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Periodic Solution for Non-linear System of Integro-differential Equations Of Volterra Type

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Abstract

In this paper, we investigate the existence and approximation of periodic solutions of non-linear systems of integro-differential equations of Volterra type by using the numerical-analytic method which were introduced by Samoilenko.

The study of such integro-differential equations leads to extend the results obtained by Butris for changing the system of non-linear integro- differential equations to periodic system of non-linear integro-differential equations of the Volterra type.

1. Introduction

In recent years, Samoilenko assumes that the numerical analytic method to study the periodic solutions for ordinary differential equations and their algorithm structure. In the original works of Samoilenko [12], the approach used and described here had been referred to as the numerical-analytic based upon successive approximations. The idea of the method , originally aimed at the investigation of periodic solutions only ,had been later applied in studies [1,4,7,9] .

Also, it should be noted that appropriate versions of the method considered can be applied in many situations for handling periodic in the case of systems of first or second order ordinary differential equations, integro-differential equations, equations with retarded arguments, systems containing unknown parameters, and countable systems of differential equations. A survey of the investigations on the subject can be found in the studies and researches [3,6,8,10] .

In this paper ,we investigate the existence and approximation of the periodic solutions for non-linear system of integro-differential equations. The numerical-analytic method is used to study the periodic solutions of ordinary differential equations introduced by Samoilenko [12] .

Consider the following system of non-linear integro-differential equations which has the form :

$$\frac{dx(t)}{dt} = (A + B(t))x(t) + f\left(t, x(t), \sum_{i=1}^{\infty}\left(\int_{-\infty}^{t} G(t,s)\dot{x}(s)ds\right)^i\right) \qquad (1.1)$$

where $x \in D \subseteq R^n$ *, D is a closed and bounded domain.*

Let the vector function $f(t,x,y) = (f_1(t,x,y), f_2(t,x,y), \ldots, f_n(t,x,y))$, *is defined and continuous on the domain ,*

$$(t,x,y)\in R^1\times D\times D_1=(-\infty,\infty)\times D\times D_1\ , \tag{1.2}$$

where D_1 *is a bounded domain subset of Euclidean space* R^m.

Let G (t, s) is $(n\times n)$ *continuous positive matrix and periodic in t, s of period T provided that :*

$$\left.\begin{aligned}&\int_{-\infty}^{t}\|G(t,s)\|ds\le K\ ,\ K>0\\&S_3=\sum_{i=1}^{\infty}K^iM_4^{i-1}\\&S_4=\sum_{i=1}^{\infty}iK^iM_4^{i-1}\end{aligned}\right\} \tag{1.3}$$

where S_3 *and* S_4 *are convergent series.*

We define the non-empty sets as follows:

$$\left.\begin{aligned}&D_f=D-M_3\\&D_{1f}=D_1-M_4\end{aligned}\right\} \tag{1.4}$$

and

$$V=[(H+L_1)B_1(t)Q+L_2S_4Q_1]<1, \tag{1.5}$$

where $M_3=\frac{T}{2}Q(H\delta_0+M)$ *and* $M_4=QN\delta_0+Q_1(H\delta_0+M)$.

By using lemma 3.1[12], we can state and prove the following lemma

Lemma 1. *Let the vector function* $f(t,x,y)$ *be defined and continuous in interval [0, T], then the inequality*

$$\|K_1(t,x_0)\|\le B_1(t)Q(H\delta_0+M) \tag{1.6}$$

Satisfies for $0\le t\le T$ *where* $B_1(t)\le\frac{T}{2}$

and

$$B_1(t) = \left[\frac{t\,(2e^{\|A\|(T-t)} - e^{\|A\|} - \|E\|) + T\,(e^{\|A\|T} - e^{\|A\|(T-t)})}{e^{\|A\|T} - \|E\|} \right]$$

Provided that:

$$K_1(t,x_0) = \int_0^t e^{A(t-s)}[B(s)x_0 + f(s,x_0) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_0 + f(s,x_0)]ds\,]ds$$

(For the proof see [1]).

2. Approximate Solution of (1.1)

In this section, we study the periodic solution for the system (1.1) by proving the following theorem .

Theorem 1. *If the system (1.1) satisfies the inequalities (1.3), (1.4) and Conditions (1.5), (1.6) has a periodic solution* $x = x(t,x_0)$, *then the sequence of functions:*

$$x_{m+1}(t,x_0) = x_0 e^{At} + \int_0^t e^{A(t-s)}[B(s)x_m(s,x_0) + f(s,x_m(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_m(\tau,x_0)d\tau)^i)$$
$$-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_m(s,x_0) + f(s,x_m(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_m(\tau,x_0)d\tau)^i)]ds\,]ds$$

(2.1)

with $x_0(t,x_0) = x_0 e^{At}$ *, m= 0 , 1, 2 ,…*

is periodic in t *of period* T*, and uniformly convergent as* $m \to \infty$ *in the domain:*

$$(t,x_0) \in [0,T] \times D_f \quad (2.2)$$

to the limit function $x_\infty(t,x_0)$ *defined in the domain (2.2) which is periodic in* t *of period* T *satisfying the system of integral equations:*

$$x(t,x_0)=x_0e^{At}+\int_0^t e^{A(t-s)}[B(s)x(s,x_0)+f(s,x(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)$$

$$-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x(s,x_0)+f(s,x(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)]ds]ds \qquad (2.3)$$

which is a unique solution on the domain (2.2) , provided that :

$$\|x_\infty(t,x_0)-x_0\|\le M_3 \qquad (2.4)$$

and

$$\|x_\infty(t,x_0)-x_m(t,x_0)\|\le B_1(t)QV_1V^{m-1}(1-V)^{-1} \qquad (2.5)$$

for all $m\ge 1$ *and* $t\in R^1$.

Proof.

