B – Spline Based Finite Element Method for Two-Dimensional Problems

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Abstract: In this work, an attempt is made to use the B-spline basis functions as the shape functions in the finite element method in contrast with the conventional Finite Element method. An open uniform knot vector is used to obtain the second degree B-Spline basis function. For the spatial discretization, the Potential Energy approach approximation method is employed. Two test cases are considered to study the effectiveness of the present method. These test cases included two dimensional elasticity problems. The results obtained by the present method are compared and found to be in good agreement with the analytical solution and the finite element method.

Keywords: B-Spline, Isogeometric Method, Potential Energy approach

INTRODUCTION

In the recent years, a new class of approximating methods which are variants of the finite element methods are proposed to solve various initial and boundary value problems. The methods that can be included in this category are Meshfree methods [1], B-Spline based Finite element method and Isogeometric methods [2]. In these methods, the approximating function provides higher order of continuity and is capable of providing accurate solutions with continuous gradients throughout the domain.

In the present work, an attempt is made to use an approximating function for the field variable based on the B-Spline basis function to solve the various boundary value problems. An open uniform knot vector is used to obtain the second degree B-Spline basis function. For the spatial discretization, the Potential Energy approach approximation method is employed. Numerical studies are performed with problems of linear elasticity in two dimensions. The two problems considered are the one end fixed a prismatic bar with axial traction on the unclamped end and two end fixed prismatic bar with body load problem.

B–SPLINE FINITE ELEMENT METHOD

The B-splines are a standard tool for describing and modelling curves and surfaces in computer aided design and computer graphics [3]. The aim of this section is to present a short description of B-splines and its associated terminology.

B-Spline Basis Function

The ***cox-de Boor*** recursion formula for the B-Spline basis functions are defined recursively starting with the zeroth degree (*p = 0*). These basis functions are given as

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad ....(1)$$

And for any polynomial degree ($p \ge 1$),

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \quad ....(2)$$

When evaluating these functions, ratios of the form 0/0 are defined as zero

In the above equations, the basis functions are defined over a parametric domain ξ. The span of the parametric domain is known as the knot vector $\{\xi_1 \quad \xi_2 \quad . \quad . \quad . \quad \xi_{n+p+1}\}$, where ξ_i is the 'i' th knot, n is the number of basis functions and p is the polynomial degree.

A knot vector is a sequence in ascending order of parameter values. A knot vector is said to be open if its first and last knots have multiplicity equal to the polynomial order plus one. An important property of open knot vectors is that the resulting basis functions are interpolatory at the ends of the parametric space. If the knots are evenly spaced then knot vector is called uniform otherwise it is non-uniform.

As examples, for polynomial degree p = 0 and 1 using uniform knot vector {0 1 2 3 4 5 ...}, the basis functions are shown in figure 1 and figure 2.

