

A COMMON FIXED POINT THEOREM FOR FOUR SELF MAPS ON AN INTUITIONISTIC FUZZY METRIC SPACE

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1. INTRODUCTION

Atanassov [3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. In 2004, Park [7] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorm. Recently, in 2006, Alaca et al. [2] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorm as a generalization of fuzzy metric space due to Kramosil and Michalek [5]. In 2006, Turkoglu et al. [9] proved Jungck's common fixed point theorem [4] in the setting of intuitionistic fuzzy metric spaces for commuting mappings. For more details, one can refer to papers ([1], [6], [10], [11]).

In this paper, we prove a common fixed point theorem for four self mappings satisfying contractive condition in intuitionistic fuzzy metric spaces. We observe that the result of Manro [12] follows as a corollary.

2. PRELIMINARIES

Definition 2.1(Zadeh, [13]): Fuzzy set A in X is a function with domain X and values in $[0, 1]$

Schweizer and Sklar [8] defined the following notion:

Definition 2.2 A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

- (i) $*$ is commutative and associative
- (ii) $*$ is continuous
- (iii) $a * 1 = a$ for all $a \in [0,1]$
- (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$

Definition 2.3 A binary operation \diamond : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-conorm if \diamond satisfies the following conditions

- (i) \diamond is commutative and associative
- (ii) \diamond is continuous

- (iii) $a \diamond 0 = a$ for all $a \in [0,1]$
- (iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$

Alaca et al. [2] defined the notion of a intuitionistic fuzzy metric space as follows:

Definition 2.4 A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is a non empty set, $*$ is a continuous t-norm, \diamond is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying

- (i) $M(x, y, t) + N(x, y, t) \leq 1 \forall x, y \in X$ and $t > 0$;
- (ii) $M(x, y, 0) = 0 \forall x, y \in X$
- (iii) $M(x, y, t) = 1 \forall x, y \in X$ and $t > 0$ if and only if $x = y$;
- (iv) $M(x, y, t) = M(y, x, t) \forall x, y \in X$ and $t > 0$;
- (v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \forall x, y \in X$ and $s, t > 0$
- (vi) $\forall x, y \in X, M(x, y, \cdot): [0, \infty) \rightarrow [0, 1]$ is left continuous;
- (vii) $\lim_{t \rightarrow \infty} M(x, y, t) = 1 \forall x, y \in X$
- (viii) $N(x, y, 0) = 1 \forall x, y \in X$
- (ix) $N(x, y, t) = 0 \forall x, y \in X$ and $t > 0$ if and only if $x = y$
- (x) $N(x, y, t) = N(y, x, t) \forall x, y \in X$ and $t > 0$;
- (xi) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s) \forall x, y \in X$ and $s, t > 0$
- (xii) $\forall x, y \in X, N(x, y, \cdot): [0, \infty) \rightarrow [0, 1]$ is right continuous;
- (xiii) $\lim_{t \rightarrow \infty} N(x, y, t) = 0 \forall x, y \in X$.

Then (M, N) is called an intuitionistic fuzzy metric on X . The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non nearness between x and y with respect to t , respectively.

Remark 2.5 Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \diamond)$ such that t-norm $*$ and t-conorm \diamond are associated as $x \diamond y = 1 - ((1-x) * (1-y)) \forall x, y \in X$

Remark 2.6 In an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$, $M(x, y, \cdot)$ is non-decreasing and $N(x, y, \cdot)$ is non-increasing $\forall x, y \in X$.