Now, by using Lemma 1 and the sequence of functions (2.1) when m= 0, we get:

$$\|x_1(t,x_0)-x_0\|=\|\; x_0e^{At}+\int_0^t e^{A(t-s)}[B(s)x_0+f(s,x_0,0)-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_0$$

$$+f(s,x_0,0)]ds]ds-x_0e^{At}\|$$

$$\le\left[\|E\|-\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\right]\int_0^t\|e^{A(t-s)}\|[\|B(s)\|\|x_0\|+\|f(s,x_0,0)\|]ds$$

$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^T\|e^{A(t-s)}\|[\|B(s)\|\|x_0\|+\|f(s,x_0,0)\|]ds$$

$$\|x_1(t,x_0)-x_0\|\le\left[\frac{t(2e^{\|A\|(T-t)}-e^{\|A\|}-\|E\|)+T(e^{\|A\|T}-e^{\|A\|(T-t)})}{e^{\|A\|T}-\|E\|}\right]Q[H\delta_0+M]$$

$$\le B_1(t)Q[H\delta_0+M]$$

$$\le\frac{T}{2}Q[H\delta_0+M]=M_3 \quad ,$$

and hence,

$$\|x_1(t,x_0)-x_0\|\leq M_3 \qquad (2.6)$$

i.e. $x_1(t,x_0)\in D$ *, for all* $t\in R^1$, $x_0\in D_f$.

Suppose that $x_p(t,x_0)\in D$ *, for all* $x_0\in D_f$, $p\in Z^+$,

when $m=p+1$ *, we find that:*

$$\|x_{p+1}(t,x_0)-x_0\|\leq M_3$$

i.e. $x_{p+1}(t,x_0)\in D$ *, for all* $t\in R^1$, $x_0\in D_f$, $p\in Z^+$.

Thus, by mathematical induction , we get the following inequality:

$$\|x_m(t,x_0)-x_0\|\leq M_3 \qquad (2.7)$$

i.e. $x_m(t,x_0)\in D$ *, for all* $x_0\in D_f$ *, and* $m=0,1,2,\ldots$

In addition to that, we find that:

$$\|\dot{x}_1(t,x_0)\|=\|\ x_0Ae^{At}+e^{A(t-s)}[B(t)x_0+f(t,x_0,0)-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_0$$

$$+f(s,x_0,0)]ds]\|$$

$$\leq\|x_0\|\|A\|\|e^{At}\|+\|e^{A(t-s)}\|[\|B(t)\|\|x_0\|+\|f(t,x_0,0)\|$$

$$+\frac{\|A\|}{e^{\|A\|T}-\|E\|}\int_0^T\|e^{A(T-s)}\|[\|B(s)\|\|x_0\|+\|f(s,x_0,0)\|]ds]$$

$$\leq\delta_0NQ+(Q+\frac{\|A\|TQ^2}{e^{\|A\|T}-\|E\|})[H\delta_0+M]$$

$$\leq\delta_0NQ+Q_1[H\delta_0+M]=M_4$$

and hence ,

$$\|\dot{x}_1(t,x_0)\|\leq M_4 \qquad (2.8)$$

i.e. $\dot{x}_1(t,x_0)\in D_1$ *, for all* $x_0\in D_f$.

where

$$Q_1 = Q + \frac{\|A\| T Q^2}{e^{\|A\|T} - \|E\|}$$

Also by induction, we find that:

$$\|\dot{x}_m(t,x_0)\| \le M_4 \qquad (2.9)$$

where

$$\dot{x}_{m+1}(t,x_0) = x_0 A e^{At} + e^{A(t-s)}[B(t)x_m(t,x_0) + f(t,x_m(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{t} G(t,s)\dot{x}_m(s,x_0)ds)^i)$$
$$-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_m(s,x_0) + f(s,x_m(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_m(\tau,x_0)d\tau)^i)]ds \qquad (2.10)$$

for all $m = 0,1,2,\ldots$

We claim that the sequence of function (2.1) is uniformly convergent on the domain (2.2).

By using Lemma 1 , and putting m=1 in (2.1) , we have:

$$\|x_2(t,x_0) - x_1(t,x_0)\| = \|x_0 e^{At} + \int_0^t e^{A(t-s)}[B(s)x_1(s,x_0) +$$
$$+f(s,x_1(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_1(\tau,x_0)d\tau)^i) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_1(s,x_0)$$
$$+f(s,x_1(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_1(\tau,x_0)d\tau)^i)]ds\,]ds - x_0 e^{At} - \int_0^t e^{A(t-s)}[B(s)x_0$$
$$+f(s,x_0,0) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_0 + f(s,x_0,0)]ds\,]ds\,\|$$

$$\|x_2(t,x_0) - x_1(t,x_0)\| \le \left[\|E\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\right]\int_0^t \|e^{A(t-s)}\|[\|B(s)\|\|x_1(s,x_0) - x_0\|$$

$$+L_1\|x_1(s,x_0)-x_0\|+L_2(\sum_{i=1}^{\infty}K^iM_4^{i-1})\|\dot{x}_1(s,x_0)\|]ds$$

$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^T\|e^{A(t-s)}\|[\|B(s)\|\|x_1(s,x_0)-x_0\|$$

$$+L_1\|x_1(s,x_0)-x_0\|+L_2(\sum_{i=1}^{\infty}K^iM_4^{i-1})\|\dot{x}_1(s,x_0)\|]ds$$

$$\|x_2(t,x_0)-x_1(t,x_0)\|\le\left[\|E\|-\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\right]\int_0^tQ[H\|x_1(s,x_0)-x_0\|+L_1\|x_1(s,x_0)-x_0\|$$

$$+L_2S_3\|\dot{x}_1(s,x_0)\|]ds+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^TQ[H\|x_1(s,x_0)-x_0\|$$

$$+L_1\|x_1(s,x_0)-x_0\|+L_2S_3\|\dot{x}_1(s,x_0)\|]ds$$

$$\|x_2(t,x_0)-x_1(t,x_0)\|\le B_1(t)Q(H+L_1)M_3+B_1(t)QL_2S_3M_4$$

or

$$\|x_2(t,x_0)-x_1(t,x_0)\|\le B_1(t)Q[(H+L_1)M_3+L_2S_3M_4]$$

let $V_1=[(H+L_1)M_3+L_2S_3M_4]$

so ,

$$\|x_2(t,x_0)-x_1(t,x_0)\|\le B_1(t)QV_1 \quad . \tag{2.11}$$

Also , when m=1 in (2.10) , we have :

$$\|\dot{x}_2(t,x_0)-\dot{x}_1(t,x_0)\|=\|x_0Ae^{At}+e^{A(t-s)}[B(t)x_1(t,x_0)$$

$$+f(t,x_1(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{t}G(t,s)\dot{x}_1(s,x_0)ds)^i)-\frac{A}{e^{AT}-E}\int_0^Te^{A(T-s)}[B(s)x_1(s,x_0)$$

