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A Theoretical Perspective of Convex Optimization and Variational Inequality Problems

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ABSTRACT
Most real life problems can be formulated into optimization problems. But, some of such problems are usually complex and as such very difficult to solve by standard optimization methods. Research has also shown that such problems can be transcribed as variational inequality problems which are easier to solve yet maintaining the optimality of the original problem. Thus, in this research, we took a theoretical survey of convex optimization and variational inequality problems.

Key Words:

1. Introduction

A convex program is an optimization problem where we seek the minimum of a convex function over a convex set (Dantzig,1963). Among these, the most commonly used is the *linear program* (LP), an optimization problem with linear objective and linear inequality constraints:

$$\text{minimize } c^T x$$
$$\text{subject to } a^T_i\, x \leq b_i, \qquad i = 1, \dots L; \qquad (1)$$

where the optimization variable is the vector x, and a_i, b_i, and c are problem parameters. Dantzig introduced the simplex method in 1948 (Dantzig,1963) which led to the widespread use of linear programming.

In recent years, convex optimization has become a computational tool of central importance in nearly all fields of study due to its ability to solve very large practical problems reliably and efficiently (Nemirovskii and Scheinberg, 1996).

Convex optimization can be described as a fusion of three disciplines: optimization, convex analysis, and numerical computation. It has recently become a tool of central importance in engineering, enabling the solution of very large, practical engineering problems reliably and efficiently (Dantzig, 1963). In some sense, convex optimization is providing new indispensable computational tools today, which naturally extend our ability to solve problems such as least squares and linear programming to a much larger and richer class of problems (Sussmann and Willems, 1997).

Our ability to solve these new types of problems comes from recent breakthroughs in algorithms for solving convex optimization problems, coupled with the dramatic improvements in computing power, both of which have happened only in the past decade or so. Today, new applications of convex optimization are constantly being reported from almost every area of engineering, including: control, signal processing, networks, circuit design, communication, information theory, computer science, operations research, economics, statistics, structural design e.t.c (Gelfand and Fomin, 1963)

Convex functions appear in many important problems in pure and applied Mathematics. Many literatures on convex analysis and convex function are usually defined only on convex domains. Moreover, Convex functions are often extended to the whole linear space by setting the value to be $+\infty$ out of the convex domain, so that the extended function is still convex. While such treatment is preferred in some applications, (e.g. Optimization, convex programming) and in some theories, it may not be the desirable thing to do if the problem is more analytic. The extension of convex functions has wide ranging applications in geometric analysis, non linear dynamics, quantum computing and Economics.

Variational Inequalities, formulated, between the end of 60' and the beginning of 70' of previous century by the Italian Mathematician G. Stampacchia provide a very general framework for a wide range of mathematical problems. Moreover, they have shown to be important models in the study of equilibrium problems, in the engineering sciences (equilibrium problems in a traffic network) and in the economic sciences (oligopolistic market equilibrium problems).

2. Purpose of the Study

The goal of this research is to give a theoretical overview of the basic concepts of convex sets, convex functions, convex optimization problems, variational inequality problems and to show the relationship between convex optimization and variational inequality problems.

3. Review of Optimization Problem and Variational Inequality Problems

While interior-point methods have been discussed for at least thirty years (see, *e.g.*, Fiacco and McCormick, 1968), the current development was launched in 1984 by Karmarkar (Karmarkar, 1984) with an algorithm for linear programming that was more efficient than the simplex method. A large body of literature now exists on interior-point methods for linear programming, and a number of books have been written (*e.g.*, Wright (1997) and Vanderbei (1997).
Numerous implementations of efficient interior-point LP solvers are now available [Czyzyk, Mehrotra and Write (1997), Vanderbei (1992), Zhang (1994)]. Of themselves, these developments did not change the traditional view that held nonlinear optimization problems to be fundamentally more difficult than linear ones.

This would later be replaced with the understanding that the fundamental division in complexity lies in convex versus non-convex programming. Some ten years after Karmarkar presented his algorithm, Nesterov and Nemirovsky (1994) noted that interior-point methods can be extended to handle many nonlinear convex optimization problems. Interior-point methods for nonlinear convex optimization problems have many of the same characteristics of the methods for linear programming. They have polynomial-time worst case complexity, and are extremely efficient in practice.

Current algorithms can solve problems with hundreds of variables and constraints in times measured in seconds, or at most a few minutes, on a personal computer. If problem structure, such as sparsity, is exploited, much larger problems can be handled. The course notes by Boyd and Vandenberghe (1997) give an accessible introduction to the field and describe a large number of applications.

A great amount of work has recently been done on some classes of nonlinear convex programs, both in terms of algorithms and applications. These include semi definite programming (SDP) Boyd and Vandenberghe (1996) and second-order cone programming (SOCP) Lobo, Vandenberghe, Boyd and Lebret (1998).
Several developments in the field of convex optimization have stimulated new and various interest in the topic. These new method allow us to solve convex optimization problems, such as semi-definite programs and second order convex programs, almost as easily as linear programs. Boyd as well discovered that convex optimization problems beyond least –squares and linear programming problems are more prevalent in practice than was previously thought. Since 1990 many applications have been discovered in area such as automatic control, system estimation and signal processing, electronic Circuit, finance, statistics, data analysis and modeling etc. Boyd interest was to develop the skills and background needed to recognized, formulate and solve convex optimization.

Lagrange developed a powerful characterization of local optima of equality constrained optimization problem in terms of the behavior of the objective function f and the constraint function g. At these points, he said that if $f\colon k \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g_i\colon \longrightarrow \mathbb{R}^k$ are C^1 functions, $i = 1, . . . , k$ and suppose x^* is local maximum or minimum of f on the set
$k = U \ \cap \{x\colon g_i(x) = k\}, U \subset \mathbb{R}^n$ is open and if the rank
$l\big(\nabla g(x^*)\big) = 0$ then we can find a vector
$\lambda^* = \lambda_1^*, . . . , \lambda_k^* \in \mathbb{R}^k$ such that $\nabla f(x^*) + \sum_{i=1}^{k} \lambda_1^* \, g_i\,(x^*) = 0$

Karush–Kuhn–Tucker was mostly interested in the optima of inequality constrained optimization problems. He discovery that Lagrange theorem can be extended to inequality-constrained problems. Kuhn –Tucker extend the theorem of Lagrange to convexity, he added that if f is concave C^1 a function mapping u into $\mathbb{R}$ open and convex for $i = 1, \ . . . , l$ and if $h_i\colon u \longrightarrow \mathbb{R}$ is also concave C^1 function and if there exists $x \in l$ such that, $h_i(x) \geq 0$, $i = 1, . . . l$. then x^* maximizes f over $k = \{x \in u\colon h_i(x) \geq 0, i = 1, . \ . \ . l\}$ if and only if there is $\lambda \in \mathbb{R}^k$ such that Kuhn-Tucker first order conditions of the theorem of Kuhn-Tucker holds. Kuhn-Tucker first order conditions are both necessary and sufficient to identify optima of a convex inequality problem.

