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Computer Aided Design of Disc of Variable Thickness:

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Abstract

A Rotating disc is a machine part of great importance, one could not Imagine machine design or production processes without rotors, pulleys, gears, fans, flywheels, grinding wheels, saw blades etc., all being one or another but always a rotating disc. The stresses in the disc are primarily radial and hoop stresses and are due to inertia. If rotors have been built as disc of uniform thickness, they could rotate with low speed only. A rotating material with the speed of 100 m/s produces stresses approximately 0.8 times the yield stress. In modern turbines and compressors for instance, the needed speed is 200 m/s to 300m/s. Thus there is the need to construct a disc that would stand these high rotations (disc of variable thickness).This paper presents the final equations for stress distribution in stepped disc and a computer base calculation supporting these have been Included.

INTRODUCTION

Analytical solutions of disc of variable thickness are very limited, and the only one efficiently solved is the disc of hyperbolic profile [1]. Two other approaches are the analysis with the use of finite difference method [6] and the step disc approach.

In this presentation an analysis have been carried out using the step disc approach [2], in designing a disc for machinery applications in turbines and compressors, other factors ought to be discussed as playing an important role like vibrations, thermal fatigue, analysis of blades etc. The starting point however, in rotating disc is always stresses due to inertia and these were primarily considered here.

DISC OF UNIFORM THICKNESS

A rotating thin disc falls into the category of axial symmetry plane stress problem. The deformation is primarily due to centrifugal force produced. If we consider an element at a radius r from the axis of rotation as shown in figure 1, such that the radial and hoop stresses σ_r and σ_t due to rotation are constant through the thickness δ. This leads to equilibrium of forces in the radial direction as;

$$\frac{d}{dr}(\sigma_r r)-\sigma_t+q_r=0 \qquad (1)$$

This equation is internally statically Indeterminate because of the two unknown stresses.

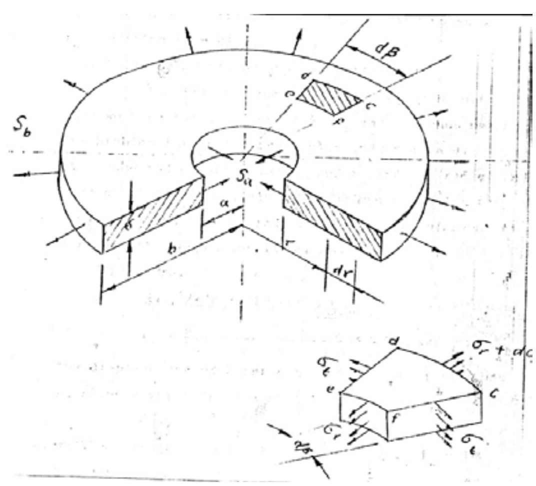

FIGURE l: An element of the disc at a radius r.

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A solution to this equation could be obtained with the use of hooks law, thus leading to the determination of the stress distributions. The general assumption made here is that the blades of the turbine rotors as a uniformly distributed mass. Hence as in the case of thin rotating ring, which rotates at a constant speed ω, a centrifugal force is set-up. This inward force may be resolved into tangential direction at each end of an element of the ring, which is subjected to circumferential stress $\sigma=\rho(\omega r)^2$, and ρ is the density of the material of the blades. Thus the displacement equation and the stress distributions are given in equations 2, 3 and 4 respectively.

$$U(r)=C_1 r+C_2.(a^2/r)-1/8\left(\frac{1-\upsilon 2}{E}\right)\rho\omega^2 r^3 \qquad (2)$$

$$\sigma_r=B_1-B_2(a^2/r^2)-\left(\frac{3+\upsilon}{8}\right)\rho\omega^2 r^2 \qquad (3)$$

$$\sigma_t=B_1+B_2(a^2/r^2)-\left(\frac{1+3\upsilon}{8}\right)\rho\omega^2 r^2 \qquad (4)$$

Where $B_l=C_l-E/(l-\upsilon)$ and $B_2=C_2-E/(l+\upsilon)$ and are determine from the appropriate boundary conditions. We may consider the case of a turbine rotor with outer radius b and inside radius a, which has been subjected to a pressure S_a on the inside surface due to rotor been shrunk onto a shaft, while on the outer surface the blades produces a centrifugal force S_b.
Hence the boundary conditions are:
$\sigma_r=S_a$ at r=a and $\sigma_t=S_b$ at r=b.