$$+f(t,x_1(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{t}G(t,s)\dot{x}_1(s,x_0)ds)^i)]ds]-x_0Ae^{At}-e^{A(t-s)}[B(t)x_0$$

$$+f(t,x_0,0)-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_0+f(s,x_0,0)]ds\,]\|$$

$$\|\dot{x}_2(t,x_0)-\dot{x}_1(t,x_0)\|\le\|e^{A(t-s)}\|[\|B(t)\|\|x_1(t,x_0)-x_0\|+L_1\|x_1(t,x_0)-x_0\|$$

$$+L_2(\sum_{i=1}^{\infty}K^iM_4^{i-1})\|\dot{x}_1(t,x_0)\|+\frac{\|A\|}{e^{\|A\|T}-\|E\|}\int_0^T\|e^{A(T-s)}\|[\|B(s)\|\|x_1(s,x_0)-x_0\|$$

$$+L_1\|x_1(s,x_0)-x_0\|+L_2(\sum_{i=1}^{\infty}K^iM_4^{i-1})\|\dot{x}_1(s,x_0)\|]ds\,]$$

so,

$$\|\dot{x}_2(t,x_0)-\dot{x}_1(t,x_0)\|\le Q[(H+L_1)M_3+L_2S_3M_4+\frac{\|A\|TQ}{e^{\|A\|T}-\|E\|}[(H+L_1)M_3+L_2S_3M_4]]$$

$$\|\dot{x}_2(t,x_0)-\dot{x}_1(t,x_0)\|\le(Q+\frac{\|A\|TQ^2}{e^{\|A\|T}-\|E\|})[(H+L_1)M_3+L_2S_3M_4]$$

and hence,

$$\|\dot{x}_2(t,x_0)-\dot{x}_1(t,x_0)\|\le Q_1V_1\ . \qquad (2.12)$$

Now , when m=2 in the sequence of functions (2.1) , we get :

$$\|x_3(t,x_0)-x_2(t,x_0)\|=\|x_0e^{At}+\int_0^t e^{A(t-s)}[B(s)x_2(s,x_0)$$

$$+f(s,x_2(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}_2(\tau,x_0)d\tau)^i)-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_2(s,x_0)$$

$$+f(s,x_2(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}_2(\tau,x_0)d\tau)^i)]ds\,]ds-x_0e^{At}-\int_0^t e^{A(t-s)}[B(s)x_1(s,x_0)$$

$$+f(s,x_1(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}_1(\tau,x_0)d\tau)^i)-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_1(s,x_0)$$

$$+f(s,x_1(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}_1(\tau,x_0)d\tau)^i)]ds\,]ds\|\ ,$$

so, by using Lemma 1, we find :

$$\|x_3(t,x_0)-x_2(t,x_0)\| \le \left[\|E\| - \left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\right]\int_0^t \|e^{A(t-s)}\|[\|B(s)\|\|x_2(s,x_0)-x_1(s,x_0)\|$$

$$+L_1\|x_2(s,x_0)-x_1(s,x_0)\| + L_2(\sum_{i=1}^{\infty} iK^i M_4^{i-1})\|\dot{x}_2(s,x_0)-\dot{x}_1(s,x_0)\|]ds$$

$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^T \|e^{A(t-s)}\|[\|B(s)\|\|x_2(s,x_0)-x_1(s,x_0)\|$$

$$+L_1\|x_2(s,x_0)-x_1(s,x_0)\| + L_2(\sum_{i=1}^{\infty} iK^i M_4^{i-1})\|\dot{x}_2(s,x_0)-\dot{x}_1(s,x_0)\|]ds$$

$$\|x_3(t,x_0)-x_2(t,x_0)\| \le \left[\|E\| - \left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\right]\int_0^t Q[(H+L_1)\|x_2(s,x_0)-x_1(s,x_0)\|$$

$$+L_2 S_4\|\dot{x}_2(s,x_0)-\dot{x}_1(s,x_0)\|]ds$$

$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^T Q[(H+L_1)\|x_2(s,x_0)-x_1(s,x_0)\|$$

$$+L_2 S_4\|\dot{x}_2(s,x_0)-\dot{x}_1(s,x_0)\|]ds \quad ,$$

so, we have:

$$\|x_3(t,x_0)-x_2(t,x_0)\| \le B_1(t)Q[B_1(t)QV_1H + B_1(t)QV_1L_1 + L_2S_4Q_1V_1]$$

$$\le B_1(t)QV_1[(H+L_1)B_1(t)Q + L_2S_4Q_1]$$

$$\le B_1(t)QV_1V \ ,$$

and hence,

$$\|x_3(t,x_0)-x_2(t,x_0)\| \le B_1(t)QV_1V \ ,$$

where $\quad S_4 = \sum_{i=1}^{\infty} iK^i M_4^{i-1} \quad , \quad V = [(H+L_1)B_1(t)Q + L_2S_4Q_1] \ ,$

and with the same , we find :

$$\|\dot{x}_3(t,x_0)-\dot{x}_2(t,x_0)\| \le \|e^{A(t-s)}\|[\|B(t)\|\|x_2(t,x_0)-x_1(t,x_0)\|+L_1\|x_2(t,x_0)-x_1(t,x_0)\|$$

$$+L_2S_4\|\dot{x}_2(t,x_0)-\dot{x}_1(t,x_0)\|+\frac{\|A\|}{e^{\|A\|T}-\|E\|}\int_0^T\|e^{A(T-s)}\|[\|B(s)\|\|x_2(s,x_0)-x_1(s,x_0)\|$$

$$+L_1\|x_2(s,x_0)-x_1(s,x_0)\|+L_2S_4\|\dot{x}_2(s,x_0)-\dot{x}_1(s,x_0)\|]ds]$$

