Periodic Solution for Non-linear System of Integro-differential Equations Of Volterra Type

[1] Raad N. Butris
University Of Duhok
Faculty of Educational Science
School of Basic Education
Mathematics department

Abstract

In this paper, we investigate the existence and approximation of periodic solutions of non-linear systems of integro-differential equations of Volterra type by using the numerical-analytic method which were introduced by Samoilenko.

The study of such integro-differential equations leads to extend the results obtained by Butris for changing the system of non-linear integro- differential equations to periodic system of non-linear integro-differential equations of the Volterra type.

1. Introduction

In recent years, Samoilenko assumes that the numerical analytic method to study the periodic solutions for ordinary differential equations and their algorithm structure. In the original works of Samoilenko [12], the approach used and described here had been referred to as the numerical-analytic based upon successive approximations. The idea of the method , originally aimed at the investigation of periodic solutions only ,had been later applied in studies [1,4,7,9] .

Also, it should be noted that appropriate versions of the method considered can be applied in many situations for handling periodic in the case of systems   of first or second order ordinary differential equations, integro-differential equations, equations with retarded arguments, systems containing unknown parameters, and countable systems of differential equations. A survey of the investigations on the subject can be found in the studies and researches [3,6,8,10] .

In this paper ,we investigate the existence and approximation of the periodic solutions for non-linear system   of integro-differential equations. The numerical-analytic method is used to study the periodic solutions of ordinary differential equations introduced by Samoilenko [12] .

Consider the following system of non-linear integro-differential equations which has the form :

$$\frac{dx(t)}{dt} = (A + B(t))x(t) + f(t, x(t), \sum_{i=1}^{\infty} (\int_{-\infty}^{t} G(t,s)\dot{x}(s)ds)^{i}) \qquad (1.1)$$

where $x \in D \subseteq R^{n}$, D is a closed and bounded domain.

Let the vector function $f(t,x,y) = (f_1(t,x,y), f_2(t,x,y),...,f_n(t,x,y))$,

is defined and continuous on the domain ,

$$(t, x, y) \in R^1 \times D \times D_1 = (-\infty, \infty) \times D \times D_1 , \qquad (1.2)$$

where D_1 is a bounded domain subset of Euclidean space R^m.

Let $G(t, s)$ is $(n \times n)$ continuous positive matrix and periodic in t, s of period T provided that :

$$\left. \begin{array}{l} \int_{-\infty}^{t} \|G(t,s)\| ds \leq K , \ K > 0 \\[2mm] S_3 = \sum_{i=1}^{\infty} K^i M_4^{i-1} \\[2mm] S_4 = \sum_{i=1}^{\infty} i K^i M_4^{i-1} \end{array} \right\} \qquad (1.3)$$

where S_3 and S_4 are convergent series.

We define the non-empty sets as follows:

$$\left. \begin{array}{l} D_f = D - M_3 \\[2mm] D_{1f} = D_1 - M_4 \end{array} \right\} \qquad (1.4)$$

and

$$V = [(H + L_1)B_1(t)Q + L_2 S_4 Q_1] < 1, \qquad (1.5)$$

where $M_3 = \dfrac{T}{2} Q(H\delta_0 + M)$ and $M_4 = QN\delta_0 + Q_1(H\delta_0 + M)$.

By using lemma 3.1[12], we can state and prove the following lemma

Lemma 1. *Let the vector function $f(t, x, y)$ be defined and continuous in interval [0, T], then the inequality*

$$\|K_1(t, x_0)\| \leq B_1(t)Q(H\delta_0 + M) \qquad (1.6)$$

Satisfies for $0 \leq t \leq T$ where $B_1(t) \leq \dfrac{T}{2}$

and

$$B_1(t) = \left[\frac{t(2e^{\|A\|(T-t)} - e^{\|A\|} - \|E\|) + T(e^{\|A\|T} - e^{\|A\|(T-t)})}{e^{\|A\|T} - \|E\|} \right]$$

Provided that:

$$K_1(t,x_0) = \int_0^t e^{A(t-s)}[B(s)x_0 + f(s,x_0) - \frac{A}{e^{AT} - E}\int_0^T e^{A(T-s)}[B(s)x_0 + f(s,x_0)]ds\,]ds$$

(For the proof see [1]).

2. Approximate Solution of (1.1)

In this section, we study the periodic solution for the system (1.1) by proving the following theorem .

Theorem 1. *If the system (1.1) satisfies the inequalities (1.3), (1.4) and Conditions (1.5), (1.6) has a periodic solution $x = x(t,x_0)$, then the sequence of functions:*

$$x_{m+1}(t,x_0) = x_0 e^{At} + \int_0^t e^{A(t-s)}[B(s)x_m(s,x_0) + f(s,x_m(s,x_0), \sum_{i=1}^{\infty}(\int_{-\infty}^s G(s,\tau)\dot{x}_m(\tau,x_0)d\tau)^i)$$

$$- \frac{A}{e^{AT} - E}\int_0^T e^{A(T-s)}[B(s)x_m(s,x_0) + f(s,x_m(s,x_0), \sum_{i=1}^{\infty}(\int_{-\infty}^s G(s,\tau)\dot{x}_m(\tau,x_0)d\tau)^i)]ds\,]ds$$

$$(2.1)$$

with $x_0(t,x_0) = x_0 e^{At}$, m= 0 , 1, 2 ,...

is periodic in t of period T , and uniformly convergent as $m \to \infty$ in the domain:

$$(t,x_0) \in [0,T] \times D_f \qquad\qquad (2.2)$$

to the limit function $x_\infty(t,x_0)$ defined in the domain (2.2) which is periodic in t of period T satisfying the system of integral equations:

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$$x(t,x_0) = x_0 e^{At} + \int_0^t e^{A(t-s)}[B(s)x(s,x_0) + f(s,x(s,x_0), \sum_{i=1}^\infty (\int_{-\infty}^s G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)$$

$$-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x(s,x_0) + f(s,x(s,x_0), \sum_{i=1}^\infty (\int_{-\infty}^s G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)]ds\,]ds$$

$$(2.3)$$

which is a unique solution on the domain (2.2) , provided that :

$$\|x_\infty(t,x_0) - x_0\| \le M_3 \tag{2.4}$$

and

$$\|x_\infty(t,x_0) - x_m(t,x_0)\| \le B_1(t)QV_1 V^{m-1}(1-V)^{-1} \tag{2.5}$$

for all $m \ge 1$ and $t \in R^1$.

Proof.