Variational inequalities, formulated, between the end of 60' and the beginning of 70' of previous century by the Italian mathematician G. Stampacchia provide a very general framework for a wide range of mathematical problems among which, rather under general hypotheses, optimization ones. Moreover, they have shown to be important models in the study of equilibrium problems, in the engineering sciences (equilibrium problems in a traffic network) and in the economic sciences (oligopolistic market equilibrium problems). Such problems, in fact, play a crucial role in the theory of complex systems and for this reason, recently, have been presented many variational formulations of these problems.

Since the appearance of the papers of Minty and ,Hartman and Stampacchia and Browder , the theory of monotone (nonlinear) operators in general and the variational inequality in particular have generated a tremendous interest amongst mathematicians. This is because of the wide applicability of the variational inequality in nonlinear elliptic boundary value problems, obstacle problems, Complementarity problems, mathematical programming, mathematical economics and in many other areas.

Kamaradian (1972) showed that the problem of Complementarity can be reduced to the variational inequality, while the relationship between mathematical programming and the variational inequality was shown by Mancino and Stampacchia (1972), between the variational inequality and convex functions by Rockafeller (1970) and Moreau (1966).

4 Unconstrained optimization problems

This is represented mathematically as

$$(P) \quad Min \quad f(x)$$
$$\text{s.t} \quad u \in X,$$

Where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n, f(x): \mathbb{R}^n \longrightarrow \mathbb{R}$, and X is an open set (usually X=$\mathbb{R}^n$).

A necessary condition for local optimality is a statement of the form. "If $\bar{x}$ is a local minimum of (P), then $\bar{x}$ must satisfy the optimality condition" such a condition helps us identify all candidates for local optima.

Theorem 1: Suppose that $f(x)$ is differentiable at $\bar{x}$. If there is a vector $d \ni \nabla f(\bar{x})^t d < 0$, $\forall \lambda > 0$ and sufficiently small, $f(\bar{x} + \lambda d) < f(\bar{x})$, and hence d is a descent direction of $f(x)$ at $\bar{x}$.

Proof

We have $f(\bar{x} + \lambda d) = f(\bar{x}) + \lambda \nabla f(\bar{x})^t d + \lambda \|d\| \propto (\bar{x}, \lambda d)$

Where $\propto (\bar{x} + \lambda d) \longrightarrow 0$ as $\lambda \longrightarrow 0$. rearranging

$$\frac{f(\bar{x} + \lambda d) - f(\bar{x})}{\lambda} = \nabla f(\bar{x})^t d + \|d\| \propto (\bar{x}, \lambda d)$$

Since $\nabla f(\bar{x})^t d < 0$ and $\propto (\bar{x}, \lambda d) \longrightarrow 0$ as $\lambda \longrightarrow 0$,

$f(\bar{x} + \lambda d) - f(\bar{x}) < 0 \quad \forall \lambda > 0$ sufficiently small.

4.1 First and Second Order Necessary Conditions for Optimality

Theorem 2: (1st-order optimality condition) Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, be differentiable at a point $\bar{x} \in \mathbb{R}^n$. If $\bar{x}$ is local solution to the problem (P), then $\nabla f(\bar{x}) = 0$

Proof:

From the definition of derivative we have that

$$f(x) = \nabla f(\bar{x}) + \nabla f(\bar{x})^t (x - \bar{x}) + 0 (\|x - \bar{x}\|)$$

Where $\lim_{x \longrightarrow \bar{x}} \frac{0 (\|x-\bar{x}\|)}{\|x-\bar{x}\|} = 0$ let $x = \bar{x} - t\nabla f(\bar{x})$.

Then,

$$0 \leq \frac{f(x - t\nabla f(\bar{x})) - f(\bar{x})}{t} = \|\nabla f(\bar{x})\|_+^2 \, 0 \frac{(t \|\nabla f(\bar{x})\|)}{t}$$

Taking the limit as $t \downarrow 0$ we obtain $0 \leq -\|\nabla f(\bar{x})\|^2 \leq 0$

Hence $\nabla f(\bar{x}) = 0$

Theorem 3: (second-order optimality conditions)

Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be twice differentiable at the point $\bar{x} \in \mathbb{R}^n$.

(1) (necessity) if $\bar{x}$ is a local solution to the problem P, then

$\nabla f(\bar{x}) = 0$ and $\nabla^2 f(\bar{x})$ is positive semi-definite.

(2) (Sufficiency) if $\nabla f(\bar{x}) = 0$ and $\nabla^2 f(\bar{x})$ is positive definite, then there is an $\propto > 0 \ni f(x) \geq f(\bar{x}) + \propto \|(x-\bar{x})\|^2 \; \forall x$ near $\bar{x}$.

Proof (1) We make use of second-order Taylor series expansion

$f(x) = f(\bar{x}) + \nabla f(\bar{x})^T (x-\bar{x}) + \frac{1}{2} f(x-\bar{x})^T \nabla^2 f(\bar{x})(x-\bar{x}) + 0(\|x-\bar{x}\|^2)$ …. (2)

Given $d \in \mathbb{R}^n$ and $t > 0$ set $= \bar{x} + td$, plugging this into (2) we find that $0 \leq \frac{f(x+td)-f(\bar{x})}{t^2} = \frac{1}{2} d^T \nabla^2 f(\bar{x}) d + \frac{0(t^2)}{t^2}$

Since $\nabla f(\bar{x}) = 0$ by taking the limit as $t \to 0$ we get that $0 \leq d^T \nabla^2 f(\bar{x}) d$.

