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MODIFIED FRACTIONAL PSEUDO SPECTRAL METHOD FOR SOLVING FRACTIONAL BVPS

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ABSTRACT: In this paper, we introduce a new formula for fractional derivative for shifted Chebyshev polynomial. Also, Fractional differentiation matrix is derived based on shifted Chebyshev polynomial. Some examples of linear Fractional Differential Equations are solved by the new formula. Numerical results are compared with the exact solution to find the error and to show the efficiently of the proposed method.

1. Introduction

Fractional differential equations have a great interest recently in last few years [5]. It is caused by the intensive development of the theory of fractional calculus. It is applied in various sciences such as physics, mechanics, chemistry, engineering, etc. Recently, there are some papers present the existence of solution of nonlinear initial fractional differential equation by the use of techniques of nonlinear analysis. Although the tools of fractional calculus have been available and applicable to various fields of study, the investigation of the theory of fractional differential equations has only been started quite recently in [1, 2, 3, 4].

The differential equations involving Riemann–Liouville differential operators of fractional order $0 < q < 1$, appear to be important in modeling several physical phenomena and therefore seem to deserve an independent study of their theory parallel to the well-known theory of ordinary differential equations.

On the other hand, Chebyshev polynomials have been proven successfully in the numerical solution of various boundary value problems [9]. Chebyshev polynomials are important in approximation theory because the roots of the Chebyshev polynomials are used as nodes in polynomial interpolation. The resulting interpolation polynomial minimizes the problem of Runge's phenomenon and provides an approximation that is close to the polynomial of best approximation to a continuous function under the maximum norm.

In this paper we aim to formulate new method for the fractional differentiation matrix based on shifted Chebyshev polynomials using it's recurrence relation and use it for solving fractional order ordinary differential equations (FODE's).

The rest of the paper is arranged as follows: In Section 2, we introduce some mathematical preliminaries of the fractional calculus and some properties of the shifted Chebyshev polynomials. In section 3, we show the meaning of differentiation matrix of any order and how to get it. In Section 4, we derived a new formula for fractional derivative of shifted Chebyshev polynomial and show the form of fractional differentiation matrix. In section 5, we solve some examples of (F.D.E) which compared with exact solution.

2. Preliminaries

In this section, firstly we define the shifted Chebyshev polynomial of first kind $T_n^*(x)$as follow:

$$T_n^*(x) = \cos(2n\theta) \quad , x = cos^2(\theta) \quad , 0 < x < 1 \tag{1}$$

with recurrence relation:

$$T_n^*(x) = 2(2x-1)T_{n-1}^*(x) - T_{n-2}^*(x) \, , \; n = 2,3, \dots \tag{2}$$

where $T_0^*(x) = 1 \, , T_1^*(x) = 2x - 1$

The shifted Chebyshev polynomial can be expanded as a series of power of x by the relation:

$$T_n^*(x) = n \sum_{k=0}^{n} \frac{(-1)^{n-k} (n+k-1)! 2^{2k}}{(n-k)!(2k)!} x^k , \qquad n > 0 \tag{3}$$

with orthogonal relation defined by:

$$\int_0^L T_{L,j}(x)T_{L,k}(x)w_L(x)\mathrm{d}x = \delta_{kj}h_k,$$

Now, we give the definition of Caputo fractional derivative as follows**:**

Let $\alpha \in R$ and $n \in N$ $such$ $that$ $n-1<\alpha<n$ the Caputo fractional derivative of order $\alpha$is defined by the following relation:

$$D_a^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(u)}{(x-u)^{\alpha-n+1}}\,du \;\;, for\; all\; n-1<\alpha<n \tag{4}$$

$$D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{(\beta-\alpha)} \tag{5}$$

Where $\quad f^{(n)} = \frac{d^n}{dx^n}f(x)$

with the following properties:

1. $D_t^\alpha\big(f(t)*g(t)\big) = \sum_{k=0}^{\infty}\binom{\alpha}{k}D_t^{\alpha-k}[f(t)]D_t^k[g(t)]$
2. $D_t^\alpha(f(t)*C) = \sum_{k=0}^{\infty}\binom{\alpha}{k}D_t^{\alpha-k}[f(t)]D_t^k[C] = CD_t^\alpha f(t)$
3. $D_t^\alpha\big(f(t)\pm g(t)\big) = D_t^\alpha(t^0 f(t)) \pm D_t^\alpha(t^0 g(t))$