   Now, by using Lemma 1 and the sequence of functions (2.1) when m= 0, we get:

$$\|x_1(t,x_0) - x_0\| = \| \ x_0 e^{At} + \int_0^t e^{A(t-s)}[B(s)x_0 + f(s,x_0,0) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_0$$

$$+f(s,x_0,0)]ds\,]ds - x_0 e^{At}\|$$

$$\le \left[\|E\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\right]\int_0^t \|e^{A(t-s)}\|[\|B(s)\|\|x_0\| + \|f(s,x_0,0)\|]ds$$

$$+\left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\int_t^T \|e^{A(t-s)}\|[\|B(s)\|\|x_0\| + \|f(s,x_0,0)\|]ds$$

$$\|x_1(t,x_0) - x_0\| \le \left[\frac{t(2e^{\|A\|(T-t)} - e^{\|A\|} - \|E\|) + T(e^{\|A\|T} - e^{\|A\|(T-t)})}{e^{\|A\|T} - \|E\|}\right]Q[H\delta_0 + M]$$

$$\le B_1(t)Q[H\delta_0 + M]$$

$$\le \frac{T}{2}Q[H\delta_0 + M] = M_3 \qquad ,$$

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and hence,

$$\|x_1(t,x_0) - x_0\| \le M_3 \qquad (2.6)$$

i.e. $x_1(t,x_0) \in D$ *, for all* $t \in R^1$, $x_0 \in D_f$ *.*

Suppose that $x_p(t,x_0) \in D$ *, for all* $x_0 \in D_f$ *,* $p \in Z^+$,

when $m = p+1$ *, we find that:*

$$\|x_{p+1}(t,x_0) - x_0\| \le M_3$$

i.e. $x_{p+1}(t,x_0) \in D$ *, for all* $t \in R^1$ *,* $x_0 \in D_f$ *,* $p \in Z^+$.

Thus, by mathematical induction , we get the following inequality:

$$\|x_m(t,x_0) - x_0\| \le M_3 \qquad (2.7)$$

i.e. $x_m(t,x_0) \in D$ *, for all* $x_0 \in D_f$ *, and* $m = 0,1,2,...$

In addition to that, we find that:

$$\|\dot{x}_1(t,x_0)\| = \| x_0 A e^{At} + e^{A(t-s)}[B(t)x_0 + f(t,x_0,0) - \frac{A}{e^{AT} - E}\int_0^T e^{A(T-s)}[B(s)x_0$$

$$+ f(s,x_0,0)]ds]\|$$

$$\le \|x_0\|\|A\|\|e^{At}\| + \|e^{A(t-s)}\|[\|B(t)\|\|x_0\| + \|f(t,x_0,0)\|$$

$$+ \frac{\|A\|}{e^{\|A\|T} - \|E\|}\int_0^T \|e^{A(T-s)}\|[\|B(s)\|\|x_0\| + \|f(s,x_0,0)\|]ds]$$

$$\le \delta_0 NQ + (Q + \frac{\|A\|TQ^2}{e^{\|A\|T} - \|E\|})[H\delta_0 + M]$$

$$\le \delta_0 NQ + Q_1[H\delta_0 + M] = M_4$$

and hence ,

$$\|\dot{x}_1(t,x_0)\| \le M_4 \qquad (2.8)$$

i.e. $\dot{x}_1(t,x_0) \in D_1$ *, for all* $x_0 \in D_f$ *.*

where

$$Q_1 = Q + \frac{\|A\| T Q^2}{e^{\|A\|T} - \|E\|}$$

Also by induction, we find that:

$$\|\dot{x}_m(t, x_0)\| \leq M_4 \qquad\qquad (2.9)$$

where

$$\dot{x}_{m+1}(t, x_0) = x_0 A e^{At} + e^{A(t-s)}[B(t)x_m(t, x_0) + f(t, x_m(t, x_0), \sum_{i=1}^{\infty} (\int_{-\infty}^{t} G(t, s)\dot{x}_m(s, x_0)ds)^i)$$

$$- \frac{A}{e^{AT} - E} \int_0^T e^{A(T-s)}[B(s)x_m(s, x_0) + f(s, x_m(s, x_0), \sum_{i=1}^{\infty} (\int_{-\infty}^{s} G(s, \tau)\dot{x}_m(\tau, x_0)d\tau)^i)]ds$$

$$(2.10)$$

for all   m= 0,1,2,...

We claim that the sequence of function (2.1) is uniformly convergent on the domain (2.2).

By using Lemma 1 , and putting m=1 in (2.1) , we have:

$$\|x_2(t, x_0) - x_1(t, x_0)\| = \|x_0 e^{At} + \int_0^t e^{A(t-s)}[B(s)x_1(s, x_0) +$$

$$+ f(s, x_1(s, x_0), \sum_{i=1}^{\infty} (\int_{-\infty}^{s} G(s, \tau)\dot{x}_1(\tau, x_0)d\tau)^i) - \frac{A}{e^{AT} - E} \int_0^T e^{A(T-s)}[B(s)x_1(s, x_0)$$

$$+ f(s, x_1(s, x_0), \sum_{i=1}^{\infty} (\int_{-\infty}^{s} G(s, \tau)\dot{x}_1(\tau, x_0)d\tau)^i)]ds]ds - x_0 e^{At} - \int_0^t e^{A(t-s)}[B(s)x_0$$

$$+ f(s, x_0, 0) - \frac{A}{e^{AT} - E} \int_0^T e^{A(T-s)}[B(s)x_0 + f(s, x_0, 0)]ds]ds\|$$

$$\|x_2(t, x_0) - x_1(t, x_0)\| \leq \left[\|E\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\right] \int_0^t \|e^{A(t-s)}\|[\|B(s)\|\|x_1(s, x_0) - x_0\|$$

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$$+L_1\|x_1(s,x_0)-x_0\|+L_2(\sum_{i=1}^{\infty}K^iM_4^{i-1})\|\dot{x}_1(s,x_0)\|]ds$$

$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^T\|e^{A(t-s)}\|[\|B(s)\|\|x_1(s,x_0)-x_0\|$$

$$+L_1\|x_1(s,x_0)-x_0\|+L_2(\sum_{i=1}^{\infty}K^iM_4^{i-1})\|\dot{x}_1(s,x_0)\|]ds$$

$$\|x_2(t,x_0)-x_1(t,x_0)\|\le\left[\|E\|-\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\right]\int_0^tQ[H\|x_1(s,x_0)-x_0\|+L_1\|x_1(s,x_0)-x_0\|$$

$$+L_2S_3\|\dot{x}_1(s,x_0)\|]ds+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^TQ[H\|x_1(s,x_0)-x_0\|$$

$$+L_1\|x_1(s,x_0)-x_0\|+L_2S_3\|\dot{x}_1(s,x_0)\|]ds$$

$$\|x_2(t,x_0)-x_1(t,x_0)\|\le B_1(t)Q(H+L_1)M_3+B_1(t)QL_2S_3M_4$$

or

$$\|x_2(t,x_0)-x_1(t,x_0)\|\le B_1(t)Q[(H+L_1)M_3+L_2S_3M_4]$$

let $\quad V_1=[(H+L_1)M_3+L_2S_3M_4]$

so ,

$$\|x_2(t,x_0)-x_1(t,x_0)\|\le B_1(t)QV_1\quad.\tag{2.11}$$

   Also , when m=1 in (2.10) , we have :

$$\|\dot{x}_2(t,x_0)-\dot{x}_1(t,x_0)\|=\|x_0Ae^{At}+e^{A(t-s)}[B(t)x_1(t,x_0)$$

$$+f(t,x_1(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^tG(t,s)\dot{x}_1(s,x_0)ds)^i)-\frac{A}{e^{AT}-E}\int_0^Te^{A(T-s)}[B(s)x_1(s,x_0)$$

$$+f(t,x_1(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^tG(t,s)\dot{x}_1(s,x_0)ds)^i)]ds]-x_0Ae^{At}-e^{A(t-s)}[B(t)x_0$$