So, by using inequalities (3.2.11) and (3.2.12), we get:

$$\|\dot{x}_3(t,x_0)-\dot{x}_2(t,x_0)\| \le Q[(H+L_1)B_1(t)QV_1+L_2S_4Q_1V_1$$

$$+\frac{\|A\|TQ}{e^{\|A\|T}-\|E\|}[(H+L_1)B_1(t)QV_1+L_2S_4Q_1V_1]]$$

$$\le Q_1V_1[(H+L_1)B_1(t)Q+L_2S_4Q_1]$$

$$\le Q_1V_1V\ ,$$

and hence ,

$$\|\dot{x}_3(t,x_0)-\dot{x}_2(t,x_0)\| \le Q_1V_1V\quad .$$

Suppose that the inequality

$$\|x_{p+1}(t,x_0)-x_p(t,x_0)\| \le B_1(t)QV_1V^{p-1}\ ,$$

and

$$\|\dot{x}_{p+1}(t,x_0)-\dot{x}_p(t,x_0)\| \le Q_1V_1V^{p-1}\ ,$$

are holds for m=p ,then we have to prove the following inequality :

$$\|x_{p+2}(t,x_0)-x_{p+1}(t,x_0)\| \le B_1(t)QV_1V^p\quad . \tag{2.13}$$

Now also by using lemma 1, we find :

$$\left\|x_{p+2}(t,x_0)-x_{p+1}(t,x_0)\right\| \le \left[\left\|E\right\|-\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\right]\int_0^t \left\|e^{A(t-s)}\right\|[\left\|B(s)\right\|\left\|x_{p+1}(s,x_0)-x_p(s,x_0)\right\|$$

$$+L_1\left\|x_{p+1}(s,x_0)-x_p(s,x_0)\right\|+L_2(\sum_{i=1}^{\infty} iK^i M_4^{i-1})\left\|\dot{x}_{p+1}(s,x_0)-\dot{x}_p(s,x_0)\right\|]ds$$

$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^T \left\|e^{A(t-s)}\right\|[\left\|B(s)\right\|\left\|x_{p+1}(s,x_0)-x_p(s,x_0)\right\|$$

$$+L_1\left\|x_{p+1}(s,x_0)-x_p(s,x_0)\right\|+L_2(\sum_{i=1}^{\infty} iK^i M_4^{i-1})\left\|\dot{x}_{p+1}(s,x_0)-\dot{x}_p(s,x_0)\right\|]ds$$

So, we have:

$$\left\|x_{p+2}(t,x_0)-x_{p+1}(t,x_0)\right\| \le B_1(t)Q[HB_1(t)QV_1V^{P-1}+L_1B_1(t)QV_1V^{P-1}+L_2S_4Q_1V_1V^{P-1}]$$

$$\le B_1(t)QV_1V^{P-1}[(H+L_1)B_1(t)Q+L_2S_4Q_1]$$

$$\le B_1(t)QV_1V^{P},$$

and hence ,

$$\left\|x_{p+2}(t,x_0)-x_{p+1}(t,x_0)\right\| \le B_1(t)QV_1V^{P} \quad , \qquad (2.14)$$

thus, by induction the following inequality holds:

$$\left\|x_{m+1}(t,x_0)-x_m(t,x_0)\right\| \le B_1(t)QV_1V^{m-1} \quad , \qquad (2.15)$$

where $V=[(H+L_1)B_1(t)Q+L_2S_4Q_1]$ *, m= 1, 2, 3,…*

From (2.15),we conclude that for k > 1 ,we have the inequality :

$$\left\|x_{m+k}(t,x_0)-x_m(t,x_0)\right\| = \sum_{j=0}^{\infty}\left\|x_{m+1+j}(t,x_0)-x_{m+j}(t,x_0)\right\| ,$$

such that

$$\left\|x_{m+k}(t,x_0)-x_m(t,x_0)\right\| \le \sum_{j=0}^{\infty}\left\|x_{m+1+j}(t,x_0)-x_{m+j}(t,x_0)\right\|$$

$$\le \sum_{j=0}^{\infty} B_1(t)QV_1V^{m-1+j}$$

$$\leq B_1(t)QV_1V^{m-1}\sum_{j=0}^{\infty}V^{j}$$

$$\leq B_1(t)QV_1(1-V)^{-1}V^{m-1},$$

and hence,

$$\left\|x_{m+k}(t,x_0)-x_m(t,x_0)\right\|\leq B_1(t)QV_1(1-V)^{-1}V^{m-1} \quad , \qquad (2.16)$$

for all $t\in R^1$ *, and* $x_0\in D_f$.

Since $V<1$ *and* $\lim_{m\to\infty}V^{m-1}=0$ *, where m= 1, 2, 3,..., , so that the right side of (3.2.16) tends to zero and, therefore, the sequence of functions (2.1) is convergent uniformly on the domain (2.2) to the limit function* $x_{\infty}(t,x_0)$ *which is defined on the same domain .*

Let

$$\lim_{m\to\infty}x_m(t,x_0)=x_{\infty}(t,x_0) \qquad (2.17)$$

Also , by the lemma 1 , and the inequality (2.16) the inequalities (2.4) and (2.5) hold for all $m\geq 1$.

By using the relation (2.17) and proceeding in (2.1) to the limit, when m → ∞, this shows that the limiting function $x_{\infty}(t,x_0)$ *is the periodic solution of the integral equation (2.3).*

Finally, we have to prove that $x(t,x_0)$ *is a unique solution of (1.1), assume that* $r(t,x_0)$ *is another solution for the system (1.1), i.e.*

$$r(t,x_0)=x_0e^{At}+\int_0^t e^{A(t-s)}[B(s)r(s,x_0)+f(s,r(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{r}(\tau,x_0)d\tau)^i)$$

$$-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)r(s,x_0)+f(s,r(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)r(\tau,x_0)d\tau)^i)]ds]ds\,.$$

Now, we find the difference between them:

$$\|x(t,x_0)-r(t,x_0)\| = \|x_0e^{At} + \int_0^t e^{A(t-s)}[B(s)x(s,x_0)$$

$$+f(s,x(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x(s,x_0)$$

$$+f(s,x(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)]ds\,]ds - x_0e^{At} - \int_0^t e^{A(t-s)}[B(s)r(s,x_0)$$

$$+f(s,r(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{r}(\tau,x_0)d\tau)^i) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)r(s,x_0)$$

$$+f(s,r(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{r}(\tau,x_0)d\tau)^i)]ds\,]ds\| \ .$$

Thus

$$\|x(t,x_0)-r(t,x_0)\| \le \left[\|E\| - \left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\right]\int_0^t \|e^{A(t-s)}\|[\|B(s)\|\|x(s,x_0)-r(s,x_0)\|$$

$$+L_1\|x(s,x_0)-r(s,x_0)\| + L_2(\sum_{i=1}^{\infty}iK^iM_4^{i-1})\|\dot{x}(s,x_0)-\dot{r}(s,x_0)\|]ds$$

