilde{X}d\tilde{Y}d\tilde{Z}d\tilde{T} \quad \ldots(3.2)$$

Applying the transformations,

$$\tilde{X}_1 = \tilde{S}^{1/2}\tilde{X}\tilde{S}^{1/2}, \tilde{Y}_1 = \tilde{S}^{1/2}\tilde{Y}\tilde{S}^{1/2}, \tilde{Z}_1 = \tilde{S}^{1/2}\tilde{Z}\tilde{S}^{1/2}, \tilde{T}_1 = \tilde{S}^{1/2}\tilde{T}\tilde{S}^{1/2}; \text{ with}$$

$$d\tilde{X}_1 = |\tilde{S}|^p d\tilde{X}, d\tilde{Y}_1 = |\tilde{S}|^p d\tilde{Y}, d\tilde{Z}_1 = |\tilde{S}|^p d\tilde{Z}, d\tilde{T}_1 = |\tilde{S}|^p d\tilde{T}; \text{ and,}$$

$|\tilde{X}_1| = |\tilde{S}||\tilde{X}|, |\tilde{Y}_1| = |\tilde{S}||\tilde{Y}|, |Z_1| = |\tilde{S}||\tilde{Z}|, |\tilde{T}_1| = |\tilde{S}||\tilde{T}|;$ to the above expression and then writing the M-transforms of the two involoved ψ_2-functions, we obtain

$$\left|\tilde{S}\right|^{-\rho_1-\rho_2-\rho_3-\rho_4}\frac{\tilde{\Gamma}_p(b_1-\rho_1-\rho_2)}{\tilde{\Gamma}_p(b_1)}\frac{\tilde{\Gamma}_p(c_1)}{\tilde{\Gamma}_p(c_1-\rho_1)}\frac{\tilde{\Gamma}_p(c_2)}{\tilde{\Gamma}_p(c_2-\rho_2)}\frac{\tilde{\Gamma}_p(c_3)}{\tilde{\Gamma}_p(c_3-\rho_3)}\frac{\tilde{\Gamma}_p(c_4)}{\tilde{\Gamma}_p(c_4-\rho_4)}\times$$

$$\frac{\tilde{\Gamma}_p(b_2-\rho_3-\rho_4)}{\tilde{\Gamma}_p(b_2)}\tilde{\Gamma}_p(\rho_1)\tilde{\Gamma}_p(\rho_2)\tilde{\Gamma}_p(\rho_3)\tilde{\Gamma}_p(\rho_4) \qquad \ldots(3.3)$$

Substituting this expression on the right side of eq.(3.1) and then integrating out $\tilde{S}$ in the resulting expression by using a Gamma integral gives $M(K_5)$ as given by eq.(2.5)

Theorem 3.2

$$K_9(a,a,a,a;b,b,c_1,c_2;e_1,e_2,d,d;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})$$

$$=\frac{1}{\tilde{\Gamma}_p(a)}\int_{\tilde{S}>0} e^{-tr(\tilde{S})}\left|\tilde{S}\right|^{a-p}\psi_2\left(b;e_1,e_2;-\tilde{S}^{1/2}\tilde{X}\tilde{S}^{1/2},-\tilde{S}^{1/2}\tilde{Y}\tilde{S}^{1/2}\right)\times$$

$$\Phi_2\left(c_1,c_2;d;-\tilde{S}^{1/2}\tilde{Z}\tilde{S}^{1/2},-\tilde{S}^{1/2}\tilde{T}\tilde{S}^{1/2}\right)d\tilde{S} \qquad \ldots(3.4)$$

for $\mathrm{Re}(a)>p-1$

Theorem 3.3

$$K_7(a,a,a,a;b,b,c_1,c_2;d_1,d_2,d_1,d_2;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})$$

$$=\frac{1}{\tilde{\Gamma}_p(b)\tilde{\Gamma}_p(c_1)\tilde{\Gamma}_p(c_2)}\int_{\tilde{S}_1>0}\int_{\tilde{S}_2>0}\int_{\tilde{S}_3>0} e^{-tr(\tilde{S}_1+\tilde{S}_2+\tilde{S}_3)}\left|\tilde{S}_1\right|^{b-p}\left|\tilde{S}_2\right|^{c_1-p}\left|\tilde{S}_3\right|^{c_2-p}\times$$

$$\psi_2\left(a;d_1,d_2;-\tilde{S}_1^{1/2}\tilde{X}\tilde{S}_1^{1/2}-\tilde{S}_2^{1/2}\tilde{Z}\tilde{S}_2^{1/2},-\tilde{S}_1^{1/2}\tilde{Y}\tilde{S}_1^{1/2}-\tilde{S}_3^{1/2}\tilde{T}\tilde{S}_3^{1/2}\right)d\tilde{S}_1 d\tilde{S}_2 d\tilde{S}_3 \qquad \ldots(3.5)$$

for $\mathrm{Re}(b,c_1,c_2)>p-1$

Theorem 3.4

$$K_1(a,a,a,a;b,b,b,c;d,e_1,e_2,d;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T})$$