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$$+f(t,x_0,0) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_0 + f(s,x_0,0)]ds \,]\|$$

$$\|\dot{x}_2(t,x_0) - \dot{x}_1(t,x_0)\| \leq \|e^{A(t-s)}\|[\|B(t)\|\|x_1(t,x_0)-x_0\| + L_1\|x_1(t,x_0)-x_0\|$$

$$+L_2(\sum_{i=1}^{\infty} K^i M_4^{i-1})\|\dot{x}_1(t,x_0)\| + \frac{\|A\|}{e^{\|A\|T}-\|E\|}\int_0^T \|e^{A(T-s)}\|[\|B(s)\|\|x_1(s,x_0)-x_0\|$$

$$+L_1\|x_1(s,x_0)-x_0\| + L_2(\sum_{i=1}^{\infty} K^i M_4^{i-1})\|\dot{x}_1(s,x_0)\|]ds\,]$$

so,

$$\|\dot{x}_2(t,x_0) - \dot{x}_1(t,x_0)\| \leq Q[(H+L_1)M_3 + L_2 S_3 M_4 + \frac{\|A\|TQ}{e^{\|A\|T}-\|E\|}[(H+L_1)M_3 + L_2 S_3 M_4]]$$

$$\|\dot{x}_2(t,x_0) - \dot{x}_1(t,x_0)\| \leq (Q + \frac{\|A\|TQ^2}{e^{\|A\|T}-\|E\|})[(H+L_1)M_3 + L_2 S_3 M_4]$$

and hence,

$$\|\dot{x}_2(t,x_0) - \dot{x}_1(t,x_0)\| \leq Q_1 V_1 \; . \tag{2.12}$$

Now , when m=2 in the sequence of functions (2.1) , we get :

$$\|x_3(t,x_0) - x_2(t,x_0)\| = \|x_0 e^{At} + \int_0^t e^{A(t-s)}[B(s)x_2(s,x_0)$$

$$+f(s,x_2(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^s G(s,\tau)\dot{x}_2(\tau,x_0)d\tau)^i) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_2(s,x_0)$$

$$+f(s,x_2(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^s G(s,\tau)\dot{x}_2(\tau,x_0)d\tau)^i)]ds\,]ds - x_0 e^{At} - \int_0^t e^{A(t-s)}[B(s)x_1(s,x_0)$$

$$+f(s,x_1(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^s G(s,\tau)\dot{x}_1(\tau,x_0)d\tau)^i) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_1(s,x_0)$$

$$+f(s,x_1(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^s G(s,\tau)\dot{x}_1(\tau,x_0)d\tau)^i)]ds\,]ds\| \quad ,$$

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so, by using Lemma 1, we find :

$$\|x_3(t,x_0) - x_2(t,x_0)\| \leq \left[\|E\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\right]\int_0^t \|e^{A(t-s)}\|[\|B(s)\|\|x_2(s,x_0) - x_1(s,x_0)\|$$

$$+ L_1\|x_2(s,x_0) - x_1(s,x_0)\| + L_2(\sum_{i=1}^{\infty} iK^i M_4^{i-1})\|\dot{x}_2(s,x_0) - \dot{x}_1(s,x_0)\|]ds$$

$$+ \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\int_t^T \|e^{A(t-s)}\|[\|B(s)\|\|x_2(s,x_0) - x_1(s,x_0)\|$$

$$+ L_1\|x_2(s,x_0) - x_1(s,x_0)\| + L_2(\sum_{i=1}^{\infty} iK^i M_4^{i-1})\|\dot{x}_2(s,x_0) - \dot{x}_1(s,x_0)\|]ds$$

$$\|x_3(t,x_0) - x_2(t,x_0)\| \leq \left[\|E\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\right]\int_0^t Q[(H + L_1)\|x_2(s,x_0) - x_1(s,x_0)\|$$

$$+ L_2 S_4\|\dot{x}_2(s,x_0) - \dot{x}_1(s,x_0)\|]ds$$

$$+ \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\int_t^T Q[(H + L_1)\|x_2(s,x_0) - x_1(s,x_0)\|$$

$$+ L_2 S_4\|\dot{x}_2(s,x_0) - \dot{x}_1(s,x_0)\|]ds \quad,$$

so, we have:

$$\|x_3(t,x_0) - x_2(t,x_0)\| \leq B_1(t)Q[B_1(t)QV_1 H + B_1(t)QV_1 L_1 + L_2 S_4 Q_1 V_1]$$

$$\leq B_1(t)QV_1[(H + L_1)B_1(t)Q + L_2 S_4 Q_1]$$

$$\leq B_1(t)QV_1 V \ ,$$

and hence,

$$\|x_3(t,x_0) - x_2(t,x_0)\| \leq B_1(t)QV_1 V \ ,$$

where $\quad S_4 = \sum_{i=1}^{\infty} iK^i M_4^{i-1} \quad , \quad V = [(H + L_1)B_1(t)Q + L_2 S_4 Q_1] \ ,$

and with the same , we find :

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$$\left\| \dot{x}_3(t,x_0) - \dot{x}_2(t,x_0) \right\| \le \left\| e^{A(t-s)} \right\| [\| B(t) \| \| x_2(t,x_0) - x_1(t,x_0) \| + L_1 \| x_2(t,x_0) - x_1(t,x_0) \|$$

$$+ L_2 S_4 \left\| \dot{x}_2(t,x_0) - \dot{x}_1(t,x_0) \right\| + \frac{\| A \|}{e^{\| A \| T} - \| E \|} \int_0^T \left\| e^{A(T-s)} \right\| [\| B(s) \| \| x_2(s,x_0) - x_1(s,x_0) \|$$

$$+ L_1 \| x_2(s,x_0) - x_1(s,x_0) \| + L_2 S_4 \left\| \dot{x}_2(s,x_0) - \dot{x}_1(s,x_0) \right\|] ds]$$