$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^T \|e^{A(t-s)}\|[\|B(s)\|\|x(s,x_0)-r(s,x_0)\|$$

$$+L_1\|x(s,x_0)-r(s,x_0)\| + L_2(\sum_{i=1}^{\infty}iK^iM_4^{i-1})\|\dot{x}(s,x_0)-\dot{r}(s,x_0)\|]ds$$

$$\|x(t,x_0)-r(t,x_0)\| \le \left[\|E\| - \left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\right]\int_0^t Q[H\,\|x(s,x_0)-r(s,x_0)\|$$

$$+L_1\|x(s,x_0)-r(s,x_0)\| + L_2S_4\|\dot{x}(s,x_0)-\dot{r}(s,x_0)\|]ds$$

$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^T Q[H\,\|x(s,x_0)-r(s,x_0)\|$$

$$+L_1\|x(s,x_0)-r(s,x_0)\| + L_2S_4\|\dot{x}(s,x_0)-\dot{r}(s,x_0)\|]ds\ ,$$

and hence,

$$\|x(t,x_0)-r(t,x_0)\| \le B_1(t)Q[(H+L_1)\|x(t,x_0)-r(t,x_0)\|+L_2S_4\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|]$$

(3.2.18)

Also, we find :

$$\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\| \le \|e^{A(t-s)}\|[\|B(t)\|\|x(t,x_0)-r(t,x_0)\|+L_1\|x(t,x_0)-r(t,x_0)\|$$

$$+L_2S_4\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|+\frac{\|A\|}{e^{\|A\|T}-\|E\|}\int_0^T\|e^{A(T-s)}\|[\|B(s)\|\|x(s,x_0)-r(s,x_0)\|$$

$$+L_1\|x(s,x_0)-r(s,x_0)\|+L_2S_4\|\dot{x}(s,x_0)-\dot{r}(s,x_0)\|]ds]$$

$$\le Q[(H+L_1)\|x(t,x_0)-r(t,x_0)\|+L_2S_4\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|$$

$$+\frac{\|A\|TQ}{e^{\|A\|T}-\|E\|}[(H+L_1)\|x(t,x_0)-r(t,x_0)\|+L_2S_4\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|]]$$

$$\le (Q+\frac{\|A\|TQ^2}{e^{\|A\|T}-\|E\|})[(H+L_1)\|x(t,x_0)-r(t,x_0)\|+L_2S_4\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|],$$

and hence,

$$\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\| \le Q_1[(H+L_1)\|x(t,x_0)-r(t,x_0)\|+L_2S_4\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|].$$

(2.19)

From inequality (2.18) and (2.19), we obtain:

$$\|x(t,x_0)-r(t,x_0)\| \le B_1(t)Q[(H+L_1)\|x(t,x_0)-r(t,x_0)\|+L_2S_4\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|]$$

$$\le B_1(t)Q\{(H+L_1)B_1(t)Q[(H+L_1)\|x(t,x_0)-r(t,x_0)\|+L_2S_4\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|]+$$

$$+L_2S_4Q_1[(H+L_1)\|x(t,x_0)-r(t,x_0)\|+L_2S_4\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|]\}$$

$$\le B_1(t)Q((H+L_1)B_1(t)Q+L_2S_4Q_1)[(H+L_1)\|x(t,x_0)-r(t,x_0)\|+$$

$$+L_2 S_4 \|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|]$$

$\le C_1 V$,

and hence ,

$\|x(t,x_0)-r(t,x_0)\| \le C_1 V$,

where $C_1 = B_1(t) Q[(H+L_1)\|x(t,x_0)-r(t,x_0)\| + L_2 S_4 \|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|]$,

so, by induction , we get :

$$\|x(t,x_0)-r(t,x_0)\| \le C_1 V^m \tag{2.20}$$

From inequality (2.20) and by using (1.4), when $m \to \infty$, $V^m \to 0$, *we obtain:*
$\|x(t,x_0)-r(t,x_0)\| \le 0$ *, thus, we find that:*

$x(t,x_0) = r(t,x_0)$,

hence, $x(t,x_0)$ *is a unique periodic solution of non-linear integro-differential equation(1.1) , for all* $t \in R^1$, $x_0 \in D_f$.

3. Existence of *a* periodic Solution of the System (1.1).

The problem of existence of periodic solution of period T of the system (1.1) is uniquely connected with the existence of zero of the function $\Delta(x_0)$
which has the form:

$\Delta : D_f \to R^n$

$$\Delta(x_0) = \frac{A}{e^{AT}-E}\int_0^T e^{A(T-t)}[B(t)x_\infty(t,x_0) + f(t,x_\infty(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{t} G(t,s)\dot{x}_\infty(s,x_0)ds)^i)]dt \tag{3.1}$$

where $x_{\infty}(t,x_0)$ *is the limiting function of the sequence of functions* $x_m(t,x_0)$.

$$\Delta_m : D_f \to R^n$$

$$\Delta_m(x_0) = \frac{A}{e^{AT} - E}\int_0^T e^{A(T-t)}[B(t)x_m(t,x_0) + f(t,x_m(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{t} G(t,s)\dot{x}_m(s,x_0)ds)^i)]dt$$

(3.2)

where m= 0, 1, 2, 3,…

Theorem2. . *Let all assumptions and conditions of theorem1. be satisfied, then the following inequality hold :*

$$\|\Delta(x_0) - \Delta_m(x_0)\| \le N_1 N_2 Q^2 (H + L_1) B_1(t) V^{m-1}(1-V)^{-1} V_1 = \rho_m \quad , \qquad (3.3)$$

where $N_1 = \dfrac{\|A\|T}{e^{\|A\|T} - \|E\|}$, $N_2 = [1 + L_2 S_4 Q_1 (1 - Q_1 L_2 S_4)^{-1}]$ *and* $Q_1 = (Q + N_1 Q^2)$