Definition 2.7 Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be (i) convergent to a point $x \in X$ if

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = 1 \text{ and } \lim_{n \rightarrow \infty} N(x_n, x, t) = 0 \forall t > 0$$

(ii) Cauchy sequence if

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = 1 \text{ and } \lim_{n \rightarrow \infty} N(x_{n+p}, x_n, t) = 0 \forall t > 0, \text{ and } p > 0$$

Definition 2.8 An intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Turkoglu et al. [10] defined the following notions:

Definition 2.9 Let A and S be self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. Then the pair (A, S) is said to be commuting if

$$M(ASx, SAx, t) = 1 \text{ and } N(ASx, SAx, t) = 0 \forall x \in X, t > 0$$

Definition 2.10 Let A and S be self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. Then the pair (A, S) is said to be weakly commuting if

$$M(ASx, SAx, t) \geq M(Ax, Sx, t) \text{ and } N(ASx, SAx, t) \leq N(Ax, Sx, t) \forall x \in X, t > 0.$$

Definition 2.11 Let A and S be self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$

Then the pair (A, S) is said to be compatible if

$\lim_{n \rightarrow \infty} M(ASx_n, SAx_n, t) = 1$ and $\lim_{n \rightarrow \infty} N(ASx_n, SAx_n, t) = 0 \forall t > 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = u$ for some $u \in X$

Definition 2.12 (D.Turgoklu et.al., [9]) Let A and S be self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$

Then the pair (A, S) is said to be point wise R -weakly commuting if given $x \in X$, there exists $R > 0$ such that

$$M(ASx, SAx, t) \geq M(Ax, Sx, t/R) \text{ and } N(ASx, SAx, t) \leq N(Ax, Sx, t/R) \forall x \in X \text{ and } t > 0.$$

Clearly, every pair of weakly commuting mappings is point wise R -weakly commuting with $R = 1$.

Definition 2.13 (S.Muralisankar, G.Kalpana, [6]) Let A and S be self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$

Then the pair (A, S) is said to be reciprocally continuous if

$ASu_n \rightarrow Az$ and $SAu_n \rightarrow Sz$, whenever $\{u_n\}$ is a sequence such that $Au_n \rightarrow z$ and $Su_n \rightarrow z$, for some $z \in X$ as $n \rightarrow \infty$.

If A and S are both continuous, then they are obviously reciprocally continuous, but converse is not true.

Lemma 2.14 ([2], [9]) Let $\{u_n\}$ be a sequence in an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. If there exists a constant $k \in (0, 1)$ such that

$$M(u_n, u_{n+1}, kt) \geq M(u_{n-1}, u_n, t) \text{ and } N(u_n, u_{n+1}, kt) \leq N(u_{n-1}, u_n, t)$$

for $n = 1, 2, 3, \dots$ then $\{u_n\}$ is a Cauchy sequence in X .

Lemma 2.15 [2] [9] Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. If there exists a constant $k \in (0, 1)$ such that

$$M(x, y, kt) \geq M(x, y, t) \text{ and } N(x, y, kt) \leq N(x, y, t) \text{ for all } x, y \in X \text{ and } t > 0, \text{ then } x = y.$$

Now we state the Lemma and Theorem of Manro [12]

Lemma 2.16 [12] Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space with $t * t \geq t$ and $(1-t) \diamond (1-t) \leq (1-t) \forall t \in [0, 1]$. Further, let A, B, S and T be four self mappings of X satisfying

(C1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(C2) there exists a constant $k \in (0, 1)$ such that

$$\begin{aligned} & [1 + aM(Sx, Ty, kt)] * M(Ax, By, kt) \\ \geq & a[M(Ax, Sx, kt) * M(By, Ty, kt) * M(By, Sx, kt)] + M(Ty, Sx, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ & * M(By, Sx, \alpha t) * M(Ax, Ty, (2-\alpha)t) \end{aligned}$$

and

$$\begin{aligned} & [1 + aN(Sx, Ty, kt)] \diamond N(Ax, By, kt) \\ \leq & a[N(Ax, Sx, kt) \diamond N(By, Ty, kt) \diamond N(By, Sx, kt)] + N(Ty, Sx, t) \diamond N(Ax, Sx, t) \diamond N(By, Ty, t) \\ & \diamond N(By, Sx, \alpha t) \diamond N(Ax, Ty, (2-\alpha)t) \end{aligned}$$

for all $x, y \in X, a \geq 0, \alpha \in (0, 2)$ and $t > 0$.