So, by using inequalities (3.2.11) and (3.2.12), we get:

$$\left\| \dot{x}_3(t,x_0) - \dot{x}_2(t,x_0) \right\| \le Q[(H + L_1)B_1(t)QV_1 + L_2 S_4 Q_1 V_1$$

$$+ \frac{\| A \| TQ}{e^{\| A \| T} - \| E \|}[(H + L_1)B_1(t)QV_1 + L_2 S_4 Q_1 V_1]]$$

$$\le Q_1 V_1[(H + L_1)B_1(t)Q + L_2 S_4 Q_1]$$

$$\le Q_1 V_1 V \ ,$$

and hence ,

$$\left\| \dot{x}_3(t,x_0) - \dot{x}_2(t,x_0) \right\| \le Q_1 V_1 V \quad .$$

Suppose that the inequality

$$\left\| x_{p+1}(t,x_0) - x_p(t,x_0) \right\| \le B_1(t)QV_1 V^{\,p-1} \ ,$$

and

$$\left\| \dot{x}_{p+1}(t,x_0) - \dot{x}_p(t,x_0) \right\| \le Q_1 V_1 V^{\,p-1} \ ,$$

are holds for m=p ,then we have to prove the following inequality :

$$\left\| x_{p+2}(t,x_0) - x_{p+1}(t,x_0) \right\| \le B_1(t)QV_1 V^{\,p} \quad . \tag{2.13}$$

Now also by using lemma 1, we find :

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$$\left\| x_{p+2}(t,x_0) - x_{p+1}(t,x_0) \right\| \le \left[\left\| E \right\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|} \right) \right] \int_0^t \left\| e^{A(t-s)} \right\| \left[\left\| B(s) \right\| \left\| x_{p+1}(s,x_0) - x_p(s,x_0) \right\| \right.$$

$$+ L_1 \left\| x_{p+1}(s,x_0) - x_p(s,x_0) \right\| + L_2 \left(\sum_{i=1}^{\infty} i K^i M_4^{i-1} \right) \left\| \dot{x}_{p+1}(s,x_0) - \dot{x}_p(s,x_0) \right\| \right] ds$$

$$+ \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|} \right) \int_t^T \left\| e^{A(t-s)} \right\| \left[\left\| B(s) \right\| \left\| x_{p+1}(s,x_0) - x_p(s,x_0) \right\| \right.$$

$$+ L_1 \left\| x_{p+1}(s,x_0) - x_p(s,x_0) \right\| + L_2 \left(\sum_{i=1}^{\infty} i K^i M_4^{i-1} \right) \left\| \dot{x}_{p+1}(s,x_0) - \dot{x}_p(s,x_0) \right\| \right] ds$$

So, we have:

$$\left\| x_{p+2}(t,x_0) - x_{p+1}(t,x_0) \right\| \le B_1(t) Q [H B_1(t) Q V_1 V^{P-1} + L_1 B_1(t) Q V_1 V^{P-1} + L_2 S_4 Q_1 V_1 V^{P-1}]$$

$$\le B_1(t) Q V_1 V^{P-1} [(H + L_1) B_1(t) Q + L_2 S_4 Q_1]$$

$$\le B_1(t) Q V_1 V^P,$$

and hence ,

$$\left\| x_{p+2}(t,x_0) - x_{p+1}(t,x_0) \right\| \le B_1(t) Q V_1 V^P \quad , \qquad (2.14)$$

thus, by induction the following inequality holds:

$$\left\| x_{m+1}(t,x_0) - x_m(t,x_0) \right\| \le B_1(t) Q V_1 V^{m-1} \qquad , \qquad (2.15)$$

where $\quad V = [(H + L_1) B_1(t) Q + L_2 S_4 Q_1] \quad , m = 1, 2, 3,...$

From (2.15), we conclude that for k > 1 ,we have the inequality :

$$\left\| x_{m+k}(t,x_0) - x_m(t,x_0) \right\| = \sum_{j=0}^{\infty} \left\| x_{m+1+j}(t,x_0) - x_{m+j}(t,x_0) \right\| ,$$

such that

$$\left\| x_{m+k}(t,x_0) - x_m(t,x_0) \right\| \le \sum_{j=0}^{\infty} \left\| x_{m+1+j}(t,x_0) - x_{m+j}(t,x_0) \right\|$$

$$\le \sum_{j=0}^{\infty} B_1(t) Q V_1 V^{m-1+j}$$

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$$\leq B_1(t)QV_1V^{m-1}\sum_{j=0}^{\infty}V^j$$

$$\leq B_1(t)QV_1(1-V)^{-1}V^{m-1},$$

and hence,

$$\left\| x_{m+k}(t,x_0) - x_m(t,x_0) \right\| \leq B_1(t)QV_1(1-V)^{-1}V^{m-1}, \qquad (2.16)$$

for all $t \in R^1$ *, and* $x_0 \in D_f$ *.*

Since $V < 1$ *and* $\lim_{m \to \infty} V^{m-1} = 0$ *, where* $m= 1, 2, 3,..., $ *, so that the right side of*

(3.2.16) tends to zero and, therefore, the sequence of functions (2.1) is convergent

uniformly on the domain (2.2) to the limit function $x_{\infty}(t,x_0)$ *which*

is defined on the same domain .

Let

$$\lim_{m \to \infty} x_m(t,x_0) = x_{\infty}(t,x_0) \qquad (2.17)$$

Also , by the lemma 1 , and the inequality (2.16) the inequalities (2.4) and (2.5) hold for all $m \geq 1$ *.*

By using the relation (2.17) and proceeding in (2.1) to the limit, when $m \to \infty$, *this shows that the limiting function* $x_{\infty}(t,x_0)$ *is the periodic solution of*

the integral equation (2.3).

Finally, we have to prove that $x(t,x_0)$ *is a unique solution of (1.1),*

assume that $r(t,x_0)$ *is another solution for the system (1.1), i.e.*

$$r(t,x_0) = x_0e^{At} + \int_0^t e^{A(t-s)}[B(s)r(s,x_0) + f(s,r(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^s G(s,\tau)\dot{r}(\tau,x_0)d\tau)^i)$$

$$-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)r(s,x_0) + f(s,r(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^s G(s,\tau)r(\tau,x_0)d\tau)^i)]ds]ds.$$

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Now, we find the difference between them:

$$\|x(t,x_0)-r(t,x_0)\| = \|x_0 e^{At} + \int_0^t e^{A(t-s)}[B(s)x(s,x_0)$$

$$+f(s,x(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x(s,x_0)$$

$$+f(s,x(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)]ds]ds - x_0 e^{At} - \int_0^t e^{A(t-s)}[B(s)r(s,x_0)$$

$$+f(s,r(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{r}(\tau,x_0)d\tau)^i) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)r(s,x_0)$$

$$+f(s,r(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{r}(\tau,x_0)d\tau)^i)]ds]ds\| \quad .$$