3. Derivation of new formula for fractional differentiation matrix:

The solution of fractional differential equation which approximated by the function$u_N(x)$ can be representing by spectral method in the following form as in [7]:

$$u_N(x) = \sum_{j=0}^{N}\theta_j a_j T_j^*(x)\;,\quad x\in[a,b], \tag{6}$$

Where $\theta_j = \frac{1}{2}, j = 0, N$ $except$ at $j = 1,\dots,N-1, \theta_j = 0$

This representation at the collocation points which given by:

$$x_j = \frac{1}{2}\left((a+b)-(b-a)\cos\left(\frac{\pi n}{N}\right)\right)\;,\;\; n = 0,1,2,\dots N$$

Using the orthogonality relation for$T_j^*(x)$ where

$$\sum_{j=0}^{N} \theta_j T_i^*(x_n) T_j^*(x_n) = \alpha_i \delta_{ij} \qquad \& \qquad \alpha_i = \begin{cases} \frac{N}{2} & , i \neq 0, N \\ N & , i = 0, N \end{cases}$$

We can compute the coefficient a_j by the relation:

$$a_j = \frac{2\theta_j}{N} \sum_{n=0}^{N} T_i^*(x_n) u(x_n) \tag{7}$$

The first and second derivative for the function$u(x_n)$ at the above collocation points with using expansion in eq. (6) and Chebyshev coefficients which defined by the eq. (7), we can approximate $u^{(1)}(x_i)$ as: $u^{(1)}(x_i) = \sum_{j=0}^{N} \theta_j a_j T_j^{*(1)}(x_j)$

$$u^{(1)}(x_i) = \sum_{n=0}^{N} \left[\frac{2\theta_n}{N} \sum_{j=0}^{N} \theta_j T_j^{*(1)}(x_i) T_j^*(x_n)\right] u(x_n)$$

The expression of differentiation matrix is given by the next equation

$$u^{(1)}(x_i) = \sum_{n=0}^{N} [A_x]_n^i * u(x) \tag{8}$$

where

$$[A_x]_n^i = \frac{2\theta_n}{N} \sum_{j=0}^{N} \theta_j T_j^{*(1)}(x_i) T_j^*(x_n) \ , \quad i, n = 0,1,2, \dots N \tag{9}$$

The first derivative for shifted Chebyshev function as follow in [8]:

$$T_j^{*(1)} = 2j \lambda \sum_{n=0,n+j}^{j-1} C_n T_n^*(x_i) \qquad , \lambda = \frac{2}{b-a}$$

The second differentiation matrix of second order of the function $u(x)$ is defined as the multiplication of first differentiation matrix by itself as: $[A_x]_n^i \times [A_x]_n^i$

Then, $u^{(2)}(x_i) = \sum_{n=0}^{N} \left([A_x]_n^i [A_x]_j^n\right) * u(x) = \sum_{n=0}^{N} \left([B_x]_j^i\right) u(x_j)$

$$B_x = (A_x)^2$$

and the elements of B_x are: $[B_x]_{i.j} = \sum_{n=0}^{N} [A_x]_n^i [A_x]_j^n, i, j = 0,1,2, \dots . N$

We want to find the fractional differentiation matrix based on shifted Chebyshev polynomial of first kind $T_n^*(x)$. From the relation which defined by the eq. (9)

$$[A_x]_n^i = \frac{2C_n}{N}\sum_{j=0}^{N} T_j^{*(1)}(x_i)T_j^*(x_n) \ , i, n = 0,1,2, \dots N$$

which defined the first differentiation matrix which defined by $T_j^{*(1)}(x_i)$, then the differentiation matrix of order (m) is defined by

$$[A_{xxx\dots m}]_n^i = \frac{2C_n}{N}\sum_{j=0}^{N} T_j^{*(m)}(x_i)T_j^*(x_n) \ , i, n = 0,1,2, \dots N$$

Also the fractional derivative of order α is defined by:

$$[A_{xx\dots\alpha x}]_n^i = \frac{2C_n}{N}\sum_{j=0}^{N} T_j^{*(\alpha)}(x_i)T_j^*(x_n) \ , i, n = 0,1,2, \dots N. \tag{10}$$

where$T_j^{*(\alpha)}(x_i)$ is fractional derivative of shifted Chebysheve polynomial of first kind.