If the pairs (A, S) and (B, T) are point wise R-weakly commuting then continuity of one of the mappings in compatible pair (A, S) or (B, T) implies their reciprocal continuity.

Theorem 2.1 [12] Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space with $t * t \geq t$ and $(1-t) \diamond (1-t) \leq (1-t) \forall t \in [0, 1]$. Further, let A, B, S and T be four self mappings of X satisfying (C1) and (C2). If the pairs (A, S) and (B, T) are compatible and point wise R-weakly commuting and one of the mappings in compatible pair (A, S) or (B, T) is continuous, then A, B, S and T have a unique common fixed point.

3. MAIN RESULTS

Lemma 3.1 Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space with $t * t \geq t$ and $(1-t) \diamond (1-t) \leq (1-t) \forall t \in [0, 1]$. Further, let A, B, S and T be four self mappings of X satisfying

(C1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(C2)^l there exist constants $k \in (0, 1)$ and $q \in (0, 1)$ such that $0 < k < q$

$$M(Ax, By, kt) \geq M(Ty, Sx, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ * M(By, Sx, (1 + q)t) * M(Ax, Ty, qt); q \in (0,1)$$

and

$$N(Ax, By, kt) \leq N(Ty, Sx, t) \diamond N(Ax, Sx, t) \diamond N(By, Ty, t) \\ \diamond N(By, Sx, (1 + q)t) \diamond N(Ax, Ty, qt)$$

for all $x, y \in X$ and $t > 0$.

If the pairs (A, S) and (B, T) are compatible then continuity of one of the mappings in compatible pair (A, S) or (B, T) implies their reciprocal continuity.

Proof: First, assume that A and S are compatible and S is continuous. We show that A and S are reciprocally continuous. Let $\{u_n\}$ be a sequence such that $Au_n \rightarrow z$ and $Su_n \rightarrow z$, for $z \in X$ as $n \rightarrow \infty$.

Since S is continuous, we have $SAu_n \rightarrow Sz$ and $SSu_n \rightarrow Sz$, for $z \in X$ as $n \rightarrow \infty$ and since (A, S) is compatible, we have

$$\lim_{n \rightarrow \infty} M(ASu_n, SAu_n, t) = 1$$

This implies that

$$\lim_{n \rightarrow \infty} M(ASu_n, SAz, t) = 1 \quad \forall t > 0$$

That is $ASu_n \rightarrow Sz$ as $n \rightarrow \infty$. By (C1) for each n , there exists $u_n \in X$ such that $ASu_n = Tv_n$. Thus we have $SSu_n \rightarrow Sz, SAu_n \rightarrow Sz, ASu_n \rightarrow Sz$ and $Tv_n \rightarrow Sz$ as $n \rightarrow \infty$, whenever $ASu_n = Tv_n$.

Now we claim that $Bv_n \rightarrow Sz$ as $n \rightarrow \infty$. In (C2)¹ put $x = Su_n$ and $y = v_n$

$$M(ASu_n, Bv_n, kt) \geq M(Tv_n, SSu_n, t) * M(ASu_n, SSu_n, t) * M(Bv_n, Tv_n, t) \\ * M(Bv_n, SSu_n, (1 + q)t) * M(ASu_n, Tv_n, qt); q \in (0,1)$$

and

$$N(ASu_n, Bv_n, kt) \leq N(Tv_n, SSu_n, t) \diamond N(ASu_n, SSu_n, t) \diamond N(Bv_n, Tv_n, t) \\ \diamond N(Bv_n, SSu_n, (1 + q)t) \diamond N(ASu_n, Tv_n, qt)$$

$$M(Sz, Bv_n, kt) \geq M(Sz, Sz, t) * M(Sz, Sz, t) * M(Bv_n, Sz, t) \\ * M(Bv_n, Sz, (1 + q)t) * M(Sz, Sz, qt); q \in (0,1)$$

and

$$N(Sz, Bv_n, kt) \leq N(Sz, Sz, t) \diamond N(Sz, Sz, t) \diamond N(Bv_n, Sz, t) \\ \diamond N(Bv_n, Sz, (1 + q)t) \diamond N(Sz, Sz, qt); q \in (0,1)$$

$$M(Sz, Bv_n, kt) \geq M(Bv_n, Sz, t) \text{ and } N(Sz, Bv_n, kt) \leq N(Bv_n, Sz, t)$$

$$M(Sz, Bv_n, kt) \rightarrow 1 \text{ and } N(Sz, Bv_n, kt) \rightarrow 0$$

Therefore $Bv_n \rightarrow Sz$. Also $ASu_n = Tv_n$ implies $Tv_n \rightarrow Sz$