Proof. *By using the relations (3.1) and (3.2) ,we have :*

$$\|\Delta(x_0) - \Delta_m(x_0)\| \le (\frac{\|A\|}{e^{\|A\|T} - \|E\|})\int_0^T \|e^{A(T-t)}\|[\|B(t)\|\|x_{\infty}(t,x_0) - x_m(t,x_0)\|$$

$$+L_1\|x_{\infty}(t,x_0) - x_m(t,x_0)\| + L_2(\sum_{i=1}^{\infty} iK^i M_4^{i-1})\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\|]dt$$

$$\|\Delta(x_0) - \Delta_m(x_0)\| \le (\frac{\|A\|}{e^{\|A\|T} - \|E\|})\int_0^T Q[(H + L_1)\|x_{\infty}(t,x_0) - x_m(t,x_0)\|$$

$$+L_2 S_4\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\|]dt$$

$$\le (\frac{\|A\|T}{e^{\|A\|T} - \|E\|})Q[(H + L_1)\|x_{\infty}(t,x_0) - x_m(t,x_0)\| + L_2 S_4\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\|]$$

$$\le N_1 Q[(H + L_1)\|x_{\infty}(t,x_0) - x_m(t,x_0)\| + L_2 S_4\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\|]$$

so,

$$\|\Delta(x_0) - \Delta_m(x_0)\| \le N_1 Q[(H + L_1)\|x_{\infty}(t,x_0) - x_m(t,x_0)\| + L_2 S_4\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\|]$$

(3.4)

Also

$$\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\| \le Q[(H + L_1)\|x_{\infty}(t,x_0) - x_m(t,x_0)\| + L_2 S_4\|\dot{x}_{\infty}(t,x_0) - \dot{x}_m(t,x_0)\|$$

$$+N_1Q[(H+L_1)\|x_\infty(t,x_0)-x_m(t,x_0)\|+L_2S_4\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\|]]$$

$$\leq (Q+N_1Q^2)[(H+L_1)\|x_\infty(t,x_0)-x_m(t,x_0)\|+L_2S_4\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\|]$$

$$\leq Q_1[(H+L_1)\|x_\infty(t,x_0)-x_m(t,x_0)\|+L_2S_4\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\|]$$

and hence,

$$\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\|\leq Q_1[(H+L_1)\|x_\infty(t,x_0)-x_m(t,x_0)\|+L_2S_4\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\|]$$

From the last inequality ,we have:

$$\|\dot{x}_\infty(t,x_0)-\dot{x}_m(t,x_0)\|\leq (1-Q_1L_2S_4)^{-1}Q_1(H+L_1)\|x_\infty(t,x_0)-x_m(t,x_0)\| \ , \tag{3.5}$$

By substituting inequality (3.5) in (3.4) , we get :

$$\|\Delta(x_0)-\Delta_m(x_0)\|\leq N_1Q(H+L_1)\|x_\infty(t,x_0)-x_m(t,x_0)\|+$$
$$+N_1Q(H+L_1)L_2S_4Q_1(1-Q_1L_2S_4)^{-1}\|x_\infty(t,x_0)-x_m(t,x_0)\|$$

And hence:

$$\|\Delta(x_0)-\Delta_m(x_0)\|\leq N_1Q(H+L_1)B_1(t)QV_1V^{m-1}(1-V)^{-1}$$
$$+N_1Q(H+L_1)L_2S_4Q_1(1-Q_1L_2S_4)^{-1}B_1(t)QV_1V^{m-1}(1-V)^{-1}$$
$$\leq N_1N_2Q^2(H+L_1)B_1(t)V_1V^{m-1}(1-V)^{-1} \ ,$$

Thus

$$\|\Delta(x_0)-\Delta_m(x_0)\|\leq N_1N_2Q^2(H+L_1)B_1(t)V_1V^{m-1}(1-V)^{-1}=\rho_m \ ,$$

I.e. the inequality (3.3) will be satisfied for all $m\geq 1$.

Theorem 3. *Let the system (1.1) be defined in the interval* $[a,b]$ *on* R^1 *and periodic in* t *of period* T *. Suppose that the sequence of functions (2.1) satisfies the inequalities:*

$$\left.\begin{array}{l} \min\limits_{a+M_3 \le x_0 \le b-M_3} \Delta_m(x_0) \le -\rho_m \quad , \\ \max\limits_{a+M_3 \le x_0 \le b-M_3} \Delta_m(x_0) \le \rho_m \quad . \end{array}\right\} \qquad (3.6)$$

then , the system (1.1) has a periodic solution $x(t,x_0)$ *for which* $x_0 \in [a+M_3, b-M_3]$,

where $\rho_m = N_1 N_2 Q^2 (H+L_1) B_1(t) V_1 V^{m-1} (1-V)^{-1}$, $M_3 = \frac{T}{2} Q(H\delta_0 + M)$.

Proof. *Let* x_1, x_2 *be any two points in the interval* $[a+M_3, b-M_3]$ *such that :*

$$\left.\begin{array}{l} \Delta_m(x_1) = \min\limits_{a+M_3 \le x_0 \le b-M_3} \Delta_m(x_0) \quad , \\ \Delta_m(x_2) = \max\limits_{a+M_3 \le x_0 \le b-M_3} \Delta_m(x_0) \quad . \end{array}\right\} \qquad (3.7)$$

By using the inequalities (3.3.3) and (3.3.6) ,we have :

$$\left.\begin{array}{l} \Delta(x_1) = \Delta_m(x_1) + (\Delta(x_1) - \Delta_m(x_1)) \le 0 \quad , \\ \Delta(x_2) = \Delta_m(x_2) + (\Delta(x_2) - \Delta_m(x_2)) \ge 0 \quad . \end{array}\right\} \qquad (3.8)$$

It follows from the inequalities (3.3.8) and the continuity of the function $\Delta(x_0)$ *that there exists an isolated singular point* $(x_\infty) = (x_0)$, $x_\infty \in [x_1, x_2]$, *such that* $\Delta(x_0) = 0$. *This means that the system (3.3.1) has a periodic solution* $x(t,x_0)$ *for which* $x_0 \in [a+M_3, b-M_3]$.

Remark 1[5]. *Theorem 2 is proved when* x_0 *is a scalar singular point which should be isolated, thus we have :*

Theorem 3. *If the function* $\Delta(x_0)$ *is defined by* $\Delta : D_f \to R^n$ *, and*

$$\Delta(x_0)=\frac{A}{e^{AT}-E}\int_0^T e^{A(T-t)}[B(t)x_\infty(t,x_0)+f(t,x_\infty(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{t}G(t,s)\dot{x}_\infty(s,x_0)ds)^i)]dt$$

(3.9)

where $x_\infty(t,x_0)$ *is a limit of the sequence of functions (2.1), then the following inequalities hold :*

$$\left\|\Delta(x_0)\right\|\le M_6 \quad , \tag{3.10}$$

where

$$M_5=(1-\frac{T}{2}QH)^{-1} \quad , \quad M_6=N_1Q(H\delta_0 QM_5+\frac{T}{2}QHMM_5+M)$$

and

$$\left\|\Delta(x_0^1)-\Delta(x_0^2)\right\|\le(F_5F_7(Q+\frac{T}{2}E_3QNF_4)+F_6)\left\|x_0^1-x_0^2\right\| \tag{3.11}$$

is satisfies for all x_0,x_0^1,x_0^2 *, where* $E_1=Q(H+L_1)$, $E_3=QL_2S_4$.