Now since d was chosen arbitrarily we have that $\nabla^2 f(\bar{x})$ is positive semi-definite.

A sufficient condition for local optimality is a statement of the form: “if $\bar{x}$ satisfies required condition, then $\bar{x}$ is a local minimum of (P)." Such a condition allows us to automatically declare that $\bar{x}$ is indeed a local minimum.

(2) From (**2**) we have that

$$\frac{f(x)-f(\bar{x})}{\|x-\bar{x}\|^2} = \frac{1}{2} \frac{(x-\bar{x})^T}{\|x-\bar{x}\|} \nabla^2 f(\bar{x}) \frac{(x-\bar{x})}{\|x-\bar{x}\|} + \frac{0(\|x-\bar{x}\|^2)}{\|x-\bar{x}\|^2} \qquad (3)$$

If $\lambda > 0$ is the smallest eigenvalue of $\nabla^2 f(\bar{x})$, choose $\varepsilon > 0$ so that

$$\left|\frac{0(\|x-\bar{x}\|^2)}{\|x-\bar{x}\|^2}\right| \leq \frac{\lambda}{4} \qquad (4)$$

Wherever $\|x-\bar{x}\| < \varepsilon$. Then $\forall \; \|x-\bar{x}\|^2 < \varepsilon$ we have from (3) and (4) that $\frac{f(x)-f(\bar{x})}{\|x-\bar{x}\|^2} \geq \frac{1}{2}\lambda + \frac{0(\|x-\bar{x}\|^2)}{\|x-\bar{x}\|^2} \geq \frac{1}{4}\lambda$

Consequently, if we set $\propto = \frac{1}{4}\lambda$, then

$f(x) \geq f(\bar{x}) + \propto \|x-\bar{x}\|^2$

Wherever $\|x-\bar{x}\| < \varepsilon$

The following result shows that the first-order condition for unconstrained optima (i.e, the condition that $\nabla f(x) = 0$) is both necessary and sufficient to identify global unconstrained maxima, when such maxima exist.

Theorem 4 *Let $D \subset \mathbf{R}^n$ be convex, and f:* D $\rightarrow$ **R** *be a concave and differentiable function on D. Then, x is an unconstrained maximum of f on D if and only if $\nabla f(x) = 0$.*

Proof

The reverse implication which requires the concavity of *f* is actually an immediate consequence of the gradient inequality which says, suppose *x* and *y are any two points in D*, by the concavity of *f*, we must have

$$f(y) - f(x) \leq \nabla f(x)(y-x).$$

if $\nabla f(x) = 0$, the right-hand side of this equation is also zero, so the equation states precisely that

$$f(x) \geq f(y).$$

Since $y \in D$ was arbitrary, *x* is a global maximum of *f* on D.

4.2 Gradients and Hessians

Let $f(x)$: $X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is open.
$f(x)$ is differentiable at $\bar{x} \in X$, if $\exists$ a vector $\nabla f(\bar{x}) \ni$ for each $x \in X$,
$f(x) = f(\bar{x}) + \nabla f(\bar{x})^t(x - \bar{x}) + \|x - \bar{x}\| \propto (\bar{x}, x - \bar{x})$,
and $\lim_{y \to 0} \propto (\bar{x}, y) = 0$, $f(x)$ is differentiable $on\ X$
If $f(x)$ is differentiable $\forall$ $\bar{x} \in X$. The gradient vector is the vector of partial derivatives.

$$\nabla f(\bar{x}) = \left(\frac{\partial f(\bar{x})}{\partial x_1}, \dots, \frac{\partial f(\bar{x})}{\partial x_n}\right)$$

The directional derivative of $f(x)\ at\ \bar{x}$ in the direction d is:

$$\lim_{\lambda \to 0} \frac{f(\bar{x} + \lambda d) - f(\bar{x})}{\lambda} = \nabla f(\bar{x})^{\mathrm{T}} d$$

The $function\ f(x)$ is twice differentiable at $\bar{x} \in X$ if $\exists$ a vector $\nabla f(\bar{x})$ and an $n \times n$ symmetric matrix $H(x) \ni$ for each $x \in X$.

$$f(x) = f(\bar{x}) + \nabla f(\bar{x})^t (x - \bar{x}) + \frac{1}{2} (x - \bar{x})^t H(x)(x - \bar{x}) + \|x - \bar{x}\|^2 (x, x - \bar{x})$$

and $\lim_{y \longrightarrow 0} (\bar{x}, y) = 0$, $f(x)$ is twice differentiable on $X\ if\ f(x)$ is twice differentiable $\forall \bar{x} \in X$.
The Hessian is the matrix of second partial derivatives:

$$H(\bar{x})_{ij} = \frac{\partial^2 f(\bar{x})}{\partial_{xi} \partial_{xj}}$$

4.3 Existence and Uniqueness Theorem

The existence and uniqueness of solution to the general optimization problem is stated below.

Theorem 5. (Weierstrass Theorem)

Let $k \subset \mathbb{R}^n$ be compact and nonempty and let $f: k \longrightarrow \mathbb{R}$ be a continuous function on k, then f has a global minimum in k, that is $\exists$ at least one point $x^* \subset k$ such that $f(x^*) = \underset{x \in k}{Inf}\ f(x)$

Proof: Let (x_n) be a minimizing sequence of f in k, that is $\lim_{n \longrightarrow \infty} f(x_n) = \underset{n \in k}{Inf}\ f(x)$
Since k is compact, it implies that k is bounded and closed, and since (x_n) is a sequence of f in k, (x_n) is bounded, By Bolzano Weierstrass theorem, (x_n) has a subsequence (x_{nk}) which converges to some point $x^* \in \mathbb{R}^n$. Since k is closed, $x^* \subset k$. Using the fact that f is continuous at x^*, it follows that

$$\lim_{k \longrightarrow \infty} f(x_{nk}) = f(x^*) \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . (5)$$

Since $f(x_{nk})$ is a subsequence of $f(x_n)$; we have

$$\lim_{k \longrightarrow \infty} f(x_n) = \underset{n \in k}{Inf} f(x^*), \quad . \quad . \quad . \qquad 6$$

From equation (5) and (6) and by the uniqueness of the limit, we have $f(x^*) = \underset{x \in k}{Inf}\ f(x)$
Therefore x^* is a global minimum of f in k.
In a case where k is closed but not necessarily bounded and f is coercive, we still conclude that solution exist, we discuss the proposition below:
Proposition (**1**) Let k be a nonempty closed subset of $\mathbb{R}^n$. If f is coercive, then;

(a) The function f is bounded below on k.
(b) Any minimizing sequence of f in k is bounded.