Letting $\sigma_p=\rho(\omega b)^2$ and solving for B_1 and B_2 we have

$$B_1=\frac{3+\upsilon}{8}\left(1+\frac{a^2}{b^2}\right)\sigma_p-\frac{a^2}{b^2-a^2}sa+\frac{b^2}{b^2-a^2}sb$$

$$B_2=\frac{3+\upsilon}{8}\sigma_p-\frac{b^2}{b^2-a^2}sa+\frac{b^2}{b^2-a^2}sb \qquad (5)$$

Thus C_1 and C_2 can also be determined, hence the displacements at the boundaries are:-

$$Ur=a=\frac{3+\upsilon}{4}\sigma p\frac{a}{E}\left(1+\frac{1-\upsilon}{3+\upsilon}x\frac{a^2}{b^2}\right)-sa\frac{a}{E}\left(\frac{b^2+a^2}{b^2-a^2}+\upsilon\right)+S_b\frac{a}{E}\left(\frac{2b^2}{b^2-a^2}\right) \qquad (6)$$

$$Ur=b=\frac{3+\upsilon}{4}\sigma p\frac{b}{E}\left(\frac{1-\upsilon}{3+\upsilon}+\frac{a^2}{b^2}\right)-S_a\frac{b}{E}\left(\frac{2b^2}{b^2-a^2}\right)+S_b\frac{b}{E}\left(\frac{b^2+a^2}{b^2-a^2}-\upsilon\right) \qquad (7)$$

DISC OF VARIABLE THICKNESS

The method used for solving the problem of disc of variable thickness in this presentation is the approximate stepped disc approach. The method involves cutting the disc into several steps, thicker near the hub and decreases down to smaller thickness toward the periphery.

COMPATIBILITY EQUATION

Figure 2 illustrate a variable disc been cut into several steps, it also indicate the reactions (intensity of forces) between the steps. r_i is the inner radius of step disc i and r_o is the radius of the central hole, while 2R is the outer diameter of the whole disc. Since X_{i-1} is the reaction (normal force intensity) between step i-1 and i in [N/m^2] and X_i the reaction between step i and i+1, it implies that the load [N/m] acting on step i would be X_i/δ_i outward and X_{i-1}/δ_i inwards.

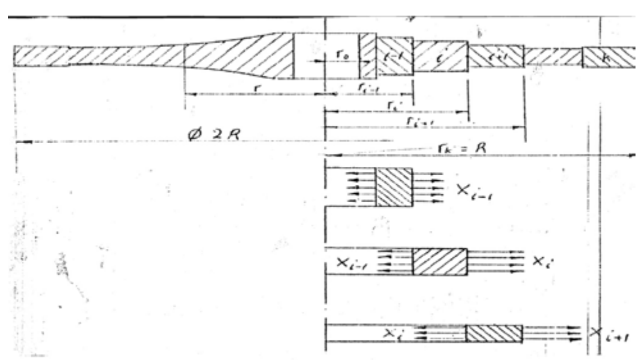


FIGURE 2: Disc of variable thickness indicating reactions (intensity of forces) between steps.