Again by (C2)¹ substitute $Su_n = z$, we get

$$M(Az, Bv_n, kt) \geq M(ASu_n, Sz, t) * M(Az, Sz, t) * M(Bv_n, ASu_n, t) \\ * M(Bv_n, Sz, (1+q)t) * M(Az, ASu_n, qt); q \in (0,1)$$

and

$$N(Az, Bv_n, kt) \leq N(ASu_n, Sz, t) \diamond N(Az, Sz, t) \diamond N(Bv_n, ASu_n, t) \\ \diamond N(Bv_n, Sz, (1+q)t) \diamond N(Az, ASu_n, qt)$$

$$M(Az, Bv_n, kt) \geq M(Az, Sz, t) * M(Az, Sz, t) * M(Bv_n, Az, t) \\ * M(Bv_n, Sz, (1+q)t) * M(Az, Az, qt)$$

and

$$N(Az, Bv_n, kt) \leq N(Az, Sz, t) \diamond N(Az, Sz, t) \diamond N(Bv_n, Az, t) \\ \diamond N(Bv_n, Sz, (1+q)t) \diamond N(Az, Az, qt)$$

$$M(Az, Bv_n, kt) \geq M(Az, Sz, t) * (Bv_n, Az, t) \text{ and } N(Az, Bv_n, kt) \leq N(Az, Sz, t) \diamond N(Bv_n, Az, t)$$

$$M(Az, Bv_n, kt) \geq M(Az, Sz, t) \text{ and } N(Az, Bv_n, kt) \leq N(Az, Sz, t)$$

By Lemma 2.14, $Az = Sz$. But $ASu_n \rightarrow Sz = Az$, implies $ASu_n \rightarrow Az$ and $SAu_n \rightarrow Sz$

Hence (A, S) is reciprocally continuous on X.

Similarly, if the pair (B, T) is compatible and T is continuous then the proof is similar. This completes the proof.

Now we prove our main theorem.

Theorem 3.1 Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space with $t * t \geq t$ and $(1-t) \diamond (1-t) \leq (1-t) \forall t \in [0,1]$. Further, let A, B, S and T be four self mappings of X satisfying (C1) and (C2)¹. If the pairs (A, S) and (B, T) are compatible and one of the mappings in compatible pairs (A, S) or (B, T) is continuous, then A, B, S and T have a unique common fixed point.

Proof By (C1) since $A(X) \subset T(X)$, for any point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point $x_1 \in X$, we can choose a point x_2 in X such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that for $n = 0, 1, 2, \dots$

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

In (C2)¹ substitute $x = x_{2n+1}$; $y = x_{2n+1}$

$$M(Ax_{2n+2}, Bx_{2n+1}, kt) \geq M(Tx_{2n+1}, Sx_{2n+2}, t) * M(Ax_{2n+2}, SBx_{2n+2}, t) * M(Bx_{2n+1}, Tx_{2n+1}, t) * M(Bx_{2n+1}, Sx_{2n+2}, (1+q)t) * M(Ax_{2n+1}, Tx_{2n+1}, qt)$$

and

$$N(Ax_{2n+2}, Bx_{2n+1}, kt) \leq N(Tx_{2n+1}, Sx_{2n+2}, t) \diamond N(Ax_{2n+2}, SBx_{2n+2}, t) \diamond N(Bx_{2n+1}, Tx_{2n+1}, t) \diamond N(Bx_{2n+1}, Sx_{2n+2}, (1+q)t) \diamond N(Ax_{2n+1}, Tx_{2n+1}, qt)$$