Proof (a)

Suppose that f is not bounded below on k. Then for all $n \in \mathbb{N}$ there exist $x_n \in k$ such that $f(x_n) < -n,$ we get a sequence (x_n) in k satisfying $f(x_n) < -n, \ \forall n \in \mathbb{N}$ . . . ………………………………. (7)

Because f is coercive, this sequence must be bounded otherwise it has a subsequence (x_{nk}) such that

$$\lim_{k \to \infty} \|x_{nk}\| = +\infty$$

From (6), It follows that

$$\lim_{k \to \infty} (x_{nk}) = -\infty$$

By Uniqueness of limit

$$\lim_{k \to \infty} (x_{nk}) = -\infty$$

Which is a contradiction, and so x_n is bounded by Bolzano-Weierstrass theorem, there exists a subsequence(x_{nk}) of (x_n) that converges to $x^* \subset k$.

Using continuity of f at x^*, we have

$$\lim_{k \to \infty} f(x_{nk}) = f(x^*)$$

From (6) we get $\lim_{k \to \infty} f(x_{nk}) = -\infty$

Therefore, by uniqueness of limit, it follows that $f(x^*) = -\infty$.

Which is a contradiction that $\lim_{k \to \infty} f(x_{nk}) = +\infty,$

So f is bounded below on k.

Proof $(\boldsymbol{b})$ let (x_n) be a minimizing sequence of f in k, that is

$$\lim_{n \to \infty} f(x_n) = \inf_{x \in k} f(x) \quad . \quad . \quad . \qquad (8)$$

We want to show that (x_n) is bounded. By contradiction, assume it is not bounded, then there exist a subsequence (x_{nk}) of (x_n) such that $\lim_{k \to \infty} \|x_{nk}\| = +\infty$

Since f is coercive, we have

$$\lim_{k \to \infty} f(x_{nk}) = +\infty \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \qquad (9)$$

Using (3.7) we have

$$\lim_{k \Rightarrow \infty} f(x_{nk}) = \inf_{x \in k} f(x)$$

$$\Rightarrow \inf f(x) = +\infty$$

Which is a contradiction that f is bounded below on k.

Theorem 6: Let k be a non-empty closed subset of $\mathbb{R}^n$ (not necessarily bounded). Assume that f is continuous on some open set containing k. Then f has a global minimum on k, that is there exists at least one point $x^* \in k$ such that $f(x^*) = \min_{x \in k} f(x^*)$

Proof: Let (x_n) be a minimizing sequence of f in k. By proposition (1), (x_n) is bounded, and by Bolzano Weierstrass Theorem, (x_n) has a subsequence (x_{nk}) which converge to some points $x^* \in k$, we have

$$\min_{k \longrightarrow \infty} f(x_{nk}) = f(x^*) \quad (10)$$

More so, since $f(x_{nk})$ is a subsequence of $f(x_n)$, we have

$$\lim_{k \longrightarrow \infty} f(x_{nk}) = \inf_{x \in k} (x) \quad (11)$$

Using equation (10) and (11) and by the Uniqueness of the limit, we have $f(x^*) = inf\ f(x)$.

Theorem 4.1: Suppose $\boldsymbol{f}: \mathbb{R} \to \mathbb{R}$ is C'. Then f is convex if and only if $f(y) + (x-y)f'(y) \leq f(x) \quad \forall\, x, y \in \mathbb{R}$. For strict convexity replace $\leq by \;<\; whenever\;\; x \neq y$.

Proof: Suppose $f(y) + (x-y)f'(y) \leq f(x) \quad \forall x, y \in \mathbb{R}$. we have to show that this implies $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \forall \lambda \in [0,1]$

And $x, y \in \mathbb{R}$. to this end choose any two points $x, y \in \mathbb{R}$ $and\ any$ $\forall \lambda \in [0,1]$.

We write $z = \lambda x + (1-\lambda)y$. From the initial assumption it follows that

$$f(z) + (x-z)f'(z) \leq f(x) \quad . \;.\; . \qquad (12)$$

$$f(z) + (y-z)f'(z) \leq f(y) \quad . \;.\; . \qquad (13)$$

Multiplying (12) by λ and (13) by $(1-\lambda)$ and adding gives

$f(z) + f'(z)[\lambda x + (1-\lambda)y - z] \leq \lambda f(x) + (1-\lambda)f(y)$.

As $z = \lambda x + (1-\lambda)y$ $then$ $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$

Since this is true $\forall \lambda \in [0,1]$ and $\forall x, y \in \mathbb{R}$ then f is convex.

$\Longleftarrow$ Now suppose f is convex. We need to show that this implies $f(y) + (x-y)f'(y) \leq f(x) \quad \forall\, x, y \in \mathbb{R}$. if $x = y$ the implication follows immediately, so suppose this is not the case. By convexity

$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \forall \lambda \in [0,1]$ and $\forall x, y \in \mathbb{R}$ or rearranging terms;

$f[y + \lambda(x-y)] - f(y) \leq \lambda\big(f(x) - f(y)\big)$ $\forall \lambda \in [0,1]$ and $\forall x, y \in \mathbb{R}$. Assuming λ differ from 0 we can get

$$(x-y)\frac{f\big(y + \lambda(x-y)\big) - f(y)}{\lambda(x-y)} \leq f(x) - f(y)$$

$\forall \lambda \in [0,1]$ and $\forall x, y \in \mathbb{R}$. substituting $\lambda(x-y)\ by\ \Delta x$

$$(x-y)\frac{f[y + \Delta x] - f(y)}{\Delta x} \leq f(x) - f(y)$$

Taking the limit when $\lambda \to 0$

$(x-y)f(y) \leq f(x) - f(y)$ or $f(y) + (x-y)f(y) \leq 0 \quad \forall x, y \in \mathbb{R}$.