4. Computing Fractional derivative of $\mathbf{T_j^*(x_i)}$

In this section, we deduce the fractional derivative for$T_j^*(x)$ using recurrence relation (2) with its expanding as a power of x by using eq. (3) and using definition of Caputo sense for fractional derivative as follows: From recurrence relation, we have

$$T_n^*(x) = 2(2x-1)(n-1)\sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}(n+k-2)!\,2^{2k}}{(n-k-1)!\,2k!}x^k$$

$$-(n-2)\sum_{k=0}^{n-2}\frac{(-1)^{n-k-2}(n+k-3)!\,2^{2k}}{(n-k-2)!\,2k!}x^k$$

then

$$T_n^*(x) = (n-1)\sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}(n+k-2)!\,2^{2k+2}}{(n-k-1)!\,2k!}x^{k+1}$$

$$-(n-1)\sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}(n+k-2)!\,2^{2k+1}}{(n-k-1)!\,2k!}x^{k}$$

$$-(n-2)\sum_{k=0}^{n-1}\frac{(-1)^{n-k-2}(n+k-3)!\,2^{2k+1}}{(n-k-2)!\,2k!}x^{k}$$

After simplifying this equation we reach to

$$T_n^*(x) = -4*(n-1)\sum_{k=0}^{n-1}\frac{(-1)^{n-k-2}(n+k-2)(n+k-3)!\,2^{2k}}{(n-k-1)(n-k-2)!\,2k!}x^{k+1}$$

$$+2*(n-1)\sum_{k=0}^{n-1}\frac{(-1)^{n-k-2}(n+k-2)(n+k-3)!\,2^{2k}}{(n-k-1)(n-k-2)!\,2k!}x^{k}$$

$$-(n-2)\sum_{k=0}^{n-1}\frac{(-1)^{n-k-2}(n+k-3)!\,2^{2k+1}}{(n-k-2)!\,2k!}x^{k}$$

Assuming Caputo definition and applying it for this equation where

$$D^{(\alpha)}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}$$

the fractional derivative of order α which defined for only terms for $k = \lceil\alpha\rceil, \lceil\alpha\rceil+1, \ldots.n$ and terms from $k = 0,1,\ldots.\lceil\alpha\rceil-1$ equal to zero then the summation convert to $\sum_{k=\lceil\alpha\rceil}^{n}\ldots.$

Assume that

$$D^{(\alpha)}T_n^*(x) = -4*(n-1)\sum_{k=\lceil\alpha\rceil}^{n-1}\frac{(-1)^{n-k-2}(n+k-2)(n+k-3)!\,2^{2k}}{(n-k-1)(n-k-2)!\,2k!}D^{\alpha}(x^{k+1})$$

$$+2*(n-1)\sum_{k=\lceil\alpha\rceil}^{n-1}\frac{(-1)^{n-k-2}(n+k-2)(n+k-3)!\,2^{2k}}{(n-k-1)(n-k-2)!\,2k!}D^{\alpha}(x^{k})$$

$$-(n-2)\sum_{k=\lceil\alpha\rceil}^{n-1}\frac{(-1)^{n-k-2}(n+k-3)!\,2^{2k+1}}{(n-k-2)!\,2k!}D^{\alpha}(x^{k})$$

$$D^{(\alpha)}T_n^*(x)=\sum_{k=\lceil\alpha\rceil}^{n-1}\left[\frac{(-1)^{n-k-2}(n+k-3)!2^{2k}}{(n-k-2)!2k!}\right]\left[\frac{2(n+k-2)}{(n-k-1)}+1\right][D^{\alpha}x^{k+1}+D^{\alpha}x^{k}] \tag{11}$$