$$M(y_{2n+2}, y_{2n+1}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n+1}, y_{2n+1}, (1+q)t) * M(y_{2n+1}, y_{2n}, qt)$$

and

$$N(y_{2n+2}, y_{2n+1}, kt) \leq N(y_{2n}, y_{2n+1}, t) \diamond N(y_{2n+2}, y_{2n+1}, t) \diamond N(y_{2n+1}, y_{2n}, t) \diamond N(y_{2n+1}, y_{2n+1}, (1+q)t) \diamond N(y_{2n+1}, y_{2n}, qt)$$

$$M(y_{2n+2}, y_{2n+1}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, qt) \rightarrow (3.1.1)$$

and

$$N(y_{2n+2}, y_{2n+1}, kt) \leq N(y_{2n}, y_{2n+1}, t) \diamond N(y_{2n+2}, y_{2n+1}, t) \diamond N(y_{2n+1}, y_{2n}, qt) \rightarrow (3.1.2)$$

$$(3.1.1) \text{ implies } M(y_{2n+2}, y_{2n+1}, kt) \geq M(y_{2n+1}, y_{2n}, qt)$$

$$\text{So that } M(y_{2n+2}, y_{2n+1}, \frac{k}{q}qt) \geq M(y_{2n+1}, y_{2n}, qt)$$

$$(3.1.2) \text{ implies } N(y_{2n+2}, y_{2n+1}, kt) \leq N(y_{2n+1}, y_{2n}, qt)$$

$$N(y_{2n+2}, y_{2n+1}, (k/q)qt) \leq N(y_{2n+1}, y_{2n}, qt)$$

Case-I: Suppose $y_n = y_{n+1}$ for some n

Then $y_0 = Ax_0 = Tx_1$ and $y_1 = Bx_1 = Sx_2$ so that $Ax_0 = Tx_1 = Bx_1 = Sx_2$

Now $y_2 = Ax_2 = Tx_3$; put $x = x_2$; $y = x_1$ in (C2)¹

$$M(Ax_2, Bx_1, kt) \geq M(Tx_1, Sx_2, t) * M(Ax_2, Sx_2, t) * M(Bx_1, Tx_1, t) * M(Bx_1, Sx_2, (1+q)t) * M(Ax_2, Tx_1, qt)$$

and

$$N(Ax_2, Bx_1, kt) \leq N(Tx_1, Sx_2, t) \diamond N(Ax_2, Sx_2, t) \diamond N(Bx_1, Tx_1, t) \diamond N(Bx_1, Sx_2, (1+q)t) \diamond N(Ax_2, Tx_1, qt)$$

$$M(y_2, y_1, kt) \geq M(y_0, y_1, t) * M(y_2, y_1, t) * M(y_1, y_0, t) * M(y_1, y_1, (1 + q)t) * M(y_2, y_1, qt)$$

and

$$N(y_2, y_1, kt) \leq N(y_0, y_1, t) \diamond N(y_2, y_1, t) \diamond N(y_1, y_0, t) \diamond N(y_1, y_1, (1 + q)t) \diamond N(y_2, y_1, qt);$$

$$M(y_2, y_1, kt) \geq M(y_2, y_1, t) * M(y_2, y_1, qt) \text{ and } N(y_2, y_1, kt) \leq N(y_2, y_1, t) \diamond N(y_2, y_1, qt)$$

$$M(y_2, y_1, kt) \geq M(y_2, y_1, qt) \text{ and } N(y_2, y_1, kt) \leq N(y_2, y_1, t)$$

Therefore $y_2 = y_1$, and so on $y_3 = y_2$; i.e the sequence converges to y_0

Since (A,S) are compatible $ASx_2 = SAx_2 \Rightarrow ABx_1 = SBx_1$ since $Sx_2 = Bx_1$ and $Ax_2 = Bx_1$