Proof. *From the properties of the function* $x_\infty(t,x_0)$ *as in the theorem 1, the function* $\Delta(x_0)$ *is continuous and bounded by* M_6 *in the domain (2.2).*

From the relation (3.9), we find that:

$$\left\|\Delta(x_0)\right\|\le\frac{\left\|A\right\|}{e^{\|A\|T}-\left\|E\right\|}\int_0^T\left\|e^{A(T-t)}\right\|[\left\|B(t)\right\|\left\|x_\infty(t,x_0)\right\| + \left\|f(t,x_\infty(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{t}G(t,s)\dot{x}_\infty(s,x_0)ds)^i)\right\|]dt$$

so, we have :

$$\left\|\Delta(x_0)\right\|\le\frac{\left\|A\right\|T}{e^{\|A\|T}-\left\|E\right\|}Q[H\left\|x_\infty(t,x_0)\right\|+M]$$

$$\left\|\Delta(x_0)\right\|\le\frac{\left\|A\right\|T}{e^{\|A\|T}-\left\|E\right\|}QH\left\|x_\infty(t,x_0)\right\|+\frac{\left\|A\right\|T}{e^{\|A\|T}-\left\|E\right\|}QM \tag{3.12}$$

Since the function $x_{\infty}(t,x_0)$ *satisfies the integral equation (2.3), then we have:*

$$\|x_{\infty}(t,x_0)\| = \|x_0\|\|e^{At}\| + \| \int_0^t e^{A(t-s)}[B(s)x_{\infty}(s,x_0)$$

$$+f(s,x_{\infty}(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_{\infty}(\tau,x_0)d\tau)^i) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x_{\infty}(s,x_0)$$

$$+f(s,x_{\infty}(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_{\infty}(\tau,x_0)d\tau)^i)]ds]ds\| \ .$$

Now , by using lemma 3.1.1 , we get :

$$\|x_{\infty}(t,x_0)\| = \delta_0 Q + \frac{T}{2}Q[H\|x_{\infty}(s,x_0)\| + M]$$

so,

$$\|x_{\infty}(t,x_0)\| = \delta_0 Q M_5 + \frac{T}{2}QMM_5 \quad . \qquad (3.13)$$

Thus, substituting inequality (3.13) in (2.12) , we get the inequality (3.10).

By using relation (3.9) , we have :

$$\|\Delta(x_0^1) - \Delta(x_0^2)\| = \| \frac{A}{e^{AT}-E}\int_0^T e^{A(T-t)}[B(t)x_{\infty}(t,x_0^1)$$

$$+f(t,x_{\infty}(t,x_0^1),\sum_{i=1}^{\infty}(\int_{-\infty}^{t} G(t,s)\dot{x}_{\infty}(s,x_0^1)ds)^i)]dt - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-t)}[B(t)x_{\infty}(t,x_0^2)$$

$$+f(t,x_{\infty}(t,x_0^2),\sum_{i=1}^{\infty}(\int_{-\infty}^{t} G(t,s)\dot{x}_{\infty}(s,x_0^2)ds)^i)]dt\|$$

$$\|\Delta(x_0^1) - \Delta(x_0^2)\| \le \frac{\|A\|T}{e^{\|A\|T} - \|E\|}Q[H\|x_{\infty}(t,x_0^1) - x_{\infty}(t,x_0^2)\| + L_1\|x_{\infty}(t,x_0^1) - x_{\infty}(t,x_0^2)\|$$

$$+L_2 S_4\|\dot{x}_{\infty}(t,x_0^1) - \dot{x}_{\infty}(t,x_0^2)\|]$$

so,

$$\|\Delta(x_0^1) - \Delta(x_0^2)\| \le N_1 E_1\|x_{\infty}(t,x_0^1) - x_{\infty}(t,x_0^2)\| + N_1 E_3\|\dot{x}_{\infty}(t,x_0^1) - \dot{x}_{\infty}(t,x_0^2)\| \ .$$

(3.14)

Now, we find that:

$$\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\| \le \|x_0^1-x_0^2\|\|A\|\|e^{At}\|+\|e^{A(t-s)}\|[\|B(t)\|\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\|$$

$$+L_1\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\|+L_2S_4\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\|]$$

$$+\frac{\|A\|}{e^{\|A\|T}-\|E\|}\int_0^T\|e^{A(T-s)}\|[\|B(s)\|\|x_\infty(s,x_0^1)-x_\infty(s,x_0^2)\|$$

$$+L_1\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\|+L_2S_4\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\|]ds$$

$$\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\| \le \|x_0^1-x_0^2\|NQ+Q[(H+L_1)\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\|$$

$$+L_2S_4\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\|+N_1Q[(H+L_1)\|x_\infty(s,x_0^1)-x_\infty(s,x_0^2)\|$$

$$+L_2S_4\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\|]]$$

$$\le \|x_0^1-x_0^2\|NQ+Q_1[(H+L_1)\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\|$$

$$+L_2S_4\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\|],$$

and hence ,

$$\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\| \le \|x_0^1-x_0^2\|NQ(1-Q_1L_2S_4)^{-1}$$

$$+Q_1(H+L_1)(1-Q_1L_2S_4)^{-1}\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\|$$

$$\|\dot{x}_\infty(t,x_0^1)-\dot{x}_\infty(t,x_0^2)\| \le \|x_0^1-x_0^2\|NQF_4+F_4Q_1(H+L_1)\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\| \;, \tag{3.15}$$

where $F_4=(1-Q_1L_2S_4)^{-1}$.