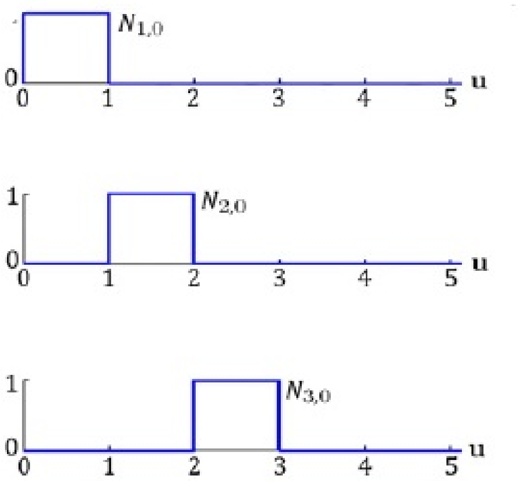

Figure 1

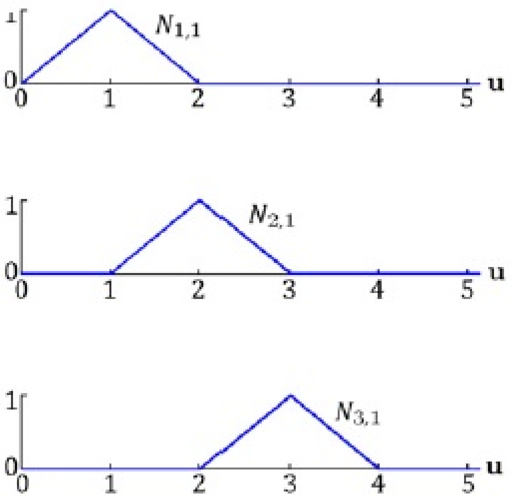

Figure 2

THE TEST PROBLEMS

Two test elasticity problems are considered to study the effectiveness of the present method. They are a prismatic bar subjected to axial traction on the unclamped end in the x direction and a two end fixed prismatic bar subjected to body load.

Prismatic bar subjected to axial traction on the unclamped end

The first problem studied is a prismatic bar subjected to axial traction on the unclamped end in the x direction. The geometry and loading conditions for this problem are shown in figure 3 [6].

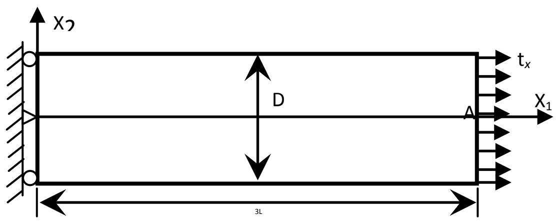

Fig 3: Prismatic bar with one end fixed problem: geometry and loading

Exact solution u is $u = \frac{(t_x)X(x)}{E}$ ……..(3)

Young's modulus (E) = 200 GPa and Poisson's Ratio (ν) = 0.3. Other dimensions of the problem are length of the beam, L = 2 mm and height of the beam, D=1 mm. The penalty parameter in enforcing essential boundary conditions is taken as 1000 times the Young's modulus. The traction on the unclamped end is 5 N/mm^2.

The axial displacement u(x) that is produced by the force is the solution. In the B-spline based Finite element method (BSFEM), the displacement field is approximated by,

$$Q(u,w) = \sum_{i=1}^{ne}\sum_{j=1}^{me} B_{i,j} N_{i,k}(u) M_{j,l}(w)$$ ……(4)

In equation (4), 'ne,me' represents the number of nodes in an elements respectively X & Y directions, $N_{i,k}$, $M_{j,l}$ is the basis functions obtained from the B-Spline basis functions for any point in the domain. In the present study, only the second degree (p =1) approximation is considered. Generally, the B-spline basis functions are not interpolatory except when the knot vector is open uniform. When an open uniform knot vector is used, the essential boundary conditions can be directly implemented. Therefore, an open uniform knot vector is taken as parametric space coinciding with the coordinates of the domain.

The domain is discretised with 63 nodes and 48 elements with four nodes per each element shown below fig.20. Knot vectors are [0 0 0.25 0.5 0.75 1.0 1.25 1.5 1.75 2.0 2.0] in X & [0 0 0.16 0.33 0.50 0.66 0.83 1.0 1.0] in Y directions for this problem

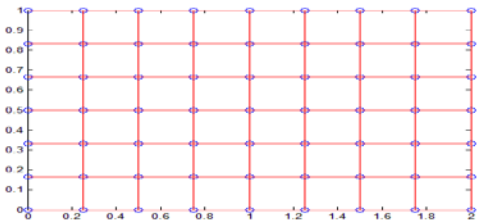

Fig 4: Nodes & Element distribution of prismatic bar (63 nodes & 48 elements)

The obtained results are compared with the exact solution are tabled in table 4 an plot in the fig 21. For 48 elements & 63 nodes error percentage in the result is less than 2%. If the number of nodes is increased to 221 the error percentage is less than 1%

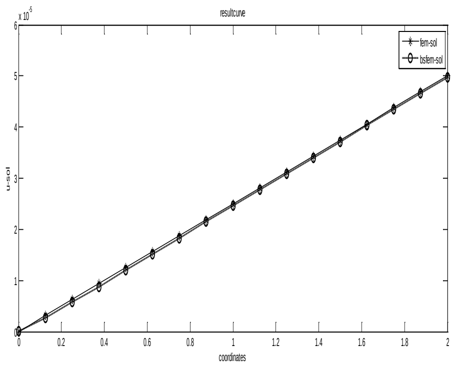

Fig 5: Compression of field variable (u) with exact solutions

where, The figure 5 shows the displacement field along the length of the bar. It can be observed from the figure that the results obtained by the present method are in very good agreement with the analytical solutions.

