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Solution of Integro-differential Equation of the first order with the operators

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Abstract

The aim of this work is to study the existence, uniqueness and stability solution of integro-differential equations with operators by using both method Picard approximation and Banach fixed point theorem. Furthermore the study of such nonlinear of integro-differential equations with operators leads us to improve and extend the above methods and thus the non-linear integro-differential equations with operators that we have introduced in this study become more general and detailed than those introduced by Butris some results.

Keywords. *Existence, uniqueness and stability solution, nonlinear system, integro-differential equations, Picard approximation method, Banach fixed point theorem.*

1. Introduction:

Integro-differential equations of various types and kinds play an important role in many branches of mathematics. Over the past thirty years substantial progress has been made in developing innovative approximate solutions techniques to a large class of integro-differential equations. In recent years, integro-differential equations arise in many problems of mathematical physics, *such as the theory of elasticity, visco elasticity,or hydrodynamics. Also, fracture mechanics, aerodynamics, theory of porous filtering, antenna problems in electromagnetic theory, viscodynamics fluids, contact problems in the theory of elasticity, mixed boundary problems in mathematical physics, biology, chemistry and engineering can be formulated as integral equations of the first, second and third kind, see [1,2,4,5,6,8,10,11].*

In this work, we study the existence, uniqueness and stability solution of integro-differential equations with the operators by using both method of Picard approximation and Banach fixed point theorem which are given in [7] and [9] respectively .

Butris [3] has been used the above methods to consider the following problem:-

$$\frac{dx}{dt} = f\left(t, x, \int_{t}^{t+T} g\big(s, x(s)\big)ds\right),$$

where $x \in D \subset R^n$ *, D is a closed and bounded domain.*

Consider the following problem:-

$$\frac{dx}{dt} = f\left(t, x, Ax, \int_{0}^{h(t)} g\Big(s, x(s), Bx(s)\Big)\, ds\right) \qquad \dots (1.1)$$

where $f(t, x, y, z)$ *and* $g(t, x, w)$ *are continuous vector functions which are defined on the domains :-*

$$\left.\begin{array}{l}(t, x, y, z) \in R^1 \times D \times D_1 \times D_2 = (-\infty, \infty) \times D \times D_1 \times D_2 \\ (t, x, w) \in R^1 \times D \times D^* = (-\infty, \infty) \times D \times D^*\end{array}\right\} \qquad \cdots (1.2)$$

where $x \in D \subset R^n$ *, D is closed and bounded domain subset of Euclidean space* R^n *and* D_1 *,* D_2 *,* D^* *are bounded domains subset of Euclidean space* R^m *.*

We assume that the operators A and B are defined in the class of continuous functions and map it into the class continuous functions.

Suppose that the vector functions $f(t,x,y,z)$,$g(t,x,w)$ *and the operators A and B satisfy the following inequalities:-*

$$\|f(t,x,y,z)\| \le M_1\,, \ \|g(t,x,w)\| \le M_2 \qquad \cdots (1.3)$$

$$\|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\| \le K_1\|x_1 - x_2\| + K_2\|y_1 - y_2\| + K_3\|z_1 - z_2\| \qquad \cdots (1.4)$$

$$\|g(t,x_1,w_1) - g(t,x_2,w_2)\| \le P_1\|x_1 - x_2\| + P_2\|w_1 - w_2\| \qquad \cdots (1.5)$$

$$\|h(t)\| \le h < \infty \qquad \ldots (1.6)$$

$$\|Ax_1 - Ax_2\| \le Q_1\|x_1 - x_2\| \qquad \ldots (1.7)$$

$$\|Bx_1 - Bx_2\| \le Q_2\|x_1 - x_2\| \qquad \ldots (1.8)$$

for all $t \in R^1$, $x, x_1, x_2 \in D$, $y, y_1, y_2 \in D_1$, $z, z_1, z_2 \in D_2$, $w, w_1, w_2 \in D^*$.

where $M_1, M_2, K_1, K_2, L_1, L_2, h,$ *are positive constants .*

We define the non-empty sets as follows:

$$\left.\begin{aligned} D_f &= D - M_1 T \\ D_{1f} &= D_1 - Q_1 M_1 T \\ D_{2f} &= D_2 - hM_1 T\,(P_1 + P_2 Q_2) \end{aligned}\right\} \qquad \ldots (1.9)$$

Furthermore, we assume that the following condition holds.

$$q = T[\,K_1 + K_2 Q_1 + K_3 h(P_1 + P_2 Q_2)\,] < 1\,. \qquad \ldots (1.10)$$

Definition 1.1[8]. *A continuous function* f *satisfy a* **Lipschitz condition** *on the domain* $G = \{(t,x): a \le t \le b, c \le x \le d\}$ *in the variable* x *on* G *if for all* $K > 0$ *and* $(t,x_1),(t,x_2) \in G$, *such that* $|f(t,x_1) - f(t,x_2)| \le K|x_1 - x_2|$.