The displacement at the external edge of step i and the internal edge of step i+1, are $(U_{ext})_i$, and $(U_{int})_{i+1}$ respectively. $(Uext)_i$ can be obtain by substituting $a=r_{i-1}, b=r_i$, $S_a=X_{i-1}/\delta_i$ and $\sigma_p=p(\omega r)^2$ into equation (7).

$$(U_{ext})_i=\frac{r_i}{E}\left[\frac{X_i}{\delta_i}\left(\frac{r_i^2+r^2_{i-1}}{r_i^2-r^2_{i-1}}-\upsilon\right)+-\frac{X_{i-1}}{\delta_i}\cdot\frac{2r^2_i}{r^2_i-r^2_{i-1}}+\frac{3+\upsilon}{4}p(\omega r_i)^2\left(\frac{1-\upsilon}{3+\upsilon}\frac{r^2_{i-1}}{r^2_i}\right)\right] \quad (8)$$

Similarly for step i+1, substituting $a=r_i$, $b=r_{i+1}$, $Sa=X_i/\delta_i$

$\sigma_p= p(\omega.r_{i+1})^2$ into equation (6).

$$(U_{int})_{1+1}=\frac{r_i}{E}\left(\frac{X_{i+1}}{\delta_{i+1}}\left(\frac{2r^2_{i+1}}{r^2_{i+1}-r^2_i}\right)-\frac{X_i}{\delta_{i+1}}\left(\frac{r^2_{i+1}+r^2_i}{r^2_{i+1}-r^2_i}+\upsilon\right)\right.$$

$$\left.+\frac{3+\upsilon}{4}p(\omega.r_{i+1})^2\left(1+\frac{1-\upsilon}{3+\upsilon}\cdot\frac{r^2_i}{r^2_{i+1}}\right)\right) \quad (9)$$

In accordance with the compatibility equation, that is rotation of step disc produces no gap between the steps: that is $(U_{ext})_i=(U_{int})_{i+1}$. To simplify the equation some dimensionless variables have been Introduced; that is $\xi_i=r_i/R$ and $n_i=\delta^*/\delta i$. Where δ^* is an arbitrary chosen step thickness. We may also take $\sigma_p^*=\rho(\omega R)^2$. Thus the compatibility equation may be written as:-

$$-\alpha_{i,i-1}X_{i-1}+\alpha_{i,i}X_i-\alpha_{i,i+1}X_{i+1}=\alpha_{i,0}\cdot\sigma_{p*}\delta^* \qquad (10)$$

Thus the constant coefficients are denoted as:-

$$\alpha_{i,I-1}=\frac{2n_i\xi^2_i}{\xi^2_i-\xi^2_{i-1}};$$

$$\alpha_{i,i}=\left[n_i\frac{\xi^2_i+\xi^2_{i-1}}{\xi^2_i-\xi^2_{i-1}}+n_{i+1}\frac{\xi^2_{i+1}+\xi^2_i}{\xi^2_{i+1}-\xi^2}+\upsilon(n_{i+1}-n_i)\right]$$

$$\alpha_{i,i+1}=\frac{2n_{i+1}\xi_{i+1}}{\xi^2_{i+1}-\xi^2_i} \text{ and } \alpha_{i,o}=\frac{3+\upsilon}{4}\left(\xi^2_{i+1}-\xi^2_{i-1}\right)$$

The rim k of the disc is substituted for the turbine blades as follows. The rim produces a reaction with the step k-1 equal to the reaction that would be produced if blades were of uniformly distributed mass, such that:-

$$X_k=m_1\left(\frac{\pi n}{30}\right)^2 r_b\frac{S}{2\pi r b1} \qquad (11)$$

m_1 - mass of single blade

n - Speed of disc [rpm]
r_b - radius to the centre of gravity of blade,
r_{b1}- radius to the last step
r_{b1}- radius to the last step
Using the compatibility equation (10) the k-1 reactions can be calculated.