Theorem 4.2: Suppose $f: \mathbb{R} \to \mathbb{R}$ is C^2. Then f is convex if and only if $f''(x) \geq 0 \quad \forall x \in \mathbb{R}$. moreover, if $f''(x) > 0 \quad \forall x \in \mathbb{R}$, then, f is strictly convex.

Concave function can be defined in a similar way. In particular $f: \mathbb{R} \to \mathbb{R}$ $is\ a$ concave function if and only if $-f: \mathbb{R} \to \mathbb{R}$ $is\ convex$

There is an alternative characterization of concave and convex functions that does not require differentiability. It is in terms of the convexity of certain sets associated with the graphs of the corresponding functions. For $f: \mathbb{R} \to \mathbb{R}$ the epigraph of f, EG_f is

$EG_f = \{(y, x) \in \mathbb{R}^2 : y \geq f(x) \quad \text{for some } x \in \mathbb{R}\}$
Visually the epigraph of for some f is the set of point in $\mathbb{R}^2$ which lies above the graph.
The next theorem characterized convex functions through their epigraph.

Theorem 4.3: A function for some $f: \mathbb{R} \to \mathbb{R}$ is convex if and only if it's epigraph is a convex set.

Proof: suppose EG_f is a convex set and choose and $x, x' \in \mathbb{R}$, let $f(x) = y$ and $f(x') = y'$. Then $(x, y) \in EG_f$ and $(x', y') \in EG_f$. Since EG_f is a convex set, we have

$$\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y' \quad \in EG_f \ \ \forall \lambda \in [0,1].$$

That is $f(\lambda x + (1-\lambda)x') \leq \lambda y + (1-\lambda)y' \quad \forall \lambda \in [0,1]$.

Since this is true for all $x, x' \in \mathbb{R}$ then f is convex.

$\Leftarrow$ Suppose f is convex and let $(x, y) \in EG_f$ $an(x', y') \in EG_f$; that is $f(x) \geq y\ and\ f(x') = y'$. Multiplying the first inequality by λ, the second one by $(1-\lambda)$, $adding$

$\lambda f(x) + (1-\lambda)f(x') \leq \lambda y + (1-\lambda)y' \quad \forall \lambda \in [0,1]$.But since f is convex $f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda)f(x') \quad \forall \lambda \in [0,1]$.

Hence $f(\lambda x + (1-\lambda)x') \leq \lambda y + (1-\lambda)y' \quad \forall \lambda \in [0,1]$.

Then $[\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y'] \in EG_f, \ \forall \lambda \in [0,1]$.

In similar way, the convexity of a function can characterized through its epigraph.

Theorem 4.4: Let $f: \mathbb{R}^n \to \mathbb{R}$ $be\ strictly$ convex. If f $has\ a$ global minimize, then it is unique.

Proof: Let $x^1 and\ x^2$ be distinct global minimizers of f. Then, for $\lambda \in (0,1)$, $f\big((1-\lambda)x^1 + \lambda x^2\big) < (1-\lambda)f(\overline{x'}) + \lambda f(x^2) = f(x^1)$ which contradicts the assumption that x' is a global minimizer.

Proposition 4.6: For a function $f: I \to \mathbb{R}$ on an interval $I \subset \mathbb{R}$, convexity is equivalent to the properties that $\frac{f(y)-f(x)}{y-x} \le \frac{f(z)-f(x)}{z-x}$ whenever $y < x < z$ . . . (14)

However, (14) yields the well known fact that convex functions are every where left and right differentiable. Indeed, the left hand side of (14) equals $S(x, y)$ hence is an increasing function of y; *and since it is bounded* from above from the right-hand side of (14) its supremum is necessarily a limit for $y \to x^-$. By definition this is the left derivative $f'_-(x)\ of\ f\ at\ x, and\ for\ z > x$

It fulfils $f'_-(x) \le \frac{f(z)-f(x)}{z-x}$ . . . (15)

Repeating the argument, it follows from this inequality that the right-hand side has $(its\ infimum\ as\ )$ a limit $f'_+(x)$ for $z \to x^+$ and that $f'_-(x) \le f'_+(x)$ . . . (16)

This completes the prove.

Proposition 4.7: A convex function $f: I \to \mathbb{R}$ on an interval $I \subset \mathbb{R}$ is differentiable both from the left and right at every interior point $x\ in\ I$ and (16) holds.

As an addendum to this proposition, it moreover follows from the proof of the existence of the one-sided derivatives that

$$f(z) \ge f(x) + f'_+(x)(z-x) \quad \forall\, z \ge x \in I \quad . \quad . \quad . (17)$$

$$f(y) \ge f(x) + f'_+(x)(y-x) \quad \forall\, y \ge x \in I \quad . \quad . \quad . (18)$$

Moreover, since $\frac{(f(z)-f(x))}{(z-x)}\ equals f'_+(x) + o(1)\ and\ similarly\ for\ f'_-(y)$,

$it\ holds\ for\ y \le x \le z$ such that

$$f(z) \ge f(x) + f'_+(x)(z-x) + o(z-x) \quad \forall\, z \ge x \in I \quad . \quad . \quad . (19)$$

$$f(y) \ge f(x) + f'_+(x)(y-x) \quad \forall\, y \ge x \in I \quad . \quad . \quad . (20)$$

Therefore the graph of f has left and right half-tangents T_+ $(at\ every\ inner\ point\ x)$, namely the graph of $y \mapsto \overline{f}(x) + f'_{\overset{+}{-}}(x)(y-x)$ for $y \lesseqgtr x$ . . . ……………………………………………………….(21)

The meaning of $(4.5) - (4.6)$ is that the graph of f lies entirely above $T_{-} and\ T_{+}$. To combine these facts, it follows by multiplying $(2.5)\ by\ y - x \geq 0\ and\ by\ y - x < 0$ that both $a = f'_{+}(x)$ and $f'_{-}(x)$ fulfils that $f(y) \geq f(x) + a(y - x)$ for every $y \in I$ $\quad (22)$

Because of the inequality here, every $a \in \mathbb{R}$ with this property is called a sub-gradient of $f\ at\ x$. Whenever a function f has a sub-gradient at x $(that\ is\ fulfils\ (22) for\ some\ a)$, then f is called sub-differentiable at x; the above shows that convex functions are always sub-differentiable in the interior of their domain.