Thus

$$\|x(t,x_0)-r(t,x_0)\| \leq \left[\|E\| - \left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\right]\int_0^t \|e^{A(t-s)}\|[\|B(s)\|\|x(s,x_0)-r(s,x_0)\|$$

$$+L_1\|x(s,x_0)-r(s,x_0)\| + L_2(\sum_{i=1}^{\infty}iK^iM_4^{i-1})\|\dot{x}(s,x_0)-\dot{r}(s,x_0)\|]ds$$

$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^T \|e^{A(t-s)}\|[\|B(s)\|\|x(s,x_0)-r(s,x_0)\|$$

$$+L_1\|x(s,x_0)-r(s,x_0)\| + L_2(\sum_{i=1}^{\infty}iK^iM_4^{i-1})\|\dot{x}(s,x_0)-\dot{r}(s,x_0)\|]ds$$

$$\|x(t,x_0)-r(t,x_0)\| \leq \left[\|E\| - \left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\right]\int_0^t Q[H\|x(s,x_0)-r(s,x_0)\|$$

$$+L_1\|x(s,x_0)-r(s,x_0)\| + L_2 S_4\|\dot{x}(s,x_0)-\dot{r}(s,x_0)\|]ds$$

$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^T Q[H\|x(s,x_0)-r(s,x_0)\|$$

$$+L_1\|x(s,x_0)-r(s,x_0)\| + L_2 S_4\|\dot{x}(s,x_0)-\dot{r}(s,x_0)\|]ds \ ,$$

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and hence,

$$\|x(t,x_0) - r(t,x_0)\| \le B_1(t)Q[(H+L_1)\|x(t,x_0) - r(t,x_0)\| + L_2S_4\|\dot{x}(t,x_0) - \dot{r}(t,x_0)\|]$$

(3.2.18)

   Also, we find :

$$\|\dot{x}(t,x_0) - \dot{r}(t,x_0)\| \le \|e^{A(t-s)}\|[\|B(t)\|\|x(t,x_0) - r(t,x_0)\| + L_1\|x(t,x_0) - r(t,x_0)\|$$

$$+ L_2S_4\|\dot{x}(t,x_0) - \dot{r}(t,x_0)\| + \frac{\|A\|}{e^{\|A\|T} - \|E\|}\int_0^T \|e^{A(T-s)}\|[\|B(s)\|\|x(s,x_0) - r(s,x_0)\|$$

$$+ L_1\|x(s,x_0) - r(s,x_0)\| + L_2S_4\|\dot{x}(s,x_0) - \dot{r}(s,x_0)\|]ds\,]$$

$$\le Q[(H+L_1)\|x(t,x_0) - r(t,x_0)\| + L_2S_4\|\dot{x}(t,x_0) - \dot{r}(t,x_0)\|$$

$$+ \frac{\|A\|TQ}{e^{\|A\|T} - \|E\|}[(H+L_1)\|x(t,x_0) - r(t,x_0)\| + L_2S_4\|\dot{x}(t,x_0) - \dot{r}(t,x_0)\|]]$$

$$\le (Q + \frac{\|A\|TQ^2}{e^{\|A\|T} - \|E\|})[(H+L_1)\|x(t,x_0) - r(t,x_0)\| + L_2S_4\|\dot{x}(t,x_0) - \dot{r}(t,x_0)\|],$$

and hence,

$$\|\dot{x}(t,x_0) - \dot{r}(t,x_0)\| \le Q_1[(H+L_1)\|x(t,x_0) - r(t,x_0)\| + L_2S_4\|\dot{x}(t,x_0) - \dot{r}(t,x_0)\|].$$

(2.19)

   From inequality (2.18) and (2.19), we obtain:

$$\|x(t,x_0) - r(t,x_0)\| \le B_1(t)Q[(H+L_1)\|x(t,x_0) - r(t,x_0)\| + L_2S_4\|\dot{x}(t,x_0) - \dot{r}(t,x_0)\|]$$

$$\le B_1(t)Q\{(H+L_1)B_1(t)Q[(H+L_1)\|x(t,x_0) - r(t,x_0)\| + L_2S_4\|\dot{x}(t,x_0) - \dot{r}(t,x_0)\|] +$$

$$+ L_2S_4Q_1[(H+L_1)\|x(t,x_0) - r(t,x_0)\| + L_2S_4\|\dot{x}(t,x_0) - \dot{r}(t,x_0)\|]\}$$

$$\le B_1(t)Q\big((H+L_1)B_1(t)Q + L_2S_4Q_1\big)[(H+L_1)\|x(t,x_0) - r(t,x_0)\| +$$

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$$+L_2 S_4 \left\| \dot{x}(t,x_0) - \dot{r}(t,x_0) \right\|]$$

$\leq C_1 V$,

and hence ,

$$\left\| x(t,x_0) - r(t,x_0) \right\| \leq C_1 V \quad ,$$

where $\quad C_1 = B_1(t) Q [(H + L_1) \left\| x(t,x_0) - r(t,x_0) \right\| + L_2 S_4 \left\| \dot{x}(t,x_0) - \dot{r}(t,x_0) \right\|]$,

so, by induction , we get :

$$\left\| x(t,x_0) - r(t,x_0) \right\| \leq C_1 V^m \qquad\qquad\qquad (2.20)$$

From inequality (2.20) and by using (1.4), when $m \to \infty$, $V^m \to 0$, *we obtain:*

$\left\| x(t,x_0) - r(t,x_0) \right\| \leq 0$, *thus, we find that:*

$x(t,x_0) = r(t,x_0)$,

hence, $x(t,x_0)$ *is a unique periodic solution of non-linear integro-differential*

equation(1.1) , for all $t \in R^1$, $x_0 \in D_f$.

3. Existence of *a* periodic Solution of the System (1.1).

The problem of existence of periodic solution of period T of the system (1.1) is uniquely connected with the existence of zero of the function $\Delta(x_0)$

which has the form:

$$\Delta : D_f \to R^n$$

$$\Delta(x_0) = \frac{A}{e^{AT} - E} \int_0^T e^{A(T-t)} [B(t) x_\infty(t,x_0) + f(t,x_\infty(t,x_0), \sum_{i=1}^\infty (\int_{-\infty}^t G(t,s)\dot{x}_\infty(s,x_0)ds)^i)]dt$$

$$(3.1)$$

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where $x_{\infty}(t, x_0)$ is the limiting function of the sequence of functions $x_m(t, x_0)$.

$$\Delta_m : D_f \rightarrow R^n$$

$$\Delta_m(x_0) = \frac{A}{e^{AT} - E} \int_0^T e^{A(T-t)}[B(t)x_m(t,x_0) + f(t, x_m(t,x_0), \sum_{i=1}^{\infty}(\int_{-\infty}^t G(t,s)\dot{x}_m(s,x_0)ds)^i)]dt$$