THE STRESS DISTRIBUTION

Having calculated X_i we can then calculate σ_r and σ_t. For simplification, calculations are carried out for each step at one point only i.e. at $(r_{i-1}\cdot r_i)^{1/2}$. Thus,

$$\sigma_r=\frac{3+\upsilon}{8}\sigma p^*(\xi_i-\xi_{i-1})^2+\frac{X_i\xi_i+X_{i-1}\xi_{i-1}}{(\xi_i+\xi_{i-1})\delta_i} \quad (12)$$

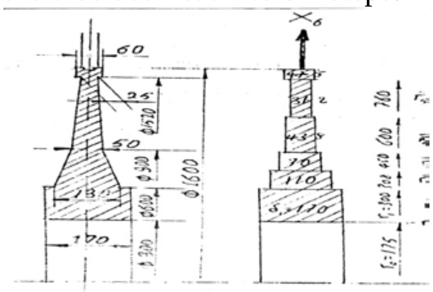
$$\sigma_t=\sigma p^*[(\xi_i\,\xi_{i-1}+\frac{3+5}{4})(\xi_i-\xi_{i-1})^2]+\frac{X_i\xi_i-X_{1-1}\xi_{i-1}}{(\xi_i-\xi_{i-1})\delta_i} \quad (13)$$

The error is <= 5% assuming that the number of steps taken is not too small (usually not less than six). Increasing the number of the steps too much is not recommended as mathematics gets too complicated, and there is no profit in "better" stress distribution.

EXAMPLARY CALCULATION

In this exemplary problem we are to determine the hoop and radial stresses of the stepped disc rotating at a speed of 3000rpm. The material of the disc has a density $\rho=7.85mg/m^2$ and $\upsilon=0.3$. Assume the number of blades to be 125 and each having a mass of 0.568Kg. Take the radius of the centre of gravity of the blades as $r_b=0.86m$. The disc and the steps are shown in figure 3. The reaction of the blades is determined from equation (11) as 1.20MN/m. For ease of computations, all the values required in the compatibility and stress equations are set in table: 1. δ*is taken as the thickness of the 5th step.

FIGURE3: Disc of variable thickness showing the disc been cut into six steps.



i	r_i	δ_i	ε_i	$\varepsilon_i^2 .10^2$	$(\varepsilon_i^2 - \varepsilon_{i-1}^2).10^2$	$(\varepsilon_i^2 + \varepsilon_{i-1}^2).10^2$
1	2	3	4	5	6	7
0	175		0.219	4.80		
1	300	170	0.375	14.06	9.26	18.86
2	368	110	0.461	21.20	7.14	35.26
3	450	70	0.562	31.58	10.38	52.78
4	605	43.8	0.756	57.15	25.57	88.40
5	760	31.2	0.950	90.25	33.10	147.40
6	800	47.5	1.000	100.00	9.75	190.25

n_i	$\frac{2n_i\varepsilon_{i-1}^2}{\varepsilon_i^2 - \varepsilon_{i-1}^2}$	$n_i\frac{((\varepsilon_i^2 + \varepsilon_{i-1}^2))}{(\varepsilon_i^2 - \varepsilon_{i-1}^2)}$	$v(n_{i+1} - n_i)$	$\alpha_{i,i}$	$\frac{2n_{i+1}\varepsilon_{i-1}^2}{(\varepsilon_{i+1}^2 - \varepsilon_i^2)}$	$\alpha_{i,i}.10^2$
8	9	10	11	12	13	14
0.1836	0.1904	0.374	0.030	1.804	1.685	13.52
0.2836	1.1160	1.400	0.048	3.711	2.710	14.45
0.4455	1.8200	2.263	0.080	4.815	3.160	29.67
0.7120	1.7580	2.472	0.086	7.018	5.452	48.40
1.000	3.4530	4.455	-0.102	17.168	13.485	35.34
0.6520		12.815				

TABLE 1: Constant coefficients in the compatibility and stress distribution equation

δ* is calculated to be 496MN/m^2. Substituting the values from table 1 into the compatibility equation (10), the following simultaneous linear equations were obtained:-