Then the fractional differentiation matrix of order (α) can be in the form

$$[A_{xxxx..\alpha}]_n=\frac{2C_n}{N}\sum_{j=0}^{N}T_j^{*(\alpha)}(x_i)T_j^*(x_n)\ \ ,i,n=0,1,2,\dots N \tag{12}$$

Finally we can define the expression of fractional differentiation matrix can be written by the equation:

$$f^{(\alpha)}(x_i)=\sum_{j=0}^{N}d_{i,j}^{(\alpha)}f(x_j)$$

where

$$d_{i,j}^{(\alpha)}=\frac{2\theta_j}{N}\sum_{n=0}^{N}\theta_n T_n^*(x_j)T_n^{*(\alpha)}(x_i) \tag{13}$$

The matrix form is:

$$\mathbf{f}^{(\alpha)}=\mathbf{D}^{(\alpha)}\mathbf{f}$$

The elements of the fractional differentiation matrix $\mathbf{D}^{(\alpha)}$ are:

$$d_{i,j}^{(\alpha)}=\frac{2}{N}\begin{bmatrix}0 & 0 & \cdots & 0\\ \frac{1}{2}\mathrm{A}_{1,0}^{(\alpha)} & \mathrm{A}_{1,1}^{(\alpha)} & \cdots & \frac{1}{2}\mathrm{A}_{1,\mathrm{N}}^{(\alpha)}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{2}\mathrm{A}_{\mathrm{N},0}^{(\alpha)} & \mathrm{A}_{\mathrm{N},1}^{(\alpha)} & \cdots & \frac{1}{2}\mathrm{A}_{\mathrm{N},\mathrm{N}}^{(\alpha)}\end{bmatrix}$$

where

$$A_{i,j}^{(\alpha)}=\sum_{n=0}^{N}\theta_n T_n^*(x_j)T_n^{*(\alpha)}(x_i)$$

5. Numerical Examples

We will introduce some of numerical examples of liner fractional differential equation and solve it by fractional differentiation matrix.

Example 1: Solve the O.F.D.E

$$D^{(\alpha)}y(x) + y(x) = \frac{2}{\Gamma(3-\alpha)}x^{2-\alpha} + \frac{1}{\Gamma(2-\alpha)}x^{1-\alpha} + x^2 - x$$

with initial conditions $y(0) = 0$ and the exact solution is

$y(x) = x^2 - x \ \ , x\epsilon[0,1]$.

Firstly the next figures shows the approximate solution and the error at different values of α.