Definition 1.2[7]. *A solution* $x(t)$ *is said to be* **stable** *if for each* $\varepsilon > 0$ *, There exists a* $\delta > 0$ *such that any solution* $\bar{x}(t)$ *which satisfies*

$\|\bar{x}(t_0) - x(t_0)\| < \delta$ *for some* t_0 *, also satisfies*

$\|\bar{x}(t) - x(t)\| < \varepsilon$ *for all* $t \ge t_0$.

Definition 1.3[7]. *Let (C [0, T] , ‖.‖) be a norm space if T maps into itself we say that T is a* ***contraction mapping*** *on C [0,T] if there exists* $\alpha \in R$ *with* $0 < \alpha < 1$, *such that* $\|Tx - Ty\| \leq \alpha\|x - y\|, (x, y) \in\ C\,[0, T]$.

Lemma 1.1[9]. *(Granwall's lemma).Suppose that* $u(t) \geq 0$ *and* $f(t) \geq 0$ *where* $t \geq t_0$ *also* $u(t), f(t) \in [t_0, \infty]$, *satisfies the conditions:*

$u(t) >\leq c + \int_{t_0}^{t} f(t_1)\, u(t_1)\ dt_1$, *then* $(t) \leq ce^{\int_{t_0}^{t} f(t_1)dt_1}$.

Theorem 1.1[7] .*Let E be a Banach space , if T is a contraction mapping on E then T has one and only one fixed point in E .*

2. Existence Solution of (1.1).

In this section, we prove the existence theorem of integro- differential equation (1.1) by using Picard approximation method.

Theorem 2.1. (Existence theorem)*. Let t the functions* $f(t, x, y, z)$ *and* $g(t, x, w)$ *be defined on the domain (1.2) continuous in* t, x, y, z, w *and satisfies the inequities (1.3) to (1.8) and condition (1.9), (1.10) has a solution* $x = x(t, x_0)$ *, then there exist a sequence of functions .*

$$x_{m+1}(t, x_0) = x_0 + \int_0^t f(s, x_m(s, x_0), Ax_m(s, x_0), \int_0^{h(s)} g(\tau, x_m(\tau, x_0), Bx_m(\tau, x_0))d\tau)ds \qquad (2.1\,)$$

with $x_0(0, x_0) = x_0$ *, m=0,1,2,…*

convergent uniformly as $m \to \infty$ *in the domain*

$$(t, x_0) \in R^1 \times D_f \qquad \text{....(2. 2)}$$

to the limit function $x\ (t, x_0)$ *difined on the domain (2.2) and satisfy the following integral equation :-*

$$x(t,x_0) = x_0 + \int_0^t f(s, x(s,x_0), Ax(s,x_0),$$

$$\int_0^{h(s)} g(\tau, x(\tau,x_0), Bx(\tau,x_0))d\tau)ds \qquad \dots (2.3)$$

which is a continuous vector solution of (1.1).

Proof.

By mathematical induction we can prove that :-

$$\|x_m(t,x_0) - x_0\| \leq M_1 T \qquad \dots (2.4)$$

Also from (2. 4) we get

$$\|Ax_m(t,x_0) - A x_0\| \leq Q_1 M_1 T .$$

That is $x_m(t,x_0) \in D$, $Ax_m(t,x_0) \in D_1$ *for all* $t \in [0,T]$ *and* $x_0 \in D_f$,

$Ax_0(t,x_0) \in D_{1f}$.

and

$$\|z_1(t,x_0) - z_0(t,x_0)\| = \left\| \int_0^{h(t)} g\big(s, x_1(s,x_0), Bx_1(s,x_0)\big)ds - \int_0^{h(t)} g(s,x_0,Bx_0)ds \right\|$$

$$\leq \int_0^{h(t)} \| \, [P_1 M_1 T + P_2 Q_2 M_1 T \,] \| \, ds$$

$$\leq hM_1T \, (P_1 + P_2 Q_2)$$

which gives $z_1(t,x_0) \in D_2$ *for all* $t \in [0,T]$ *and* $z_0 \in D_{2f}$.