$1.804X_1 = 2.10*10^6$

$-1.I16X_1 + 3.711X_2 - 2.271X_3 = 2.24*10^6$

$-1.820X_2 + 4.815X_3 - 3.160X_4 = 4.59*10^6$

$-1.758X_3 + 7.013X_4 - 5.452X_5 = 7.50*10^6$

$-3.452X_4 + 17.168X_5 - 13.485X_6 = 5.50*10^6$

It is obvious that solving this type of linear equations becomes more tedious as the number of steps is increased. Thus the use of computer becomes necessary. A computer program which uses the method of Gaussian Elimination has been utilized. This program enables solving any given set of equations for the stepped disc and greatly reduces the amount of work when used. The solution obtained is as follows:-

X_1=9.26MN/m, X_2 =8.67MN/m, X_3 =7.24MN/m
X_4=4.58MN/m,X_5=2.18MN/m,
X_6=X_b=1.20MN/m

With the use of the stress distribution equations (12) and (13), the stresses are calculated for each step. As an illustration the stresses for the fourth step, i.e. at i=4, X_4=4.56*10^6 N/m and X_3=7.24*10^6 N/m. From table 1: ε_4=0.754, ε_3=0.562 and ε_4-ε_3=0.194,

σ_t=496[0.568(0.756)+3.3/4(0.194)^2]+[(4.58)(0.756)-(7.23)(0.562)]/(0.194)(43.8*10^{-3})= 149MN/m^2 and

σ_r=496(3.3/8)(0.194)^2+[(4.58)(0.576)+(2.23)(0.562)]/[(0.756+0.562)(43.8*10^{-3})] = 137MN/m^2.

COMPARISON OF STRESS DISTRIBUTION

In the numerical problem above, if we assume the disc to be of uniform thickness, then

σ_r=735.3 - 22.2/r^2 - 319.7r^2

σ_t=735.3 + 22.2/r^2 - 184.1r^2

It is clearly seen that the thickness of the disc plays no role in determining the stresses. The stress distribution diagrams in figures 4.1 and 4.2 shows that making the disc of variable thickness greatly reduces the stresses not only at the dangerous section, but throughout the disc as a whole, thus allowing for higher rotations. On the average, stresses in the uniform disc are three times larger.

As stresses depend on ω^2, it results that approximately variable discs can rotate $(3)^{1/2}$ time faster. For critical dangerous cases of hoop stress at the inner radius the profit is even higher (2.5 times higher rotations can be applied).

Fig. 4.1: Radial stress distribution for both uniform and variable rotor.

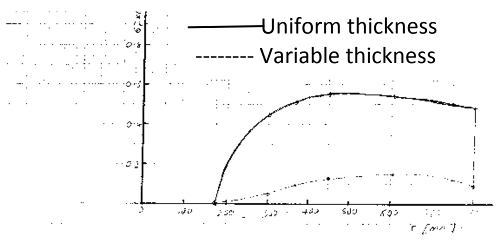
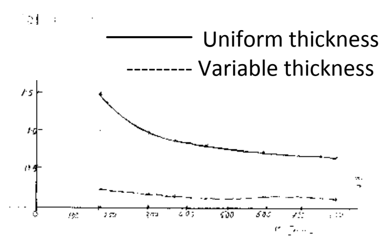


Fig. 4.2: Hoop stress distribution for both uniform and variable rotor.



Finally having computerized calculations of the interface reactions X_i, it becomes very ease to write a program for obtaining final results for σ_r and σ_t for any number of steps. The ways the calculations are organized enable this greatly.

CONCLUSION

Analytical solutions to the problems of disc of variable thickness are very limited, known only for the case of hyperbolic profile. The analyzed solution which involves the usage of the stepped disc approach enables us to obtain an efficient solution with the aid of a computer program.

For further or future Investigations in case of analyzing turbine and compressor rotors, other Influences ought to be considered, like vibration and fatigue. For both cases, computer aided calculations (design) are of great importance. Commercially available programs (software) would accelerate all calculations.

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