At $\alpha = 0.3 , 0.5 , 0.7 , 0.9$ and $N = 20$

$\alpha = 0.3, N = 20$

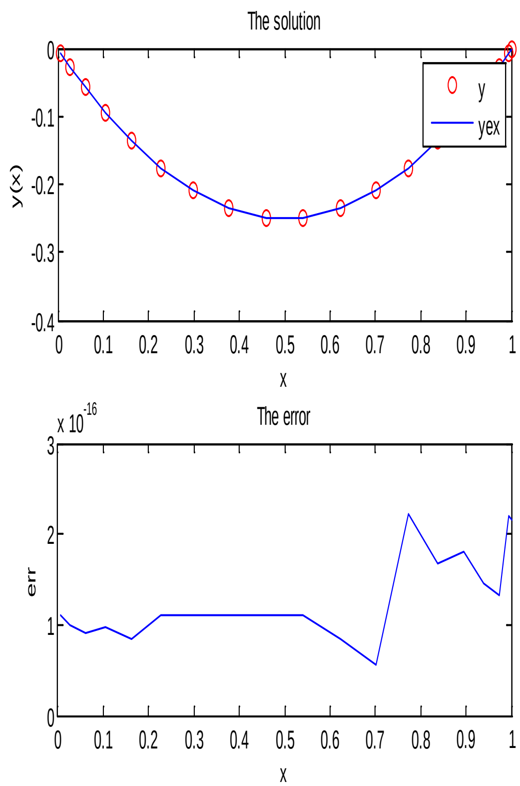


Figure 1

$\alpha = 0.5, N = 20$

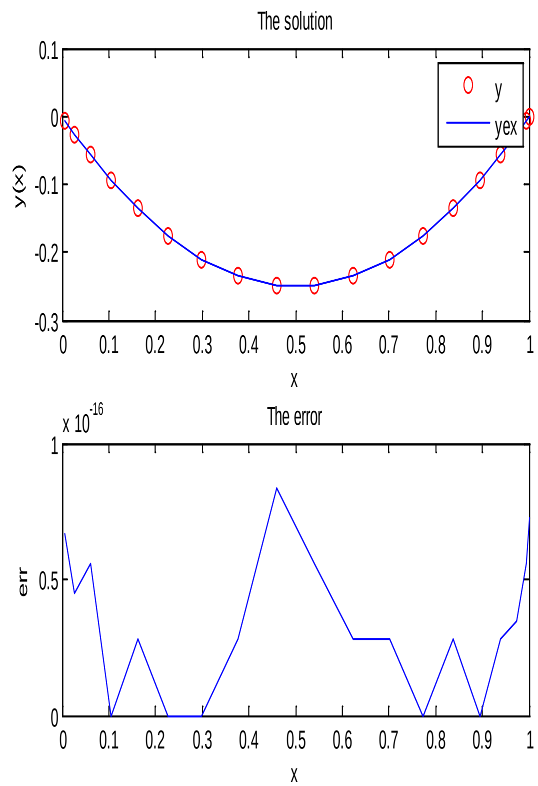


Figure 2

The next table shows the maximum errors at different values of α, N :

Table of maximum errors at different α and **N**

α	N=10	N=15	N=20	N=25
0.1	1.8735e-16	1.1101e-16	1.3878e-16	1.3878e-16
0.3	1.6653e-16	1.6653e-16	2.2204e-16	2.0817e-16
0.5	9.7145e-17	1.1796e-16	8.3267e-17	1.4573e-16
0.7	1.4572e-16	1.1102e-16	2.0123e-16	1.5959e-16
0.9	1.6653e-16	1.6653e-16	8.9338e-17	1.9429e-16

Example 2:

Solve the following D.E:

$$D^{(\alpha)}y(x) = x^2 + \frac{2}{\Gamma(3-\alpha)}x^{2-\alpha} + y(x)$$

and the exact solution is

$$y(x) = x^2$$

with initial condition

$$y(0) = 0, x \in [0,1]$$

the following figures for different values of N at $\alpha = 0.5$

N=10	N=15

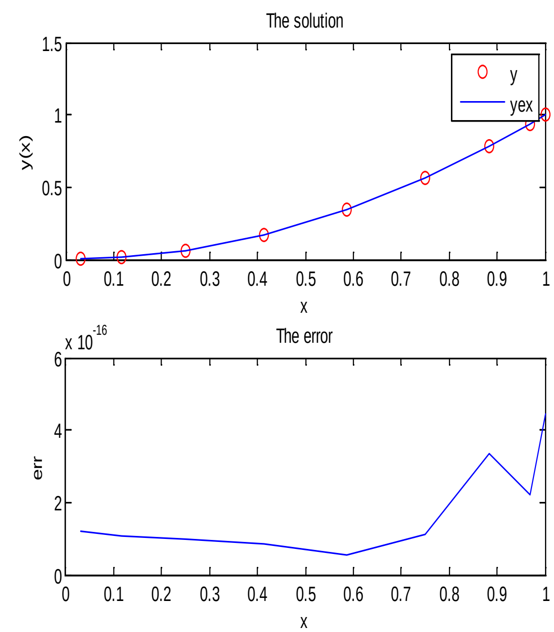
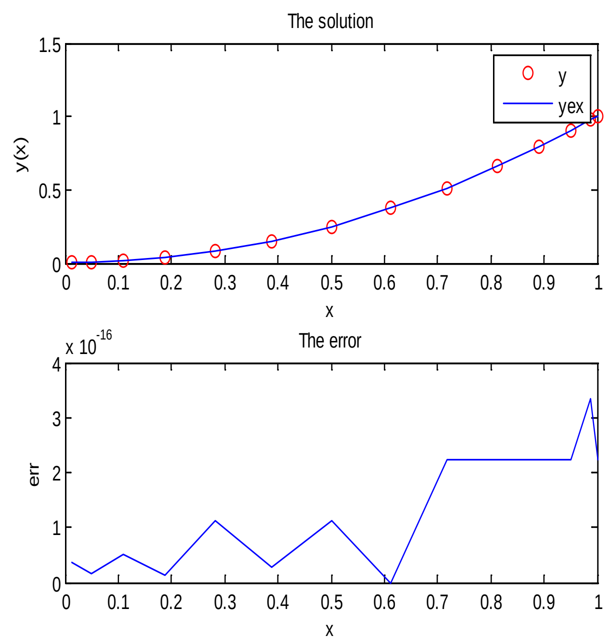