(3.2)

where $m= 0, 1, 2, 3,...$

Theorem2. . *Let all assumptions and conditions of theorem1. be satisfied, then the following inequality hold :*

$$\|\Delta(x_0) - \Delta_m(x_0)\| \leq N_1 N_2 Q^2 (H + L_1) B_1(t) V^{m-1}(1-V)^{-1} V_1 = \rho_m \quad , \quad (3.3)$$

where $\quad N_1 = \dfrac{\|A\|T}{e^{\|A\|T} - \|E\|} \quad , \quad N_2 = [1 + L_2 S_4 Q_1 (1 - Q_1 L_2 S_4)^{-1}]$ and $Q_1 = (Q + N_1 Q^2)$

Proof. *By using the relations (3.1) and (3.2) ,we have :*

$$\|\Delta(x_0) - \Delta_m(x_0)\| \leq (\frac{\|A\|}{e^{\|A\|T} - \|E\|})\int_0^T \|e^{A(T-t)}\|[\|B(t)\|\|x_{\infty}(t,x_0) - x_m(t,x_0)\|$$

$$+ L_1\|x_{\infty}(t,x_0) - x_m(t,x_0)\| + L_2(\sum_{i=1}^{\infty} iK^i M_4^{i-1})\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\|]dt$$

$$\|\Delta(x_0) - \Delta_m(x_0)\| \leq (\frac{\|A\|}{e^{\|A\|T} - \|E\|})\int_0^T Q[(H + L_1)\|x_{\infty}(t,x_0) - x_m(t,x_0)\|$$

$$+ L_2 S_4\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\|]dt$$

$$\leq (\frac{\|A\|T}{e^{\|A\|T} - \|E\|})Q[(H + L_1)\|x_{\infty}(t,x_0) - x_m(t,x_0)\| + L_2 S_4\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\|]$$

$$\leq N_1 Q[(H + L_1)\|x_{\infty}(t,x_0) - x_m(t,x_0)\| + L_2 S_4\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\|]$$

so,

$$\|\Delta(x_0) - \Delta_m(x_0)\| \leq N_1 Q[(H + L_1)\|x_{\infty}(t,x_0) - x_m(t,x_0)\| + L_2 S_4\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\|]$$

(3.4)

  Also

$$\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\| \leq Q[(H + L_1)\|x_{\infty}(t,x_0) - x_m(t,x_0)\| + L_2 S_4\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\|$$

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$$+N_1Q[(H+L_1)\|x_\infty(t,x_0)-x_m(t,x_0)\|+L_2S_4\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\|]]$$

$$\leq (Q+N_1Q^2)[(H+L_1)\|x_\infty(t,x_0)-x_m(t,x_0)\|+L_2S_4\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\|]$$

$$\leq Q_1[(H+L_1)\|x_\infty(t,x_0)-x_m(t,x_0)\|+L_2S_4\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\|]$$

and hence,

$$\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\|\leq Q_1[(H+L_1)\|x_\infty(t,x_0)-x_m(t,x_0)\|+L_2S_4\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\|]$$

From the last inequality ,we have:

$$\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\|\leq (1-Q_1L_2S_4)^{-1}Q_1(H+L_1)\|x_\infty(t,x_0)-x_m(t,x_0)\| \ ,$$

$$(3.5)$$

By substituting inequality (3.5) in (3.4) , we get :

$$\|\Delta(x_0)-\Delta_m(x_0)\|\leq N_1Q(H+L_1)\|x_\infty(t,x_0)-x_m(t,x_0)\|+$$

$$+N_1Q(H+L_1)L_2S_4Q_1(1-Q_1L_2S_4)^{-1}\|x_\infty(t,x_0)-x_m(t,x_0)\|$$

And hence:

$$\|\Delta(x_0)-\Delta_m(x_0)\|\leq N_1Q(H+L_1)B_1(t)QV_1V^{m-1}(1-V)^{-1}$$

$$+N_1Q(H+L_1)L_2S_4Q_1(1-Q_1L_2S_4)^{-1}B_1(t)QV_1V^{m-1}(1-V)^{-1}$$

$$\leq N_1N_2Q^2(H+L_1)B_1(t)V_1V^{m-1}(1-V)^{-1} \ ,$$

Thus

$$\|\Delta(x_0)-\Delta_m(x_0)\|\leq N_1N_2Q^2(H+L_1)B_1(t)V_1V^{m-1}(1-V)^{-1}=\rho_m \ ,$$

I.e. the inequality (3.3) will be satisfied for all $m\geq 1$.

Theorem 3. *Let the system (1.1) be defined in the interval $[a,b]$ on R^1 and periodic in t of period T . Suppose that the sequence of functions (2.1) satisfies the inequalities:*

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$$\left. \begin{array}{l} \min_{a+M_3 \leq x_0 \leq b-M_3} \Delta_m(x_0) \leq -\rho_m \quad , \\[2ex] \max_{a+M_3 \leq x_0 \leq b-M_3} \Delta_m(x_0) \leq \rho_m \quad . \end{array} \right\} \qquad (3.6)$$

then , the system (1.1) has a periodic solution $x(t,x_0)$ *for which*

$x_0 \in [a+M_3, b-M_3]$,

where $\rho_m = N_1 N_2 Q^2 (H+L_1) B_1(t) V_1 V^{m-1} (1-V)^{-1}$, $M_3 = \dfrac{T}{2} Q(H\delta_0 + M)$.

Proof. *Let* x_1, x_2 *be any two points in the interval* $[a+M_3, b-M_3]$ *such that :*

$$\left. \begin{array}{l} \Delta_m(x_1) = \min_{a+M_3 \leq x_0 \leq b-M_3} \Delta_m(x_0) \quad , \\[2ex] \Delta_m(x_2) = \max_{a+M_3 \leq x_0 \leq b-M_3} \Delta_m(x_0) \quad . \end{array} \right\} \qquad (3.7)$$

By using the inequalities (3.3.3) and (3.3.6) ,we have :

$$\left. \begin{array}{l} \Delta(x_1) = \Delta_m(x_1) + (\Delta(x_1) - \Delta_m(x_1)) \leq 0 \quad , \\[1ex] \Delta(x_2) = \Delta_m(x_2) + (\Delta(x_2) - \Delta_m(x_2)) \geq 0 \quad . \end{array} \right\} \qquad (3.8)$$

It follows from the inequalities (3.3.8) and the continuity of the function

$\Delta(x_0)$ *that there exists an isolated singular point* $(x_\infty) = (x_0)$, $x_\infty \in [x_1, x_2]$,

such that $\Delta(x_0) = 0$. *This means that the system (3.3.1) has a periodic solution*

$x(t,x_0)$ *for which* $x_0 \in [a+M_3, b-M_3]$.