The set of sub-gradients of $f\ at\ x$ is usually denoted by $\partial f(x)$; it is called the sub-differential of $f\ at\ x$. It is straight forward to show that the sub-differential, as a subset $\partial f(x) \subset \mathbb{R}$, is always non-empty $(for\ n = 1), closed\ and\ convex; in\ fact$

$$\partial f(x) = [f'_{-}(x), f'_{+}(x)] \quad . \quad . \quad . \qquad (23)$$

Consequently $\partial f(x) = f'(x)$ at each point x is differentiable on f.

As an appetizing example, consider $f(x) = |x|\ for\ x \in \mathbb{R}$. At x_0, the sub-differential is simply determine from (23) as

$$\partial f(x_0) = [-1,1] \quad . \quad . \quad . \qquad (24)$$

Although f is not differentiable at x_0 this function f has a minimum (as it is convex) at $x_0 = 0$ precisely because $0 \in [-1,1]$.

5.1 Variational Inequality Theory

Variational inequality theory provides us with a tool for formulating a variety of equilibrium problems; qualitatively analyzing the problems in terms of existence and uniqueness of solutions, stability and sensitivity analysis, and providing us with algorithms with accompanying convergence analysis for computational purposes.

It contains, as special cases, such well-known problems in mathematical programming as: systems of nonlinear equations, optimization problems, Complementarity problems, and is also related to fixed point problems.

5.2 Stampacchia and Minty Scalar Variational Inequalities

We introduce Stampacchia and Minty variational inequalities in finite dimensional spaces.

Definition 5.1: *Let K be a nonempty subset of* R^n *and let F be a function from* R^n *to* R^n. *A Stampacchia variational inequality* (for short *SV I*(*F*,*K*)) *is the problem to find an* $x^* \in K$ *such that:*

$$SV\,I\,(F,K) \qquad < F(x^*), y - x^* > \geq 0 \qquad ; \forall\ y \in K \qquad\qquad (25)$$

where $< \cdot\ ,\ \cdot >$ *denotes the inner product in* R^n.

The problem was first introduced and studied by Stampacchia in 1964; subsequently, however, many other papers have appeared dealing with theoretical aspects and with applications of this problem .The vector x^*, solution of *SV I* (*F*,*K*), is called *Stampacchia equilibrium point of the map F on K.*

We now introduce some equivalent formulations of *SV I* (*F*,*K*) in the case in which the domain is an open set or a convex and closed set. If the domain *K* is an open set, then the solution of *SVI*(*F*,*K*) is equivalent to that of a system of equations, as shows the following result:

Proposition 5.1: *Let* $K \subseteq R^n$ *be an open set and let be* $F : K \rightarrow R^n$. *The vector* $x^* \in K$ *is a solution of SV I* (*F*,*K*) *if and only if* x^* *solves the system of equation* $F(x^*) = 0$.

Proof: If $F(x^*) = 0$, then *SV I* (*F*,*K*) holds with equality

$$< F(x^*), x - x^* > = 0 \qquad \forall\ x \in K$$

Conversely if x^* is a solution of *SV I*(*F*,*K*) and *K* is an open set, exists $\delta > 0$ such that $\beta(x^*, \delta) \subset K$ and so, by supposition,

$$< F(x^*), x - x^* > \geq 0 \qquad \forall\ x \in \beta(x^*, \delta)$$

But $\forall\ x \in \beta(x^*, \delta)$ also $(2x^* - x) \in \beta(x^*, \delta)$ and then

$$< F(x^*), 2x^* - x - x^* > = < F(x^*), x^* - x > \geq 0 \qquad \forall\ x \in \beta(x^*, \delta)$$

Therefore $< F(x^*), x - x^* >= 0 \qquad \forall\ x \in \beta(x^*, \delta)$

and that is equivalent to condition x^* solves $F(x^*) = 0$.

Many classical economic equilibrium problems have been formulated as systems of equations, since market clearing conditions necessarily equate the total supply with the total demand. Note that systems of equations, however, preclude the introduction of inequalities, which may be needed, for example, in the case of non negativity assumptions on certain variables such as price. If, instead, *K* is a convex and closed set, an equivalent geometric formulation of *SV I* (*F*,*K*) can be given introducing the concepts of normal cone and generalized equation.

Definition 5.2: *If* $C \subseteq R^n$ *is a convex and closed set, the normal cone to at a point* $x^* \in C$ is:

$$N_C(x^*) = \{x \in R^n : < x, y - x^* > \leq 0 \quad \forall\ y \in C\}$$

It is easily seen that the normal cone is closed and convex. Then, if *K* is convex, $x^* \in K$ is a solution of *SV I* (*F*,*K*) if and only if:

$$-F(x^*) \in N_K(x^*),$$

that is, if and only if $0 \in F(x^*) + N_K(x^*)$ and so *SVI* (*F*,*K*) is equivalent to a generalized equation.

An alternative formulation of the Stampacchia variational inequality (equivalent only under monotonicity and continuity hypotheses) has been proposed by G.J. Minty. The variational inequality which he formulated is known as *Minty variational inequality*.

Definition 5.3 *Let K be a nonempty subset of* R^n *and let F be a function from K to* R^n. *A Minty variational inequality (for short MVI*(*F*,*K*)) *is the following problem: to find an* $x^* \in K$ *such that:*

$$< F(y), x^* - y > \leq 0 \qquad \forall\ y \in K$$

Any solution of *MVI*(*F*,*K*)is called a *Minty equilibrium point* of the map *F* over *K*.

It is important to underline that, while in *MVI*(*F, K*) is considered the value assumed by *F* in every $y \in K$, in *SVI*(*F, K*) the function *F* is estimated only in the given point $x^* \in K$.

A well-known Lemma, formulated by Minty in 1967, states the equivalence of the two alternative formulations (the one presented by Stampacchia and the one introduced by Minty) under continuity and monotonicity assumptions of involved function. In other words Minty's lemma gives a complete characterization of the solutions of $MVI(F,K)$ in terms of the solution of $SVI(F,K)$, when the set K is convex and the operator F is continuous and monotone.