Also, by mathematical induction we can prove that :-

$$\|z_m(t,x_0) - z_0(t,x_0)\| \leq hM_1T \, (P_1 + P_2 Q),$$

that is $z_m(t,x_0) \in D_2$ *for all* $t \in [0,T]$ *and* $z_0 \in D_{2f}$.

Next, we shall to prove that the sequence of functions (2.1) uniformly converges on the domain (2.2).

From (2.1) when m =1, we have

$\|x_2(t,x_0) - x_1(t,x_0)\|$

$\leq \int_0^t [\, K_1 \|x_1(s,x_0) - x_0\| + K_2\, Q_1 \|x_1(s,x_0) - x_0\|$

$+ K_3 h\, (P\, \|x_1(s,x_0) - x_0\| - P_2 Q_2\, \|x_1(s,x_0) - x_0\|\,)\,]\, ds$

$\leq T\, [\, K_1 + K_2 Q_1 + K_3 h(\, P_1 + P_2 Q_2)\,] \|x_1(t,x_0) - x_0\|$

$\leq q\, \|x_1(t,x_0) - x_0\|$ ... (2. 5)

Now, for all $m \geq 1$ *, the following inequality holds*

$\|x_{m+1}(t,x_0) - x_m(t,x_0)\| \leq q^m \|x_1(t,x_0) - x_0\|$ ... (2.6)

From (2. 6), we can clued that from p ≥ 1, we obtain that

$\left\|x_{m+p}(t,x_0) - x_m(t,x_0)\right\| = \left\|x_{m+p}(t,x_0) - x_{m+p-1}(t,x_0)\right\| +$

$\left\|x_{m+p-1}(t,x_0) - x_{m+p-2}(t,x_0)\right\| + \ldots +$

$\|x_{m+1}(t,x_0) - x_m(t,x_0)\|$

$\leq q^{m+p-1}\, \|x_1(t,x_0) - x_0\| +$

$q^{m+p-2}\, \|x_1(t,x_0) - x_0\|$

$+ \ldots + q^m\, \|x_1(t,x_0) - x_0\|.$

Therefore

$\left\|x_{m+p}(t,x_0) - x_m(t,x_0)\right\| \leq q^m\, (1 - q)^{-1}\, \|x_1(t,x_0) - x_0\|$ ... (2. 7)

for all $t \in [0,T]$ *and* $x_0 \in D_f$.

Since $q < 1$*, then* $\lim_{m\to\infty} q^m = 0$*, So that the right side of (2. 7) tends to zero.*

There for the sequence of functions (2.1) is convergent uniformly on the domain (2.2) to the limit function $x(t, x_0)$ *which is defined on the same domain .Let*

$$\lim_{m\to\infty} x_m(t, x_0) = x\ (t, x_0) \qquad ...(2.\ 8)$$

Finally, we show that $x(t, x_0) \in D$ *, for all* $t \in [0,T]$ *.Taking*

$$\left\| \int_0^t f(s, x_m(s, x_0), Ax_m(s), \int_0^{h(S)} g\big(\tau, x_m(\tau, x_0), Bx_m(\tau, x_0)\big)d\tau)ds - \int_0^t f(s, x(s, x_0), Ax(s), \int_0^{h(S)} g(\tau, x(\tau, x_0), Bx(\tau, x_0))d\tau)ds \right\|$$

$$\leq \int_0^t [\, K_1 \|x_m(s, x_0) - x(s, x_0)\| + K_2\, Q_1 \|x_m(s, x_0) - x(s, x_0)\|$$
$$+ K_3 h\ (P_1 \|x_m(s, x_0) - x(s, x_0)\| - P_2 Q_2 \|x_m(s, x_0) - x(s, x_0)\|\,)\,]\, ds$$

$$\leq T\ [\, K_1 + K_2 Q_1 + K_3 h(\, P_1 + P_2 Q_2)\,]\ \|x_m(t, x_0) - x(t, x_0)\|$$

From (2. 8), we get

$$\|x_m(t, x_0) - x(t, x_0)\| \leq \epsilon_1$$

Putting $\epsilon_1 = \dfrac{\epsilon}{T\ [\, K_1 + K_2 Q_1 + K_3 h(\, P_1 + P_2 Q_2)\,]}$ *we have*

$$\left\| \int_0^t f(s, x_m(s, x_0), Ax_m(s), \int_0^{h(S)} g\big(\tau, x_m(\tau, x_0), Bx_m(\tau, x_0)\big)d\tau)ds - \int_0^t f(s, x(s, x_0), Ax(s), \int_0^{h(S)} g(\tau, x(\tau, x_0), Bx(\tau, x_0))d\tau)ds \right\|$$

$$\leq T\ [\, K_1 + K_2 Q_1 + K_3 h(\, P_1 + P_2 Q_2)\,]\ \frac{\epsilon}{T[\, K_1 + K_2 Q_1 + K_3 h(\, P_1 + P_2 Q_2)\,]}$$