	

Figure 3	Figure 4

N=20

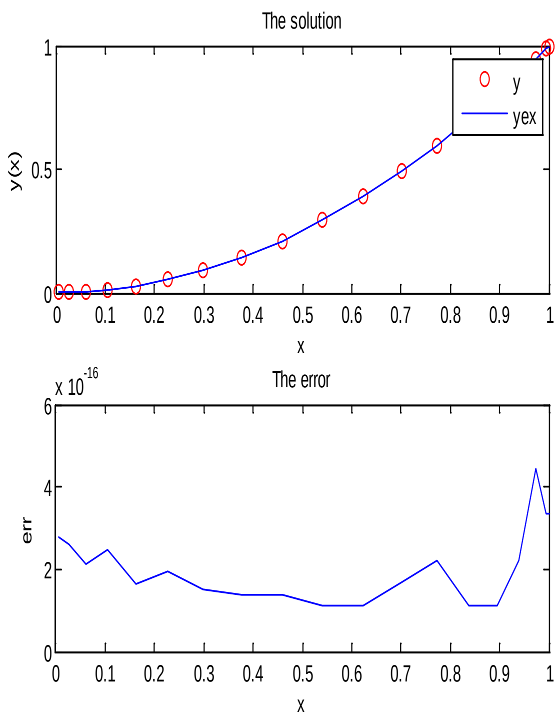


Figure 5

N=25

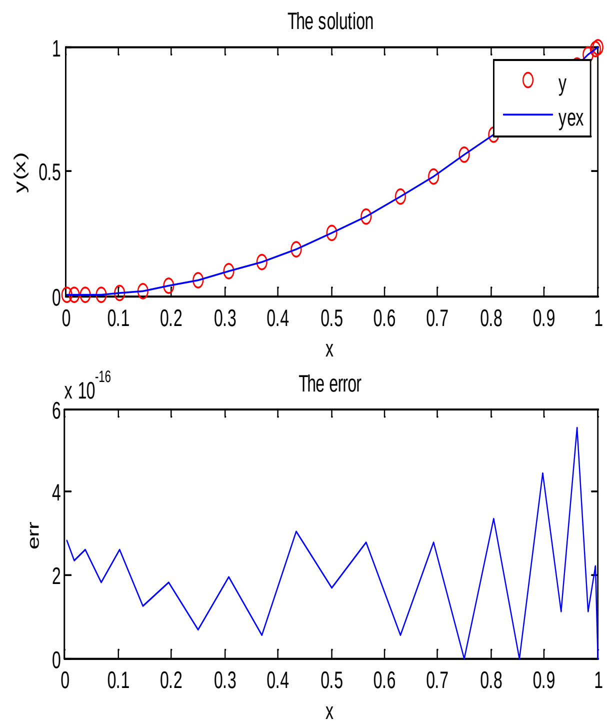


Figure 6

The next table shows the maximum errors at different values of α, N:

α	N=10	N=15	N=20	N=25
0.3	4.4409e-016	6.6613e-016	7.7716e-016	6.6763e-016
0.5	4.4409e-016	3.3307e-016	4.4409e-016	5.5511e-016
0.7	9.6570e-016	6.6613e-016	5.5511e-016	3.9145e-016
0.9	6.2699e-016	1.1858e-015	6.2815e-016	3.8131e-016

Example 3:

Solving of (Bagley-Torvik equation)

$$y^{(2)}(x) + D^{\frac{3}{2}}y(x) + y(x) = g(x) \quad , x \in [0,5]$$

and,

$$g(x) = x^2 + 4\sqrt{x/\pi} + 2$$

with boundary conditions:

$$y(0) = 0 \quad , y(5) = 25$$

The following figures show the solution and absolute errors at different number of points:

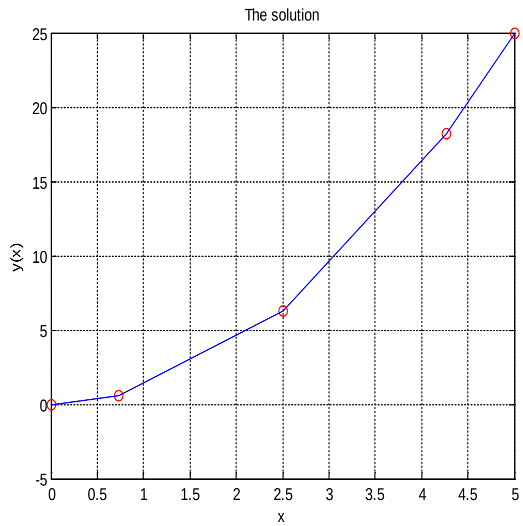
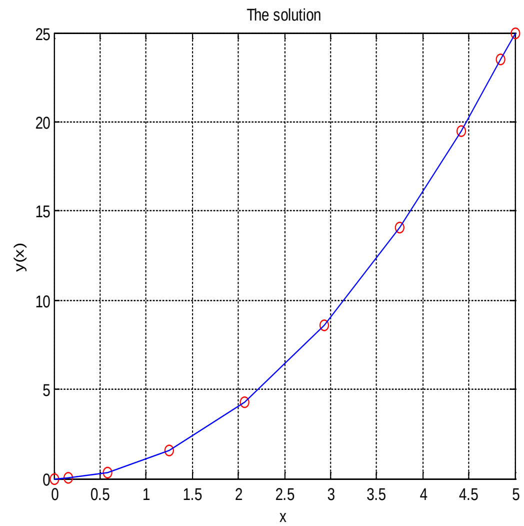
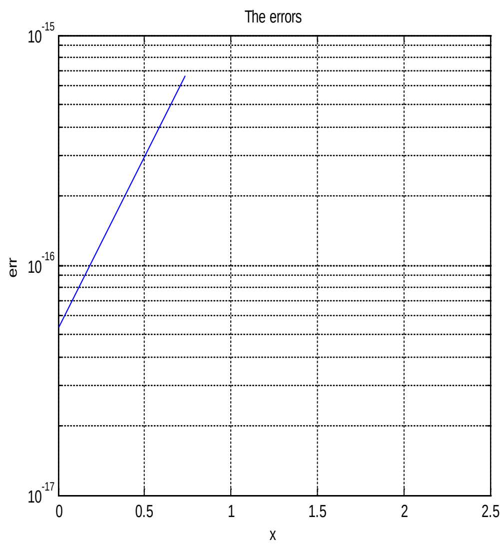
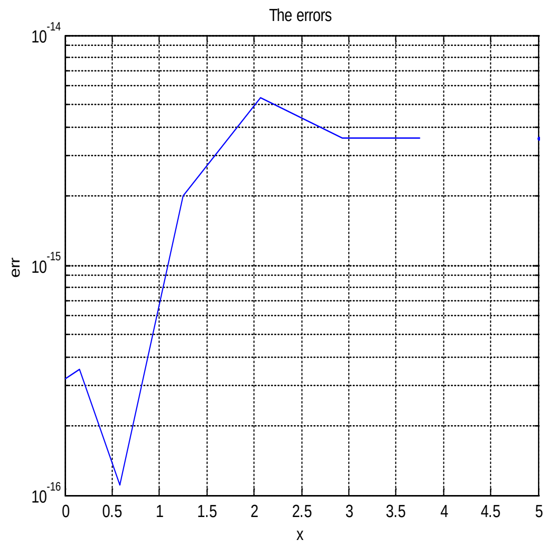
$N = 5$	$N = 10$
	
	