Two end fixed prismatic bar subjected to body load.

The second problem studied is two end fixed prismatic bar subjected to body load. The geometry and loading conditions for this problem[6] are shown in figure 6

Young's modulus (E) = 200 GPa and Poisson's Ratio (ν) = 0.3. Other dimensions of the problem are length of the beam, L = 3L mm and height of the beam, D=1 mm. The penalty parameter in enforcing essential boundary conditions is taken as 1000 times the Young's modulus. The body force is 2 N/mm^3

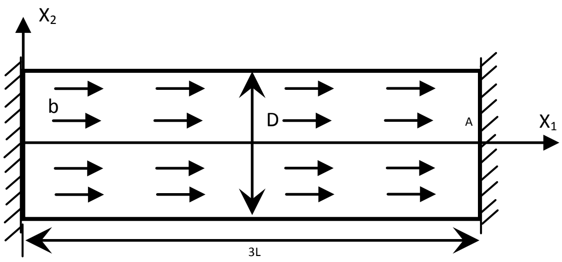

Fig 6: Two end fixed prismatic bar subjected to body load: geometry and loading

Exact solution given as

$$u = bx\frac{3L_x - x}{2E}.......(5)$$

The domain is discretised with 130 nodes and 108 elements with four nodes per each element shown below fig.23. Knot vectors are [0 0 0.25 0.5 0.75 1.0 1.25 1.5 1.75 2.0 2.25 2.5 2.75 3.0 3.0] in X-direction [0 0 0.11 0.22 0.33 0.44 0.55 0.66 0.77 0.88 1.0 1.0] in Y-direction for this problem

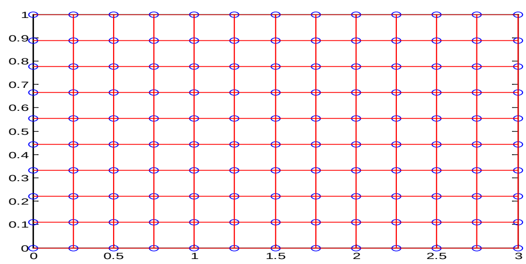

Fig 7: Nodes & Element distribution of simple supported beam (130 nodes and 108 elements)

The obtained results are compared with the exact solution are an plot in the fig 8. For 108 elements & 130 nodes error percentage in the result is less than 2%. If the number of nodes is increased to 336 the error percentage is less than 1%

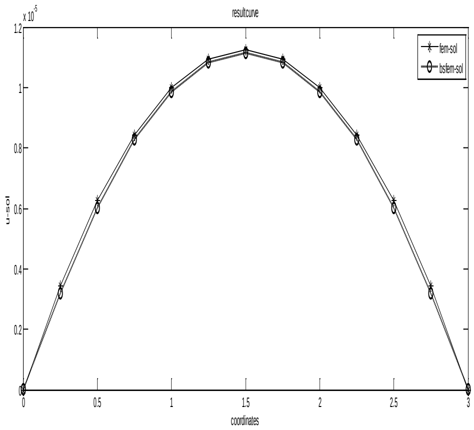

Fig 8: Compression of field variable (u) with exact solutions

CONCLUSION

In this work, an attempt is made to use the B-spline basis functions as the shape functions in the finite element method. An open uniform knot vector is used to obtain the first degree and second degree B-Spline basis function. For the spatial discretization, the potential energy approach is employed for two dimensional problems. Two test cases have been performed to study the effectiveness of the current method. The results obtained by the present method are compared and found to be in good agreement with the analytical solution (exact solution) as well as the finite element method. The maximum error found that 2%

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