$\leq \in$, *for all* $m \geq 0$,

that is $\lim_{m\to\infty} \int_0^t f(s, x_m(s,x_0), Ax_m(s,x_0), \int_0^{h(s)} g\big(\tau, x_m(\tau,x_0), Bx_m(\tau,x_0)\big)d\tau)ds$

$= \int_0^t f(s, x(s,x_0), Ax(s,x_0), \int_0^{h(s)} g(\tau, x(\tau,x_0), Bx(\tau,x_0))d\tau)ds$

So $x(t,x_0) \in D$ *and hence* $x(t,x_0)$ *is a solution of (1.1) .*

Theorem 2.2. (Uniqueness Theorem). *with the hypotheses and all conditions and inequalities of the theorem 2.1 the solution* $x(s,x_0)$ *of the problem (1.1) is a unique on the domain (1.2).*

Proof . *Let* $y(t,x_0)$ *be another solution of the integro- differential equation (1.2), then*

$$y\,(t,x_0) = x_0 + \int_0^t f(s, y(s,x_0), Ay(s,x_0), \int_0^{h(s)} g(\tau, y(\tau,x_0), By(\tau,x_0))d\tau)ds$$

Thus

$\|x(t,x_0) - y\,(t,x_0)\,\|$

$$\leq \int_0^t \left\| f(s, x(s,x_0), Ax(s,x_0), \int_0^{h(s)} g\big(\tau, x(\tau,x_0), Bx(\tau,x_0)\big)d\tau \right.$$
$$\left. - f(t, y(s), Ay(s), \int_0^{h(s)} g(\tau, y(\tau,x_0), By(\tau,x_0))d\tau \right\| ds$$

$$\leq \int_0^t [\, K_1 \|x(t,x_0) - y\,(t,x_0)\,\| + K_2\, Q_1 \|x(s,x_0) - y\,(s,x_0)\,\|$$
$$+ K_3 h\,(P_1 \,\|x(s,x_0) - y\,(s,x_0)\,\| - P_2 Q_2 \,\|x(s,x_0) - y\,(s,x_0)\,\|\,)\,]\, ds$$

So that

$\|x(t,x_0) - y\,(t,x_0)\,\| \leq q\, \|x(t,x_0) - y\,(t,x_0)\,\|$

By iteration we find that

$\|x(t,x_0) - y(t,x_0)\| \le q^m \|x(t,x_0) - y(t,x_0)\|$.

But from the condition (1.10) we get $q^m \longrightarrow 0$ *as* $m \longrightarrow \infty$ *and hence*

$\|x(t,x_0) - y(t,x_0)\| < \|x(t,x_0) - y(t,x_0)\|$

This is contradiction, that is

$\|x(t,x_0) - y(t,x_0)\| \longrightarrow 0.$

And hence

$x(t,x_0) = y(t,x_0).$

Therefore $x(t,x_0)$ *is a unique solution of the problem (1.1).*

3. Stability solution of (1.1).

In this section, we study the stability solution of the problem (1.1) by the following theorem.

Theorem3.1.(Stability theorem) . *If the inequalities (1.3) to (1.8) are satisfied, and* $z(t,x_0)$ *which is defined bellow as different solution for the problem (1.1) then the solution is stabile if satisfy the inequality*

$\|x(t,x_0) - z(t,x_0)\| \le \epsilon\ ,\ \ \epsilon > 0$

where

$$x(t,x_0) = x_0 + \int_0^t f\left(s, x(s,x_0), Ax(s,x_0), \int_0^{h(s)} g(\tau, x(\tau,x_0), Bx(\tau,x_0))d\tau\right)ds$$

and

$$z(t,x_0) = z_0 + \int_0^t f\left(s, z(s,x_0), Az(s,x_0), \int_0^{h(s)} g(\tau, z(\tau,x_0), Bz(\tau,x_0))d\tau\right)ds$$