Minty's Lemma: *Let $F : K \rightarrow R^n$ with $K \subseteq R^n$.*

i) *If F is continuous on K and K is convex, then every $x^* \in K$ which solves $MVI\ (F,K)$ is also a solution of $SVI\ (F,K)$.*

ii) If, instead, F is monotone on the convex set K, that is if

$$< F(x) - F(y), x - y > \geq 0 \qquad \forall\ x, y \in K$$

then every $x^* \in K$ which solves $SVI\ (F,K)$ is also a solution of $MVI(F,K)$.

Remark 5.1: It can be observed that for the implication $MVI \Rightarrow SVI$ only the convexity of K and the continuity of F are used, while for the reverse implication only the monotonicity of F is exploited. Such hypothesis, contained in the point ii), can be weakened with the concept of pseudomonotonicity; in other words the implication

$SVI \Rightarrow MVI$ is still true if F it is pseudomonotone, i.e.:

$$< F(x), x - y > \leq 0 \Rightarrow\ < F(y), x - y > \leq 0 \qquad \forall\ x, y \in K$$

One of the crucial problems in variational inequalities theory, on which is focused an important part of research, is the existence of a solution. Many classical results ensure that S, the solution set of $SVI(F,K)$, is a nonempty set. The following theorem by *Hartman and Stampacchia* requires a convex and compact set K and a continuous function F.

Theorem 5.2: *If K is a nonempty convex and compact subset of R^n*

and $F : K \rightarrow R^n$ *is a continuous function, there exists an $x_0 \in K$ solution of $SVI\ (F,K)$, i.e. $S \neq \emptyset$.*

In general, the variational inequality problem $SVI\ (F, K)$ can have more than one solution. If instead F is strictly monotone, then the problem $SVI\ (F, K)$ can have at most one solution.

Theorem 5.3: *If K is a nonempty convex and compact subset of R^n and $F : K \rightarrow R^n$ is strictly monotone on K, then the problem $SVI\ (F,K)$ has at most one solution.*

The hypotheses of continuity of F and of compactness of K do not ensure, instead, the existence of a solution for $MVI\ (F, K)$; they don't ensure, that is, that M, the solution set of $MVI\ (F, K)$, is a nonempty set.

5.3 Relations Between *SVI*, *MVI* and Extremal Problems

It is interesting the study of the relations between variational inequalities and optimization problems. Variational inequalities are, in fact, considered as related to a scalar optimization problem in which the objective function is a primitive of the operator involved in the inequality itself. In other words, definitions 5.1 and 5.3 can be put in relationship with problems of the type:

$$P\ (f,K) \qquad \min_{x \in K} f(x)$$

where $K \subseteq R^n$ and $f : R^n \rightarrow R$.

We recall that a point $x^* \in K$ is a solution of $P\ (f,K)$ if:

$$f(x) - f(x^*) \geq 0 \qquad \forall\ x \in K$$

while a point $x^* \in K$ is a strong or strict solution of $P\ (f,K)$ if:

$$f(x) - f(x^*) > 0 \qquad \forall\ x \in K \setminus \{ x^* \}$$

The connections between minimum problems and the variational inequalities $SV\ I\ (F,K)$ and $MV\ I\ (F,K)$ have been widely studied in the case in which K is a convex set and the objective function $f : R^n \rightarrow R$, defined and differentiable on a open set containing K, is a primitive of F, that is $f^{/}(x) = F(x)$. In other words, the easiest way to relate the variational inequalities of Stampacchia and Minty to minimization problems is to consider variational inequalities of differential type. It is possible, indeed, to consider the following variational inequalities:

- To find a point $x^* \in K$ such that:

$$< \nabla f\,(x^*)\,,\, y - x^* > \geq 0 \qquad \forall\; y \in K$$

- To find a point $x^* \in K$ such that:

$$< \nabla f\,(y)\,,\, x^* - y > \geq 0 \qquad \forall\; y \in K$$

Such problems are denoted, respectively with:
$SVI(\nabla f, K)$ and $MVI(\nabla f, K)$.

In the scalar case several results which state relations between solutions of a Stampacchia or Minty variational inequality of differential type and the underlying minimization problem are known. We recall, briefly, that if $x^* \in K \subseteq R^n$, with K convex and nonempty, is a solution of the primitive minimization problem:

$$P\,(f, K) \qquad \square \min_{x \in K} f\,(x)$$

for some function $f : R^n \rightarrow R$, differentiable on an open set containing the convex set K, then x^* solves $SVI(\nabla f, K)$, as stated by the following result:

Proposition 5.3 :*Let K be a convex subset of* R^n *and let* $f : R^n \rightarrow R$ *be differentiable on an open set containing K.*

i) If $x^* \in K$ *is a solution of* $P\,(f,K)$, *then* x^* *solves* $SVI(\nabla f, K)$.

ii) If f is convex and $x^* \in K$ *solves* $SVI(\nabla f, K)$, *then* x^* *is a solution of* $P\,(f, K)$.

In other words if $F(x)$ is the gradient of the differentiable function $f : R^n \rightarrow R$ and if K is convex , then $SVI(\nabla f, K)$ is a necessary optimality condition for the minimization of the function f over the set K, condition which becomes also sufficient if f is convex.

If, instead, $x^* \in K$ is a solution of $MVI(\nabla f, K)$ then x^* is also solution of $P\,(f, K)$. More precisely, $MVI(\nabla f, K)$ is a sufficient optimality condition which becomes necessary if f is convex.

Proposition 5.4: *Let K be a convex subset of* R^n *and let* $f : R^n \rightarrow R$ *be differentiable on a open set containing K.*

i) If $x^* \in K$ *is a solution of* $MVI(\nabla f, K)$, *then* x^* *is a solution of* $P\,(f, K)$.

ii) If f is convex and x^* *is a solution of* $P\,(f,K)$, *then* x^* *solves* $MVI(\nabla f, K)$.

Remark 5.2: If, in point i) of Proposition 5.4, we suppose that x^* is a “strict solution” of $MVI(\nabla f, K)$,i.e.:

$$< \nabla f\,(y)\,,\, y - x^* > \; > 0 \qquad \forall\; y \in K\; y = x^*$$

then it is possible to prove that x^* is the unique solution of $P\,(f,K)$.

Remark 5.3: In both propositions the convexity of *f* is necessary to prove only one of the implications. Such hypothesis can be weakened with the pseudo-convexity.