Put $x = Bx_1, y = x_1$ in (C2)¹

$$M(ABx_1, Bx_1, kt) \geq M(Tx_1, SBx_1, t) * M(ABx_1, SBx_1, t) * M(Bx_1, Tx_1, t) * M(Bx_1, SBx_1, (1 + q)t) * M(ABx_1, Tx_1, qt)$$

and

$$(ABx_1, Bx_1, kt) \leq N(Tx_1, SBx_1, t) \diamond N(ABx_1, SBx_1, t) \diamond N(Bx_1, Tx_1, t) \diamond N(Bx_1, SBx_1, (1 + q)t) \diamond N(ABx_1, Tx_1, qt)$$

$$M(ABx_1, Bx_1, kt) \geq M(Bx_1, ABx_1, t) * M(ABx_1, ABx_1, t) * M(Bx_1, Bx_1, t) * M(Bx_1, ABx_1, (1 + q)t) * M(ABx_1, Bx_1, qt)$$

and

$$N(ABx_1, Bx_1, kt) \leq N(Bx_1, ABx_1, t) \diamond N(ABx_1, ABx_1, t) \diamond N(Bx_1, Bx_1, t) \diamond N(Bx_1, ABx_1, (1 + q)t) \diamond N(ABx_1, Bx_1, qt)$$

$$M(ABx_1, Bx_1, kt) \geq M(ABx_1, Bx_1, qt) \text{ And } N(ABx_1, Bx_1, kt) \leq N(ABx_1, Bx_1, qt) \text{ since } 0 < k < q$$

Therefore $ABx_1 = Bx_1$. Hence Bx_1 is fixed point of A.

Since (B, T) are compatible $BBx_1 = TBx_1$ (since $Bx_1 = Tx_1$)

Therefore $SBx_1 = ABx_1 = Bx_1 \Rightarrow Bx_1$ is a fixed point of S. But $Ax_0 = Tx_1 = Bx_1 = Sx_2 = Ax_2$ and $ABx_1 = Bx_1$.

$$\Rightarrow ATx_1 = Tx_1, ASx_2 = Sx_2; AAx_2 = Ax_2, AAx_0 = Ax_0$$

Now put $y = Bx_1$ and $x = Bx_1$, in (C2)¹ put $x = y$, then

$$M(ABx_1, BBx_1, kt) \geq M(TBx_1, SBx_1, t) * M(ABx_1, SBx_1, t) * M(BBx_1, T, Bx_1, t) * M(BBx_1, SBx_1, (1 + q)t) * M(ABx_1, TBx_1, qt)$$

and

$$N(ABx_1, BBx_1, kt) \leq N(TBx_1, SBx_1, t) \diamond N(ABx_1, SBx_1, t) \diamond N(BBx_1T, Bx_1, t) \diamond N(BBx_1, SBx_1, (1+q)t) \diamond N(ABx_1, TBx_1, qt)$$

$$M(Bx_1, BBx_1, kt) \geq M(TBx_1, SBx_1, t) * M(Bx_1, SBx_1, t) * M(BBx_1T, Bx_1, t) * M(BBx_1, SBx_1, (1+q)t) * M(Bx_1, TBx_1, qt)$$

and

$$N(Bx_1, BBx_1, kt) \leq N(TBx_1, SBx_1, t) \diamond N(Bx_1, SBx_1, t) \diamond N(BBx_1T, Bx_1, t) \diamond N(BBx_1, SBx_1, (1+q)t) \diamond N(Bx_1, TBx_1, qt)$$

$$M(Bx_1, BBx_1, kt) \geq M(Bx_1, TBx_1, qt) \text{ and } N(Bx_1, BBx_1, kt) \leq N(Bx_1, TBx_1, qt)$$

$$M(Bx_1, BBx_1, kt) \geq M(Bx_1, BBx_1, qt) \text{ and } N(Bx_1, BBx_1, kt) \leq N(Bx_1, BBx_1, qt)$$

Therefore $BBx_1 = Bx_1$. Hence Bx_1 is a fixed point of B. Similarly Bx_1 is affixed point of T.