Proof. *Taking*

$\|x(t,x_0) - z(t,x_0)\|$

$$\le \|x_0 - z_0\| + \int_0^t \Big\| f\Big(s, x(s,x_0), Ax(s,x_0), \int_0^{h(s)} g(\tau, x(\tau,x_0), Bx(\tau,x_0))d\tau$$

$$-f(s, z(s, x_0), Az(s, x_0), \int_0^{h(s)} g(\tau, z(\tau, x_0), Bz(\tau, x_0))d\tau\Big\| \; ds$$

$$\leq \|x_0 - z_0\| + \int_0^t [\, K_1 \, \|x(s, x_0) - z\,(s, x_0)\,\| + K_2 \, Q_1 \|x(s, x_0) - z\,(s, x_0)\,\|$$

$$+ K_3 h\,(P_1 \, \|x(s, x_0) - z\,(s, x_0)\,\| - P_2 Q_2 \, \|x(s, x_0) - z\,(s, x_0)\,\|\,)\,]\, ds$$

$$\leq \|x_0 - z_0\| + \int_0^t \|[\, K_1 + K_2 \, Q_1 + K_3 h\,(P_1 - P_2 Q_2]\| \, \|x(s, x_0) - z\,(s, x_0)\,\| \; ds$$

By using lemma1.1 ,we have

$$\|x(t, x_0) - z\,(t, x_0)\,\| \leq \|x_0 - z_0\| \; e^{\int_0^t [\, K_1 + K_2 \, Q_1 + K_3 h\,(P_1 - P_2 Q_2]\, ds}$$

And by the definition of the stability for $\|x_0 - z_0\| \leq \delta$ *we get*

$$\|x(t, x_0) - z\,(t, x_0)\,\| \leq \delta \, e^{Tq}$$

and hence

$\|x(t, x_0) - z\,(t, x_0)\,\| \leq \in$ *, where choosing* $\delta = \in e^{-Tq}$ *.*

So that the vector solution of (1.1) is stable for all $t \in [0, T]$ *.*

4 . Existence and Uniqueness Solution of (1.1):

In this section, we prove the existence uniqueness theorem of the problem (1.1) by using Banach fixed point theorem.

Theorem 4.1. (Banach Fixed Point Theorem). *Let the vector functions* $f(t, x, y, z)$ *and* $g(t, x, w)$ *in the problem (1.1) are defined and continuous on the domain (1.2) and satisfies assumptions and all conditions of theorem (1.1) then the problem (1.1) has a unique continuous solution on the domain (1.2).*

Proof. *Let (C [0, T],* $\|.\|$ *) be a Banach space and* T^* *be a mapping on C[0,T] as follows:-*

$$T^* x(t, x_0) = x_0 + \int_0^t f(s, x(s, x_0), Ax(s, x_0),$$

$$\int_0^{h(s)} g(\tau, x(\tau, x_0), Bx(\tau, x_0))d\tau)ds$$

Since $\int_0^{h(t)} g(s, x(s, x_0), Bx(s, x_0))ds$ *) is continuous on the interval [0,T], and*

$$\int_0^t f(s, x(s, x_0), Ax(s, x_0), \int_0^{h(s)} g(\tau, x(\tau, x_0), Bx(\tau, x_0))d\tau)ds$$

is also continuous on the same interval ,then $T^* x(t, x_0) \in C\ [0, T]$

Thus $T^* : C[0, T] \longrightarrow C[0, T]$.

Now, we claim that T^* *is a contraction mapping on* $[0, T]$.

Let $x(t, x_0), z(t, x_0)$ *be any two vector functions on C [0,T] , then*

$$\|T^* x(t, x_0) - T^* z(t, x_0)\|$$

$$\leq \max_{t\in[0,T]} \{\int_0^t |f(s, x(s, x_0), Ax(s, x_0), \int_0^{h(s)} g\big(\tau, x(\tau, x_0), Bx(\tau, x_0)\big)d\tau$$

$$-f\left(s, z(s, x_0), Az(s, x_0), \int_0^{h(s)} g\big(\tau, z(\tau, x_0), Bz(\tau, x_0)\big)d\tau\right| ds$$

$$\leq T[K_1 + K_2Q_1 + K_3h(P_1 + P_2Q_2)] \max_{t\in[0,T]} |x(t, x_0) - z(t, x_0)|$$

$$\|T^* x(t, x_0) - T^* z(t, x_0)\| \leq q\ \|x(t, x_0) - z(t, x_0)\|$$

So T^* *is a contraction mapping if* $0 < q < 1$ *.Thus, by Banach fixed point theorem, there exists a fixed point* $x(t, x_0)$ *in C [0, T] such that*

$T^* x(t, x_0) = x(t, x_0)$, *there fore*

$$x(t, x_0) = x_0 + \int_0^t f(s, x(s, x_0), Ax(s, x_0),$$

$$\int_0^{h(s)} g\big(\tau, x(\tau, x_0), Bx(\tau, x_0)\big)d\tau)ds.$$

Hence $x(t, x_0)$ *is a unique continuous solution of the problem (1.1).*

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