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ON THE BINARY QUADRATIC DIOPHANTINE EQUATION

$$x^2 - 3xy + y^2 + 33x = 0$$

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Abstract

The binary quadratic equation $x^2 - 3xy + y^2 + 33x = 0$ represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.

Keywords: Binary quadratic equation, Integral solutions.

MSC subject classification: 11D09.

1. Introduction

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety $[1-6]$. In $[7-16]$ the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of an another interesting binary quadratic equation given by $x^2 - 3xy + y^2 + 33x = 0$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

2. Method of analysis:

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$x^2 - 3xy + y^2 + 33x = 0 \qquad (1)$$

Note that (1) is satisfied by the following non-zero integer pairs

(33,33), (33,66), (-3,6), (-33,-99), (66,132),(6,-12),(-99,-297),(176,880)

However, we have other solutions for (1), which are illustrated below:

Solving (1) for x, we've

$$x = \frac{1}{2}(3y - 33 \pm \sqrt{5y^2 - 198y + 33^2}) \qquad (2)$$

Let $\alpha^2 = 5y^2 - 198y + 33^2$

which is written as

$$(5y - 99)^2 = 5\alpha^2 + 66^2$$

$$\Rightarrow Y^2 = 5\alpha^2 + 66^2 \qquad (3)$$

where

$$Y = 5y - 99 \qquad (4)$$

The least positive integer solution of (3) is

$$\alpha_0 = 264 \ , \ Y_0 = 594$$

Now, to find the other solution of (3), consider the Pellian equation

$$Y^2 = 5\alpha^2 + 1 \qquad (5)$$

whose fundamental solution is $(\tilde{\alpha}_0, \tilde{Y}_0) = (4,9)$

The other solutions of (5) can be derived from the relations $\tilde{Y}_s = \frac{f_s}{2}$ $\qquad \tilde{\alpha}_s = \frac{g_s}{2\sqrt{5}}$

where

$$f_s = [(9 + 4\sqrt{5})^{s+1} + (9 - 4\sqrt{5})^{s+1}]$$

$$g_s = [(9+4\sqrt{5})^{s+1} - (9-4\sqrt{5})^{s+1}] \quad ,$$

s=0,2,4,……..

Applying the lemma of Brahmagupta

between (α_0, Y_0) & $(\tilde{\alpha}_s, \tilde{Y}_s)$,

the other solutions of (3) can be obtained from the relations

$$\alpha_{s+1} = 132 f_s + \frac{297 g_s}{\sqrt{5}}$$

$$Y_{s+1} = 297 f_s + 132\sqrt{5} f_s \qquad (7)$$

Using (4), (6) & (7), the non-zero distinct integer solutions of the hyperbola (1) are obtained as follows,

$$x_{s+1} = \frac{1}{2}(3y_{s+1} - 33 \pm \alpha_{s+1}) \qquad (8)$$

$$y_{s+1} = \frac{1}{5}(Y_{s+1} + 99) \qquad (9)$$

The recurrence relations satisfied by x_{s+1} , y_{s+1} are respectively

$$x_{s+1} - 322x_{s+3} + x_{s+5} = -4224$$

$$y_{s+1} - 322y_{s+3} + y_{s+5} = -6336$$

For simplicity considering positive sign on the R.H.S of (8) a few numerical examples are given in Table 1.

Table:1 Examples

s	x_{s+1}	y_{s+1}
0	5577	2145
2	1791537	684321
4	576865113	220342881

2.1 Properties :

Both the values of x, y are positive and odd.

- $10x_{s+1} - y_{s+3} - 55y_{s+1} \equiv 0 \pmod{792}$
- $144x_{s+3} + y_{s+1} - 8y_{s+3} \equiv 0 \pmod{5544}$
- $17424x_{s+5} + 14688553y_{s+3} - 45617y_{s+1} = 0(\mathrm{mod}\ 2897$ (6)
- $55y_{s+5} + 144x_{s+1} - 1711y_{s+3} \equiv 0 \pmod{347688}$
- $y_{s+5} - 144x_{s+3} + 55y_{s+3} = 0 \pmod{792}$
- $377y_{s+5} - 144x_{s+5} - y_{s+3} \equiv 0 \pmod{5544}$
- $-20736y_{s+3} + 7920x_{s+3} - 8x_{s+1} \equiv 0 \pmod{304128}$
- $-1391104y_{s+3} + 165x_{s+5} + 1131x_{s+1} \equiv 0 \pmod{2737152}$
- $-342144y_{s+3} + 895752x_{s+3} - 2376x_{s+5} \equiv 0(\mathrm{mod}\ 50181$
- $\frac{1}{594}(1615y_{3s+3} - 5y_{3s+5} - 31878 + 1782 f_n)$ is a cubical integer
- $\frac{1}{594}(1615y_{2s+2} - 5y_{2s+4} - 30690)$ is a perfect square

3.Remarkable observations:

1) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the hyperbola.

$$(594)^2 U^2 - (264\sqrt{5})^2 V^2 = -491825157120$$

where

$$U = -1605y_{s+1} + 5y_{s+3} + 31680$$
$$V = 1615y_{s+1} - 5y_{s+3} - 31680$$

$$U_1^2 - 3820V_1^2 = -1393920$$

Where

$$U_1 = -1605y_{s+1} - 5y_{s+3} + 31680)$$

$$V_1 = \frac{1}{594}(-1615y_{3s+3} + 5y_{3s+5} + 31878 + 594 f_n^{\ 3})$$

2) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the parabola.

$$(594)^2 N^2 = (264\sqrt{5})^2 M - 1393920$$

where

$$M = (1615y_{2s+2} - 5y_{2s+4} - 30690)$$

$$N = (-1605y_{s+1} + 5y_{s+3} + 31680)$$

3.1 NOTE:

Treat (1) as a quadratic in y and solve for y .Employing the procedure similar to the above, the other sets of non-zero distinct integer solutions to (1) are obtained as

$$x_{s+1} = \frac{1}{5}(Y_{s+1} + 66)$$

$$y_{s+1} = \frac{1}{2}(3x_{s+1} \pm \alpha_{s+1})$$

4. Conclusion:

In this paper,we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude, one may search for other choices of solutions to the considered binary quadratic equation and further quadratic equations with multi-variables.

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