By substituting inequality (3.15) in (3.14) , we get :

$$\|\Delta(x_0^1)-\Delta(x_0^2)\| \le N_1E_1\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\|+N_1E_3NQF_4\|x_0^1-x_0^2\|$$

$$+N_1E_3F_4Q_1(H+L_1)\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|$$

$$\le (N_1E_1+N_1E_3Q_1F_4(H+L_1))\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|+N_1E_3NQF_4\left\|x_0^1-x_0^2\right\|.$$

Putting

$$F_5=(N_1E_1+N_1E_3Q_1F_4(H+L_1)) \quad \text{and} \quad F_6=N_1E_3NQF_4$$

so, the last inequality becomes :

$$\left\|\Delta(x_0^1)-\Delta(x_0^2)\right\|\le F_5\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|+F_6\left\|x_0^1-x_0^2\right\| , \tag{3.16}$$

where $x_\infty(t,x_0^1)$ *and* $x_\infty(t,x_0^2)$ *are the solutions of the integral equation :*

$$x(t,x_0^k)=x_0^k e^{At}+\int_0^t e^{A(t-s)}[B(s)x(s,x_0^k)+f(s,x(s,x_0^k),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0^k)d\tau)^i)$$

$$-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}[B(s)x(s,x_0^k)+f(s,x(s,x_0^k),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0^k)d\tau)^i)]ds]ds \tag{3.17}$$

with

$x_0^k(t,x_0)=x_0^k$ *, where k =1, 2 .*

From the equation (3.17) and by using lemma 1, we have:

$$\left\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\right\|\le\left\|x_0^1-x_0^2\right\|\left\|e^{At}\right\|$$

$$+\left[\left\|E\right\|-\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\left\|E\right\|}\right)\right]\int_0^t\left\|e^{A(t-s)}\right\|[\left\|B(s)\right\|\left\|x_\infty(s,x_0^1)-x_\infty(s,x_0^2)\right\|$$

$$+L_1\left\|x_\infty(s,x_0^1)-x_\infty(s,x_0^2)\right\|+L_2S_4\left\|\dot{x}_\infty(s,x_0^1)-\dot{x}_\infty(s,x_0^2)\right\|]ds$$

$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\left\|E\right\|}\right)\int_t^T\left\|e^{A(t-s)}\right\|[\left\|B(s)\right\|\left\|x_\infty(s,x_0^1)-x_\infty(s,x_0^2)\right\|$$

$$+L_1\left\|x_\infty(s,x_0^1)-x_\infty(s,x_0^2)\right\|+L_2S_4\left\|\dot{x}_\infty(s,x_0^1)-\dot{x}_\infty(s,x_0^2)\right\|]ds$$

so that :

$$\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\| \le \|x_0^1-x_0^2\|Q+\frac{T}{2}Q(H+L_1)\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\|$$

$$+\frac{T}{2}QL_2S_4\|\dot{x}_\infty(s,x_0^1)-\dot{x}_\infty(s,x_0^2)\| \quad . \qquad (3.18)$$

Now, by substituting inequality (3.15) in (3.18), we get:

$$\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\| \le \|x_0^1-x_0^2\|Q+\frac{T}{2}Q(H+L_1)\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\|$$

$$+\frac{T}{2}Q^2L_2S_4NF_4\|x_0^1-x_0^2\|+\frac{T}{2}L_2S_4F_4Q_1Q(H+L_1)\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\|$$

$$\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\| \le (\frac{T}{2}L_2S_4F_4Q_1E_1+\frac{T}{2}E_1)\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\|$$

$$+(\frac{T}{2}E_3QNF_4+Q)\|x_0^1-x_0^2\|$$

so,

$$\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\| \le (1-(\frac{T}{2}L_2S_4F_4Q_1E_1+\frac{T}{2}E_1))^{-1}(\frac{T}{2}E_3QNF_4+Q)\|x_0^1-x_0^2\| \ ,$$

$$\|x_\infty(t,x_0^1)-x_\infty(t,x_0^2)\| \le F_7(\frac{T}{2}E_3QNF_4+Q)\|x_0^1-x_0^2\| \quad . \qquad (3.19)$$

where

$$F_7=(1-(\frac{T}{2}L_2S_4F_4Q_1E_1+\frac{T}{2}E_1))^{-1}$$

Also, substituting inequality (3.19) in (3.16) , we get the inequality (3.11).

References

[1] Ali, A. R., Periodic solutions for some classes of non-linear systems of integro-differential equations, M. Sc. Thesis, college of Science, Duhok University, Iraq, (2010).

[2]. Butris R.N., Existence, Uniqueness and Stability Solution of Non-linear System of Integro-Differential Equation of Volterra type, University of Zahok,Iraq , International Journal of Innovative Science, Engineering & Technology, Vol. 2 Issue 7, (2015).

[3]. Butris, R. N. , Periodic solution of non-linear system of Integro - differential equations depending on the Gamma distribution, India, Gen. Math . Notes Vol.15, No.1, (2013).

[4] .Benaicha, S. , Periodic boundary value problems for second order ordinary differential equations, University of Oran , Es-senia, Algeria, Applied Mathematical Sciences, Vol. 3, No.6 ,(2009).

[5] . Mitropolsky, Yu. A. and Martynyuk, D. I. , For Periodic Solutions For the Oscillations System with Retarded Argument, Kiev , Ukraine, General School ,(1979).

[6] .Narjanov, O. D., On a Periodic solution for integro-differential equation, Math. J. , Kiev, Ukraine, Tom.2. (1977).

[8]. Naima, D. M. , Periodic solution of non-autonomous second–order differential equations and boundary value problem, India, Issue. 6, (1999).

[9]. Perestyuk, N. A. and Martynyuk, D. I., Periodic solutions of a certain class systems of differential equations, Math. J. Univ. of Kiev, Kiev, Ukraine, Tom 3, (1976).

[10] . Rafeq, A. Sh. , Periodic solutions for some of non-linear systems of integro – differential equations, M. Sc. Thesis, college of Science, Duhok University, Duhok, (2009).

[11] . Shslapk, Yu. D. , Periodic solution of first order nonlinear differential equations unsolvable for the derivation, Math. J. Ukraine, Kive , (1980).

[12] . Samoilenko, A. M. and Ronto, N. I., A Numerical – Analytic Methods for Investigating of Periodic Solutions, Kiev, Ukraine, (1976) .

[13]. Zavizion, G.V. , Periodic solutions of integro-differential equations, J. of Diff. Equations, Vol.45, No. 2, pp.240-248, (2009).