The result of the proposition 5.4 leads to some deeper relationships between the solutions of *MVI* (∇f ,*K*) and the corresponding primitive minimization problem. It seems that an equilibrium modeled through a *MV I* (∇f ,*K*) is more regular that one modeled through a *SV I* (∇f ,*K*). This conclusion leads to argue that if *MVI*(∇f ,*K*) admits a solution and the operator *F* admits a primitive *f*(∇f = *F*),then *f* has some regularity property.

5.4 The General Variational Inequality Problem

The finite - dimensional variational inequality problem, VI(F *,K), is to determine a vector* $x^* \in K \subset R^n$, *such that*

$$F(x^*)^T.(x - x^*) \geq 0, \quad \forall x \in K$$

or, equivalently,

$$\left\langle F(x^*)^T, (x - x^*) \right\rangle \geq 0, \quad \forall x \in K \qquad (26)$$

where F *is a given continuous function from K to* R^n*, K is a given closed convex set, and denotes* $\langle .,. \rangle$ *the inner product in n dimensional Euclidean space.*

In geometric terms, the variational inequality (26) states that $F(x^*)^T$ is "orthogonal" to the feasible set *K* at the point x^*. This formulation, as shall be demonstrated, is particularly convenient because it allows for a unified treatment of equilibrium problems and optimization problems.

Indeed, many mathematical problems can be formulated as variational inequality problems, and several examples applicable to optimization problems.

Theorem 5.4

Let K = R^n *and let* F : $R^n \rightarrow R^n$ *be a given function. A vector x solves VI(*F *,* R^n*) if and only if* F $(x^*) = 0$.

Proof:

If F $(x^*) = 0$, then inequality (1.4) holds with equality.

Conversely, if x^* satisfies (1.4), let $x = x^* - F(x^*)$, which implies that

$$F(x^*)^T.(-F(x^*)) \geq 0, \qquad \text{or} \qquad -\|F(x)\|^2 \geq 0 \qquad (27)$$

and, therefore, $F(x^*) = 0$.

The subsequent two theorems identify the relationship between an optimization problem and a variational inequality problem.

Theorem 5.5

Let x^* *be a solution to the optimization problem:*

Minimize f(*x*) (28)

subject to: $x \in K$,

where f is continuously differentiable and K is closed and convex. Then x^* *is a solution of the variational inequality problem:*

$$\nabla f(x^*)^T.(x-x^*) \geq 0, \quad \forall x \in K \tag{29}$$

Proof: Let $f(t) = f(x^* + t(x - x^*))$, for $t \in [0, 1]$. Since $f(t)$ achieves its minimum at $t = 0$, $0 \leq f'(0) = \nabla f(x^*)^T.(x-x^*)$, that is, x^* is a solution of **(28)**

Theorem 5.5

If $\phi(x)$ *is a convex function and* x^* *is a solution to VI(*∇F *, K), then* x^* *is a solution to the optimization problem (5.3).*

Proof: Since $f(x)$ is convex,

$$f(x) \geq f(x^*) + \nabla f(x^*)^T.(x-x^*), \quad \forall x \in K \tag{30}$$

But $\nabla f(x^*)^T.(x - x^*) \geq 0$, since x^* is a solution to VI(∇F , K). Therefore, from (30) one concludes that that is, x^* is a minimum point of the mathematical programming problem (28).
If the feasible set $K = R^n$, then the unconstrained optimization problem is also a variational inequality problem.

On the other hand, in the case where a certain symmetry condition holds, the variational inequality problem can be reformulated as an optimization problem. In other words, in the case that the variational inequality formulation of the equilibrium conditions underlying a specific problem is characterized by a function with a symmetric Jacobian, then the solution of the equilibrium conditions and the solution of a particular optimization problem are one and the same.
We first introduce the following definition and then fix this relationship in a theorem.

Theorem 5.6

Assume that $F(x)$ *is continuously differentiable on K and that the Jacobian matrix*

$$\nabla F(x) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

is symmetric and positive semidefinite. Then there is a real-valued convex function $f : K \mapsto R^1$ *satisfying* $\nabla f(x) = F(x)$
with x^* *the solution of VI*(F,K) *also being the solution of the mathematical programming problem:*

Minimize $f(x)$

subject to: $x \in K$. (31)

Proof: Under the symmetry assumption it follows from Green's Theorem that

$$f(x) = \int F(x)^T dx \qquad (32)$$

where $\int$ is a line integral. The conclusion follows since *if* $f(x)$ *is a convex function and* x^* *is a solution to VI(*∇F *,K), then* x^* *is a solution to the optimization problem (31).*

6.1 Summary and Conclusion

We discovered in the course of this research that convexity has certain unique and important roles in optimization. In particular, we discovered from already established results; that for convex functions, a local minimum is a global minimum, that if the minimum exist it must be unique.

This study also reveals the following as some of the established implications of convex/concave functions

i. Every concave or convex function must also be continuous on the interior of its domain
ii. Every concave or convex function must possess minimal differentiability properties. Among other things, every directional derivative must be well-defined at all points in the domain of a concave or convex function,
iii. The concavity or convexity of an everywhere differentiable function f can be completely characterized in terms of the behavior of its derivative ∇, and the concavity or convexity of a C^2 function f can be completely characterized In terms of the behavior of its second derivative $\nabla^2 f$.

To-date, problems which have been formulated and studied as variational inequality problems include:

i. traffic network equilibrium problems
ii. spatial price equilibrium problems
iii. oligopolistic market equilibrium problems
iv. financial equilibrium problems
v. migration equilibrium problems, as well as
vi. environmental network problems, and
vii. knowledge network problems.

There are several algorithms and iterative methods for solving these variational inequality problems which include;

i. General Iterative Scheme of Dafermos which induces such algorithms as: The Projection Method and The Relaxation Method.
ii. The Modified Projection Method of Korpelevich which converges under less restrictive conditions than the general iterative scheme.
iii. A variety of Decomposition Algorithms, both serial and parallel.

Although, variational inequality problem encompasses the optimization problem, we also discovered from already established results that a solution to a convex optimization problem also solves the variational inequality problem.

It is in the light of the above, we suggest that instead of spending time finding solution to a convex optimization problem that is complex, it will be better to transcribe such problem into a variational inequality problem which may be easier to solve.

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