Therefore Bx_1 is a common fixed point for A, B, S and T and we are through.

CASE II Assume that $y_n \neq y_{n+1} \forall n$. Then

$$M(y_{m+1}, y_m, kt) \geq M(y_m, y_{m-1}, qt) \text{ and } N(y_{m+1}, y_m, kt) \leq N(y_m, y_{m-1}, qt)$$

By continuing, from Lemma 2.1.3 $\{y_n\}$ is convergent. Let $\{y_n\} \rightarrow z$.

Now suppose (A, S) is compatible and S is continuous then by Lemma 3.1 A and S are reciprocally continuous so that $ASx_n \rightarrow Az$ and $SAx_n \rightarrow Sz$

As (A, S) is compatible, we have

$$\lim_{n \rightarrow \infty} M(ASx_n, SAx_n, t) = 1 \text{ and } \lim_{n \rightarrow \infty} N(ASx_n, SAx_n, t) = 0$$

$$\lim_{n \rightarrow \infty} M(Az, Sz, t) = 1 \text{ and } \lim_{n \rightarrow \infty} N(Az, Sz, t) = 0$$

Hence $Az = Sz$. Since $A(X) \subset T(X)$ there exists $p \in X$ such that $Az = Tp = Sz$

In (C2)¹ put $x = z$ and $y = p$

$$M(Az, Bp, kt) \geq M(Tp, Sz, t) * M(Az, Sz, t) * M(Bp, Tp, t) * M(Bp, Sz, (1+q)t) * M(Az, Tp, qt)$$

and

$$N(Az, Bp, kt) \geq N(Tp, Sz, t) \diamond N(Az, Sz, t) \diamond N(Bp, Tp, t) \diamond N(Bp, Sz, (1+q)t) \diamond N(Az, Tp, qt)$$

$$M(Az, Bp, kt) \geq M(Az, Bp, t) \text{ and } N(Az, Bp, kt) \leq N(Az, Bp, t) \forall t > 0$$

Therefore $Az = Bp$, thus $Az = Bp = Sz = Tp$.

In (C2)¹ put $x = Az$; $y = p$

$$\begin{aligned} M(AAz, Bp, kt) &\geq M(Tp, SAz, t) * M(AAz, Sz, t) * M(Bp, Tp, t) * M(Bp, SAz, (1 + q)t) \\ &* M(AAz, Tp, qt) \end{aligned}$$

and $N(AAz, Bp, kt) \geq N(Tp, SAz, t) \diamond N(AAz, SAz, t) \diamond N(Bp, Tp, t) \diamond N(Bp, SAz, (1 + q)t) \diamond N(AAz, Tp, qt)$

$$M(AAz, Bp, kt) \geq M(Tp, SAz, t) * M(AAz, Bp, qt)$$

$$N(AAz, Bp, kt) \leq N(Tp, SAz, t) \diamond N(AAz, Bp, qt)$$

$$M(AAz, Bp, kt) \geq M(AAz, Bp, qt) \text{ and } N(AAz, Bp, kt) \leq N(Bp, AAz, t)$$

Then $AAz = Bp = Az$ implies $AAz = Az = SAz$. Therefore Az is a common fixed point for A and S

Similarly by (C2)¹ we get that Az is a common fixed point of B and T . Hence Az is a common fixed point of A, B, S and T .

Uniqueness: Let x and y be two fixed points of A, B, S and T . Then by (C2)¹

$$M(Ax, By, kt) \geq M(Ty, Sx, t) \text{ \& } N(Ax, By, kt) \leq N(Sx, Ty, qt)$$

Therefore $x = y$.

Now we show that the result of Manro [12] follows as a corollary of our theorem

Corollary 3.1: If (C2) in Theorem 2.1 is replaced by (C2)¹ in theorem 3.1, then A, B, S and T have unique fixed point.

Proof: Since (C2) implies (C2)¹ the result follows.

Note: It may be noted that the pointwise R-weak commutativity of the pairs (A, S) and (B, T) imposed in theorem 2.1 is successfully avoided.

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