$$=\frac{1}{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b)}\int_{\tilde{R}_1>0}\int_{R_2>0} e^{-tr(\tilde{R}_1+\tilde{R}_2)}\left|\tilde{R}_1\right|^{a-p}\left|\tilde{R}_2\right|^{b-p}\times$$

$${}_0F_1\left(;e_1;-\tilde{R}_2^{1/2}\tilde{R}_1^{1/2}\tilde{Y}\tilde{R}_1^{1/2}\tilde{R}_2^{1/2}\right){}_0F_1\left(;e_2;-\tilde{R}_2^{1/2}\tilde{R}_1^{1/2}\tilde{Z}\tilde{R}_1^{1/2}\tilde{R}_2^{1/2}\right)\times$$

$$\Phi_3\left(c;d;-\tilde{R}_1^{1/2}\tilde{T}\tilde{R}_1^{1/2},-\tilde{R}_2^{1/2}\tilde{R}_1^{1/2}\tilde{X}\tilde{R}_1^{1/2}\tilde{R}_2^{1/2}\right)d\tilde{R}_1 d\tilde{R}_2 \qquad \ldots(3.6)$$

for $\operatorname{Re}(a,b) > p-1$

Theorem 3.5

$$K_8\left(a,a,a,a;b,b,c_1,c_2;d,e_1,d,e_2;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T}\right)$$

$$= \frac{1}{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b)\tilde{\Gamma}_p(c_1)} \int_{\tilde{R}_1>0}\int_{\tilde{R}_2>0}\int_{\tilde{R}_3>0} e^{-\operatorname{tr}\left(\tilde{R}_1+\tilde{R}_2+\tilde{R}_3\right)} \left|\tilde{R}_1\right|^{a-p}\left|\tilde{R}_2\right|^{b-p}\left|\tilde{R}_3\right|^{c_1-p} \times$$

$${}_0F_1\left(;d;-\tilde{R}_2^{1/2}\tilde{R}_1^{1/2}\tilde{X}\tilde{R}_1^{1/2}\tilde{R}_2^{1/2}-\tilde{R}_3^{1/2}\tilde{R}_1^{1/2}\tilde{Z}\tilde{R}_1^{1/2}\tilde{R}_3^{1/2}\right){}_0F_1\left(;e_1;-\tilde{R}_2^{1/2}\tilde{R}_1^{1/2}\tilde{Y}\tilde{R}_1^{1/2}\tilde{R}_2^{1/2}\right)\times$$

$${}_1F_1\left(c_2;e_2;-\tilde{R}_1^{1/2}\tilde{T}\tilde{R}_1^{1/2}\right)d\tilde{R}_1 d\tilde{R}_2 d\tilde{R}_3 \qquad \ldots(3.7)$$

Theorem 3.6

$$K_{10}\left(a,a,a,a;b,b,c_1,c_2;d_1,d_2,d_3,d_4;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T}\right)$$

$$= \frac{1}{\tilde{\Gamma}_p(a)} \int_{\tilde{S}>0} e^{-\operatorname{tr}(\tilde{S})}\left|\tilde{S}\right|^{a-p}\Psi_2\left(b;d_1,d_2;-\tilde{S}^{1/2}\tilde{X}\tilde{S}^{1/2},-\tilde{S}^{1/2}\tilde{Y}\tilde{S}^{1/2}\right)$$

$${}_1F_1\left(c_1;d_3;-\tilde{S}^{1/2}\tilde{Z}\tilde{S}^{1/2}\right){}_1F_1\left(c_2;d_4;-\tilde{S}^{1/2}\tilde{T}\tilde{S}^{1/2}\right)d\tilde{S} \qquad \ldots(3.8)$$

for $\operatorname{Re}(a) > p-1$

Theorem 3.7

$$K_2\left(a,a,a,a;b,b,b,c;d_1,d_2,d_3,d_4;-\tilde{X},-\tilde{Y},-\tilde{Z},-\tilde{T}\right)$$

$$= \frac{1}{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b)} \int_{\tilde{R}_1>0}\int_{\tilde{R}_2>0} e^{-\operatorname{tr}\left(\tilde{R}_1+\tilde{R}_2\right)}\left|\tilde{R}_1\right|^{a-p}\left|\tilde{R}_2\right|^{b-p}{}_0F_1\left(;d_1;-\tilde{R}_2^{1/2}\tilde{R}_1^{1/2}\tilde{X}\tilde{R}_1^{1/2}\tilde{R}_2^{1/2}\right)\times$$

$${}_0F_1\left(;d_2;-\tilde{R}_2^{1/2}\tilde{R}_1^{1/2}\tilde{Y}\tilde{R}_1^{1/2}\tilde{R}_2^{1/2}\right){}_0F_1\left(;d_3;-\tilde{R}_2^{1/2}\tilde{R}_1^{1/2}\tilde{Z}\tilde{R}_1^{1/2}\tilde{R}_2^{1/2}\right)$$

$${}_1F_1\left(c;d_4;-\tilde{R}_1^{1/2}\tilde{T}\tilde{R}_1^{1/2}\right)d\tilde{R}_1 d\tilde{R}_2 \qquad \ldots(3.9)$$

for $\operatorname{Re}(a,b) > p-1$

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