Remark 1[5]. *Theorem 2 is proved when* x_0 *is a scalar singular point which should be isolated, thus we have :*

Theorem 3. *If the function* $\Delta(x_0)$ *is defined by* $\Delta : D_f \to R^n$, *and*

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$$\Delta(x_0) = \frac{A}{e^{AT} - E} \int_0^T e^{A(T-t)} [B(t)x_\infty(t,x_0) + f(t,x_\infty(t,x_0), \sum_{i=1}^\infty (\int_{-\infty}^t G(t,s)\dot{x}_\infty(s,x_0)ds)^i)]dt$$

(3.9)

where $x_\infty(t,x_0)$ is a limit of the sequence of functions (2.1), then the following

inequalities hold :

$$\|\Delta(x_0)\| \le M_6 \quad , \tag{3.10}$$

where

$$M_5 = (1 - \frac{T}{2}QH)^{-1} \quad , \quad M_6 = N_1Q(H\delta_0QM_5 + \frac{T}{2}QHMM_5 + M)$$

and

$$\|\Delta(x_0^1) - \Delta(x_0^2)\| \le (F_5 F_7(Q + \frac{T}{2}E_3QNF_4) + F_6)\|x_0^1 - x_0^2\| \tag{3.11}$$

is satisfies for all x_0, x_0^1, x_0^2 , where $E_1 = Q(H + L_1)$, $E_3 = QL_2S_4$.

Proof. *From the properties of the function $x_\infty(t,x_0)$ as in the theorem 1, the*

function $\Delta(x_0)$ is continuous and bounded by M_6 in the domain (2.2).

From the relation (3.9), we find that:

$$\|\Delta(x_0)\| \le \frac{\|A\|}{e^{\|A\|T} - \|E\|} \int_0^T \|e^{A(T-t)}\|[\|B(t)\|\|x_\infty(t,x_0)\|$$

$$+ \|f(t,x_\infty(t,x_0), \sum_{i=1}^\infty (\int_{-\infty}^t G(t,s)\dot{x}_\infty(s,x_0)ds)^i)\|]dt$$

so, we have :

$$\|\Delta(x_0)\| \le \frac{\|A\|T}{e^{\|A\|T} - \|E\|}Q[H\|x_\infty(t,x_0)\| + M]$$

$$\|\Delta(x_0)\| \le \frac{\|A\|T}{e^{\|A\|T} - \|E\|}QH\|x_\infty(t,x_0)\| + \frac{\|A\|T}{e^{\|A\|T} - \|E\|}QM \tag{3.12}$$

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Since the function $x_\infty(t, x_0)$ satisfies the integral equation (2.3), then we have:

$$\|x_\infty(t, x_0)\| = \|x_0\|\|e^{At}\| + \| \int_0^t e^{A(t-s)}[B(s)x_\infty(s, x_0)$$

$$+f(s, x_\infty(s, x_0), \sum_{i=1}^\infty (\int_{-\infty}^s G(s, \tau)\dot{x}_\infty(\tau, x_0)d\tau)^i) - \frac{A}{e^{AT} - E}\int_0^T e^{A(T-s)}[B(s)x_\infty(s, x_0)$$

$$+f(s, x_\infty(s, x_0), \sum_{i=1}^\infty (\int_{-\infty}^s G(s, \tau)\dot{x}_\infty(\tau, x_0)d\tau)^i)]ds]ds\| .$$

Now , by using lemma 3.1.1 , we get :

$$\|x_\infty(t, x_0)\| = \delta_0 Q + \frac{T}{2}Q[H\|x_\infty(s, x_0)\| + M]$$

so,

$$\|x_\infty(t, x_0)\| = \delta_0 Q M_5 + \frac{T}{2}Q M M_5 \quad . \qquad (3.13)$$

Thus, substituting inequality (3.13) in (2.12) , we get the inequality (3.10).

By using relation (3.9) , we have :

$$\|\Delta(x_0^1) - \Delta(x_0^2)\| = \| \frac{A}{e^{AT} - E}\int_0^T e^{A(T-t)}[B(t)x_\infty(t, x_0^1)$$

$$+f(t, x_\infty(t, x_0^1), \sum_{i=1}^\infty (\int_{-\infty}^t G(t, s)\dot{x}_\infty(s, x_0^1)ds)^i)]dt - \frac{A}{e^{AT} - E}\int_0^T e^{A(T-t)}[B(t)x_\infty(t, x_0^2)$$

$$+f(t, x_\infty(t, x_0^2), \sum_{i=1}^\infty (\int_{-\infty}^t G(t, s)\dot{x}_\infty(s, x_0^2)ds)^i)]dt\|$$

$$\|\Delta(x_0^1) - \Delta(x_0^2)\| \le \frac{\|A\|T}{e^{\|A\|T} - \|E\|}Q[H\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\| + L_1\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\|$$

$$+L_2 S_4\|\dot{x}_\infty(t, x_0^1) - \dot{x}_\infty(t, x_0^2)\|]$$

so,

$$\|\Delta(x_0^1) - \Delta(x_0^2)\| \le N_1 E_1\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\| + N_1 E_3\|\dot{x}_\infty(t, x_0^1) - \dot{x}_\infty(t, x_0^2)\| \quad .$$

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$$(3.14)$$

Now, we find that:

$$\left\| \dot{x}_{\infty}(t,x_0^1) - \dot{x}_{\infty}(t,x_0^2) \right\| \le \left\| x_0^1 - x_0^2 \right\| \left\| A \right\| \left\| e^{At} \right\| + \left\| e^{A(t-s)} \right\| [\left\| B(t) \right\| \left\| x_{\infty}(t,x_0^1) - x_{\infty}(t,x_0^2) \right\|$$

$$+ L_1 \left\| x_{\infty}(t,x_0^1) - x_{\infty}(t,x_0^2) \right\| + L_2 S_4 \left\| \dot{x}_{\infty}(t,x_0^1) - \dot{x}_{\infty}(t,x_0^2) \right\|]$$

$$+ \frac{\left\| A \right\|}{e^{\|A\|T} - \|E\|} \int_0^T \left\| e^{A(T-s)} \right\| [\left\| B(s) \right\| \left\| x_{\infty}(s,x_0^1) - x_{\infty}(s,x_0^2) \right\|$$

$$+ L_1 \left\| x_{\infty}(t,x_0^1) - x_{\infty}(t,x_0^2) \right\| + L_2 S_4 \left\| \dot{x}_{\infty}(t,x_0^1) - \dot{x}_{\infty}(t,x_0^2) \right\|] ds$$

$$\left\| \dot{x}_{\infty}(t,x_0^1) - \dot{x}_{\infty}(t,x_0^2) \right\| \le \left\| x_0^1 - x_0^2 \right\| NQ + Q[(H + L_1) \left\| x_{\infty}(t,x_0^1) - x_{\infty}(t,x_0^2) \right\|$$