Figure 7	Figure 8

$N = 15$	$N = 20$

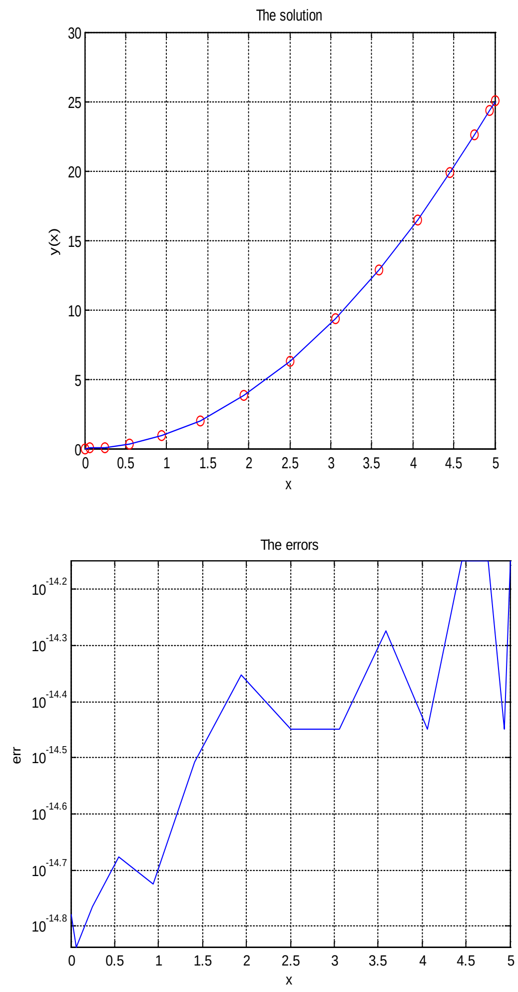


Figure 9

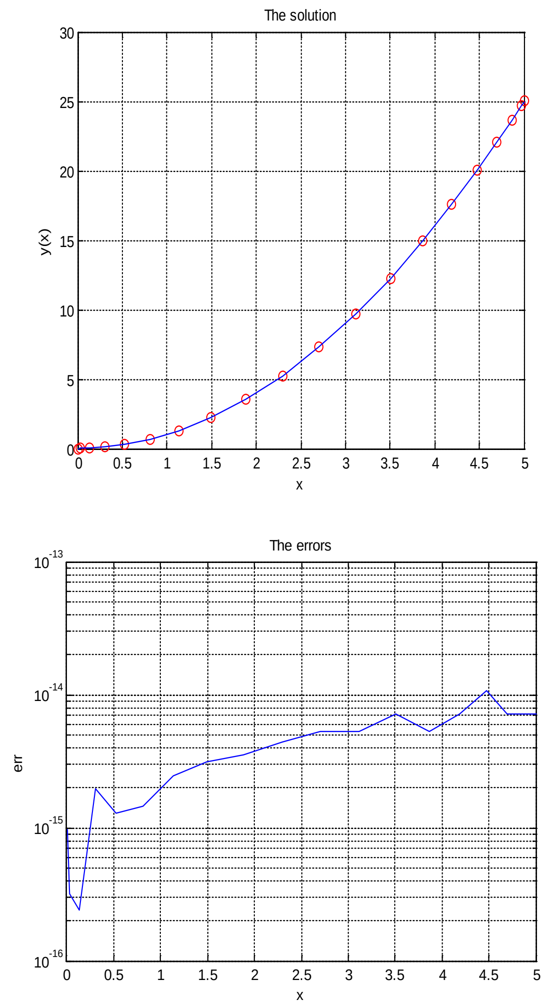


Figure 10

6. Conclusion:

In this paper, the main goal is to solve Linear Fractional Differential Equations (L.F.D.E) depends on Fractional Differentiation Matrix (F.D.M). We derived a new method to calculate Fractional Differentiation Matrix from recurrence relation of shifted Chebyshev polynomial. We proposed numerical algorithm for solving (L.F.D.E) using GL points of shifted Chebyshev polynomial and approximated the solution using spectral method. We solve some examples and compared the approximation solution by the exact solution and compute the maximum error to know the convergence of our solution.

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