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ON THE BINARY QUADRATIC DIOPHANTINE EQUATION

$$x^2 - 3xy + y^2 + 16x = 0$$

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Abstract

The binary quadratic equation $x^2 - 3xy + y^2 + 16x = 0$ represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.

Keywords: **Binary quadratic equation, Integral solutions.**

MSC subject classification: 11D09.

1. Introduction

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety $[1-6]$. In $[7-16]$ the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of an another interesting binary quadratic equation given by $x^2 - 3xy + y^2 + 16x = 0$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

METHOD OF ANALYSIS:

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$x^2 - 3xy + y^2 + 16x = 0 \quad (1)$$

Note that (1) is satisfied by the following non-zero distinct integer pairs

However, we have the solutions for (1), which is illustrated below:

solving (1) for x, we've

$$x = \frac{1}{2}[3y - 16 \pm \sqrt{5y^2 - 96y + 256}] \quad (2)$$

Let $\alpha^2 = 5y^2 - 96y + 256$

which is written as ,

$$(5y - 48)^2 = 5\alpha^2 + 32^2 \quad (3)$$

$$Y^2 = 5\alpha^2 + 1024$$

where

$$Y = 5y - 48 \quad (4)$$

the least positive integer solution of (3) is

$$\alpha_0 = 128 \ , \ Y_0 = 288$$

Now to find the other solution of (3),

Consider the Pellian equation

$$Y^2 = 5\alpha^2 + 1 \quad (5)$$

whose fundamental solution is $(\tilde{\alpha}_0, \tilde{Y}_0) = (4,9)$

The other solutions of (5) can be derived from the relations

$$\tilde{Y}_n = \frac{f_n}{2} \qquad \tilde{\alpha}_n = \frac{g_n}{2\sqrt{5}}$$

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where

$$f_n = [(9+4\sqrt{5})^{n+1} + (9-4\sqrt{5})^{n+1}]$$

$$g_n = [(9+4\sqrt{5})^{n+1} - (9-4\sqrt{5})^{n+1}] \quad ,$$

n=0,2,4,……..

Applying the lemma of Brahmagupta between $(\alpha_0, Y_0) \& (\tilde{\alpha}_s, \tilde{Y}_s)$

The other solutions of (3) can be obtained from the relations

$$\alpha_{n+1} = 64 f_n + \frac{144 g_n}{\sqrt{5}}$$

$$Y_{n+1} = 144 f_n + 64\sqrt{5} g_n \qquad (7)$$

Taking positive sign on the R.H.S of (2) and using (4), (6) & (7) the non-zero distinct integer solution of the hyperbola (1) are obtained as follows,

$$x_{n+1} = \frac{1}{2}(3y_{n+1} - 16 + \alpha_{n+1}) \qquad (8)$$

$$y_{n+1} = \frac{1}{2}(y_{n+1} + 48)$$

,n=0,2,4 (9)

The recurrence relations satisfied by x_{n+1} , y_{n+1} are respectively

$$x_{n+5} - 322x_{n+3} + x_{n+1} = -2048$$

$$y_{n+5} - 322y_{n+3} + y_{n+1} = -3072$$

A few numerical examples are presented in the table below.

n	x_{n+1}	y_{n+1}
0	2704	1040
2	868624	331792
4	279692176	106832912

A few interesting relations among the solutions are presented below:

[1] $x_{n+1} \& y_{n+1}$ are always even

[2] $x_{n+1} \equiv 0 \pmod{2}$ (6)

[3] $\frac{3}{8}[105y_{2n+2} - 40x_{2n+2} - 752] + 12$ is a nasty number.

[4] $\frac{1}{16}[105y_{2n+2} - 40x_{2n+2} - 752] + 2$ is a quadratic integer.

[5] $\frac{1}{16}[105y_{3n+3} - 40x_{3n+3} - 752] + 3\left[\frac{1}{16}(105y_{n+1} - 40x_{n+1} - 752)\right]$ is a cubic integer.

[6] $\frac{1}{5120}[180x_{n+1} - 470y_{n+1} + 3360]^2 (105y_{n+1} - 40x_{n+1} - 752) = \frac{1}{4}[105y_{n+1} - 40x_{n+1} - 752]$

[7] $y_{n+3} - 144x_{n+1} - 55y_{n+1} = -384$

[8] $y_{n+5} - 46368x_{n+1} + 17711y_{n+1} = -126720$

[9] $x_{n+3} - 377x_{n+1} + 144y_{n+1} = -1024$

[10] $x_{n+5} - 121393x_{n+1} - 46368y_{n+1} = -331776$

[11] $322y_{n+3} - y_{n+5} - y_{n+1} = 3072$

[12] $144x_{n+1} - 17711y_{n+3} + 55y_{n+5} = -168576$

[13] $377y_{n+3} - 144x_{n+3} - y_{n+1} = 2688$

[14] $144y_{n+3} - 55x_{n+3} - x_{n+1} = 1024$

[15] $121393y_{n+3} - 144y_{n+5} - 377y_{n+1} = 1160832$

[16] $46368y_{n+3} - 55x_{n+5} - 377x_{n+1} = 442368$

[17] $377y_{n+5} - 46368x_{n+3} + 55y_{n+1} = -292608$

[18] $121393x_{n+3} - 377x_{n+5} - 144y_{n+1} = 773120$

[19] $$\frac{1}{16}[105y_{2n+2} - 40x_{2n+2} - 752] + 2 = \left[\frac{1}{16}(105y_{n+1} - 40x_{n+1} - 752)\right]^2$$

[20] $$\frac{1}{16}[105y_{3n+3} - 40x_{3n+3} - 752] + 3\left[\frac{1}{16}(105y_{n+1} - 40x_{n+1} - 752)\right] = \left[\frac{1}{16}(105y_{n+1} - 40x_{n+1} - 752)\right]^3$$

Remarkable observations:

1)By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the hyperbola.

- $512U^2 - V^2 = 2048$

Where

$$U = \frac{1}{16}(105y_{n+1} - 40x_{n+1} - 752)$$

$$V = (18x_{n+1} - 47y_{n+1} - 336)$$

$$181925120U_1^2 - V_1^2 = 7276500480$$

where

$$U_1 = \frac{1}{16}(105y_{n+1} - 40x_{n+1} - 752)$$

$$V_1 = 90x_{n+1} - 75635y_{n+1} + 725520$$

$$81920U_2^2 - V_2^2 = 327680$$

Where

$$U_2 = \frac{1}{16}(105y_{n+1} - 40x_{n+1} - 752)$$

$$V_2 = \frac{1}{128\sqrt{5}}(5y_{n+3} - 1605y_{n+1} + 15360)$$

2. By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the parabola.

$$N^2 = 320M - 10240$$

Where

$$M = \frac{1}{16}[105y_{2n+2} - 40x_{2n+2} - 752]$$

$$N = 180x_{n+1} - 470y_{n+1} + 3360$$

$$N_1^2 = 5120M_1 - 163840$$

where

$$M_1 = \frac{1}{16}[105y_{2n+2} - 40x_{2n+2} - 752]$$

$$N_1 = 5y_{n+3} - 1605y_{n+1} + 15360$$

CONCLUSION :

In this paper , we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude , one may search for other choices of solutions to the considered binary equation and further , quadratic equations with multi-variables.

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