$$+ L_2 S_4 \left\| \dot{x}_{\infty}(t,x_0^1) - \dot{x}_{\infty}(t,x_0^2) \right\| + N_1 Q[(H + L_1) \left\| x_{\infty}(s,x_0^1) - x_{\infty}(s,x_0^2) \right\|$$

$$+ L_2 S_4 \left\| \dot{x}_{\infty}(t,x_0^1) - \dot{x}_{\infty}(t,x_0^2) \right\|]]$$

$$\le \left\| x_0^1 - x_0^2 \right\| NQ + Q_1 [(H + L_1) \left\| x_{\infty}(t,x_0^1) - x_{\infty}(t,x_0^2) \right\|$$

$$+ L_2 S_4 \left\| \dot{x}_{\infty}(t,x_0^1) - \dot{x}_{\infty}(t,x_0^2) \right\|],$$

and hence ,

$$\left\| \dot{x}_{\infty}(t,x_0^1) - \dot{x}_{\infty}(t,x_0^2) \right\| \le \left\| x_0^1 - x_0^2 \right\| NQ(1 - Q_1 L_2 S_4)^{-1}$$

$$+ Q_1(H + L_1)(1 - Q_1 L_2 S_4)^{-1} \left\| x_{\infty}(t,x_0^1) - x_{\infty}(t,x_0^2) \right\|$$

$$\left\| \dot{x}_{\infty}(t,x_0^1) - \dot{x}_{\infty}(t,x_0^2) \right\| \le \left\| x_0^1 - x_0^2 \right\| NQF_4 + F_4 Q_1(H + L_1) \left\| x_{\infty}(t,x_0^1) - x_{\infty}(t,x_0^2) \right\| ,$$

$$(3.15)$$

where $F_4 = (1 - Q_1 L_2 S_4)^{-1}$.

By substituting inequality (3.15) in (3.14) , we get :

$$\left\| \Delta(x_0^1) - \Delta(x_0^2) \right\| \le N_1 E_1 \left\| x_{\infty}(t,x_0^1) - x_{\infty}(t,x_0^2) \right\| + N_1 E_3 NQF_4 \left\| x_0^1 - x_0^2 \right\|$$

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$$+ N_1 E_3 F_4 Q_1 (H + L_1) \left\| x_\infty (t, x_0^1) - x_\infty (t, x_0^2) \right\|$$

$$\leq (N_1 E_1 + N_1 E_3 Q_1 F_4 (H + L_1)) \left\| x_\infty (t, x_0^1) - x_\infty (t, x_0^2) \right\| + N_1 E_3 N Q F_4 \left\| x_0^1 - x_0^2 \right\|.$$

Putting

$$F_5 = (N_1 E_1 + N_1 E_3 Q_1 F_4 (H + L_1)) \quad and \quad F_6 = N_1 E_3 N Q F_4$$

so, the last inequality becomes :

$$\left\| \Delta (x_0^1) - \Delta (x_0^2) \right\| \leq F_5 \left\| x_\infty (t, x_0^1) - x_\infty (t, x_0^2) \right\| + F_6 \left\| x_0^1 - x_0^2 \right\| , \qquad (3.16)$$

where $x_\infty (t, x_0^1)$ and $x_\infty (t, x_0^2)$ are the solutions of the integral equation :

$$x(t, x_0^k) = x_0^k e^{At} + \int_0^t e^{A(t-s)} [B(s) x(s, x_0^k) + f(s, x(s, x_0^k), \sum_{i=1}^\infty (\int_{-\infty}^s G(s, \tau) \dot{x}(\tau, x_0^k) d\tau)^i)$$

$$- \frac{A}{e^{AT} - E} \int_0^T e^{A(T-s)} [B(s) x(s, x_0^k) + f(s, x(s, x_0^k), \sum_{i=1}^\infty (\int_{-\infty}^s G(s, \tau) \dot{x}(\tau, x_0^k) d\tau)^i)] ds] ds$$

$$(3.17)$$

with

$$x_0^k (t, x_0) = x_0^k \quad , \text{ where } k = 1, 2 .$$

From the equation (3.17) and by using lemma 1, we have:

$$\left\| x_\infty (t, x_0^1) - x_\infty (t, x_0^2) \right\| \leq \left\| x_0^1 - x_0^2 \right\| \left\| e^{At} \right\|$$

$$+ \left[\left\| E \right\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|} \right) \right] \int_0^t \left\| e^{A(t-s)} \right\| [\left\| B(s) \right\| \left\| x_\infty (s, x_0^1) - x_\infty (s, x_0^2) \right\|$$

$$+ L_1 \left\| x_\infty (s, x_0^1) - x_\infty (s, x_0^2) \right\| + L_2 S_4 \left\| \dot{x}_\infty (s, x_0^1) - \dot{x}_\infty (s, x_0^2) \right\|] ds$$

$$+ \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|} \right) \int_t^T \left\| e^{A(t-s)} \right\| [\left\| B(s) \right\| \left\| x_\infty (s, x_0^1) - x_\infty (s, x_0^2) \right\|$$

$$+ L_1 \left\| x_\infty (s, x_0^1) - x_\infty (s, x_0^2) \right\| + L_2 S_4 \left\| \dot{x}_\infty (s, x_0^1) - \dot{x}_\infty (s, x_0^2) \right\|] ds$$

so that :

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$$\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\| \le \left\|x_0^1-x_0^2\right\|Q + \frac{T}{2}Q(H+L_1)\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|$$

$$+\frac{T}{2}QL_2S_4\left\|\dot{x}_\infty(s,x_0^1)-\dot{x}_\infty(s,x_0^2)\right\| \quad . \qquad (3.18)$$

Now, by substituting inequality (3.15) in (3.18), we get:

$$\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\| \le \left\|x_0^1-x_0^2\right\|Q + \frac{T}{2}Q(H+L_1)\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|$$

$$+\frac{T}{2}Q^2L_2S_4NF_4\left\|x_0^1-x_0^2\right\| + \frac{T}{2}L_2S_4F_4Q_1Q(H+L_1)\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|$$

$$\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\| \le (\frac{T}{2}L_2S_4F_4Q_1E_1 + \frac{T}{2}E_1)\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|$$

$$+(\frac{T}{2}E_3QNF_4+Q)\left\|x_0^1-x_0^2\right\|$$

so,

$$\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\| \le (1-(\frac{T}{2}L_2S_4F_4Q_1E_1 + \frac{T}{2}E_1))^{-1}(\frac{T}{2}E_3QNF_4+Q)\left\|x_0^1-x_0^2\right\| ,$$

$$\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\| \le F_7(\frac{T}{2}E_3QNF_4+Q)\left\|x_0^1-x_0^2\right\| \quad . \qquad (3.19)$$

where

$$F_7 = (1-(\frac{T}{2}L_2S_4F_4Q_1E_1 + \frac{T}{2}E_1))^{-1}$$

Also, substituting inequality (3.19) in (3.16) , we get the inequality   (3.11).

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