International Journal of Scientific Engineering and Applied Science (IJSEAS) - Volume-1, Issue-4, June 2015
ISSN: 2395-3470
www.ijseas.com

ON THE BINARY QUADRATIC DIOPHANTINE EQUATION

$$x^2 - 3xy + y^2 + 33x = 0$$

S. Vidhyalakshmi [1], **M.A.Gopalan** [2], **A.Kavitha** [3], **D.Mary Madona** [4]

[1,2,3] Professor, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu,

[4] M.Phil Scholar, Department of Mathematics,SIGC,Trichy-620002,Tamilnadu

Abstract

The binary quadratic equation $x^2 - 3xy + y^2 + 33x = 0$ represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.

Keywords: Binary quadratic equation, Integral solutions.

MSC subject classification: 11D09.

1. Introduction

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety $[1-6]$. In $[7-16]$ the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of an another interesting binary quadratic equation given by $x^2 - 3xy + y^2 + 33x = 0$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

2. Method of analysis:

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$x^2 - 3xy + y^2 + 33x = 0 \qquad (1)$$

Note that (1) is satisfied by the following non-zero integer pairs

(33,33), (33,66), (-3,6), (-33,-99), (66,132),(6,-12),(-99,-297),(176,880)

However, we have other solutions for (1), which are illustrated below:

Solving (1) for x, we've

$$x = \frac{1}{2}(3y - 33 \pm \sqrt{5y^2 - 198y + 33^2}) \qquad (2)$$

Let $\alpha^2 = 5y^2 - 198y + 33^2$

which is written as

$$(5y - 99)^2 = 5\alpha^2 + 66^2$$

$$\Rightarrow Y^2 = 5\alpha^2 + 66^2 \qquad (3)$$

where

$$Y = 5y - 99 \qquad (4)$$

The least positive integer solution of (3) is

$$\alpha_0 = 264 \ , \ Y_0 = 594$$

Now, to find the other solution of (3), consider the Pellian equation

$$Y^2 = 5\alpha^2 + 1 \qquad (5)$$

whose fundamental solution is $(\tilde{\alpha}_0, \tilde{Y}_0) = (4,9)$

The other solutions of (5) can be derived from the relations $\tilde{Y}_s = \dfrac{f_s}{2}$ $\quad \tilde{\alpha}_s = \dfrac{g_s}{2\sqrt{5}}$

where

$$f_s = [(9+4\sqrt{5})^{s+1} + (9-4\sqrt{5})^{s+1}]$$

$$g_s = [(9+4\sqrt{5})^{s+1} - (9-4\sqrt{5})^{s+1}] \quad ,$$

s=0,2,4,……..

Applying the lemma of Brahmagupta

between (α_0, Y_0) & $(\tilde{\alpha}_s, \tilde{Y}_s)$,

the other solutions of (3) can be obtained from the relations

$$\alpha_{s+1} = 132 f_s + \frac{297 g_s}{\sqrt{5}}$$

$$Y_{s+1} = 297 f_s + 132\sqrt{5} f_s \qquad (7)$$

Using (4), (6) & (7), the non-zero distinct integer solutions of the hyperbola (1) are obtained as follows,

$$x_{s+1} = \frac{1}{2}(3y_{s+1} - 33 \pm \alpha_{s+1}) \qquad (8)$$

$$y_{s+1} = \frac{1}{5}(Y_{s+1} + 99) \qquad (9)$$

The recurrence relations satisfied by x_{s+1} , y_{s+1} are

respectively

$$x_{s+1} - 322x_{s+3} + x_{s+5} = -4224$$

$$y_{s+1} - 322y_{s+3} + y_{s+5} = -6336$$

For simplicity considering positive sign on the R.H.S of (8) a few numerical examples are given in Table 1.

Table:1 Examples

s	x_{s+1}	y_{s+1}
0	5577	2145
2	1791537	684321
4	576865113	220342881

2.1 Properties :

Both the values of x, y are positive and odd.

- $10x_{s+1} - y_{s+3} - 55y_{s+1} \equiv 0 \pmod{792}$
- $144x_{s+3} + y_{s+1} - 8y_{s+3} \equiv 0 \pmod{5544}$
- $17424x_{s+5} + 14688553y_{s+3} - 45617y_{s+1} = 0(\mathrm{mod}\, 2897($ (6)
- $55y_{s+5} + 144x_{s+1} - 1711y_{s+3} \equiv 0 \pmod{347688}$
- $y_{s+5} - 144x_{s+3} + 55y_{s+3} = 0 \pmod{792}$
- $377y_{s+5} - 144x_{s+5} - y_{s+3} \equiv 0 \pmod{5544}$
- $-20736y_{s+3} + 7920x_{s+3} - 8x_{s+1} \equiv 0 \pmod{304128}$
- $-1391104y_{s+3} + 165x_{s+5} + 1131x_{s+1} \equiv 0 \pmod{2737152}$
- $-342144y_{s+3} + 895752x_{s+3} - 2376x_{s+5} \equiv 0(\mathrm{mod}\, 50181$
- $\frac{1}{594}(1615y_{3s+3} - 5y_{3s+5} - 31878 + 1782 f_n)$ is a cubical integer
- $\frac{1}{594}(1615y_{2s+2} - 5y_{2s+4} - 30690)$ is a perfect square

3.Remarkable observations:

1) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the hyperbola.

$$(594)^2 U^2 - (264\sqrt{5})^2 V^2 = -491825157120$$

where

$$U = -1605y_{s+1} + 5y_{s+3} + 31680$$

$$V = 1615y_{s+1} - 5y_{s+3} - 31680$$

$$U_1^2 - 3820V_1^2 = -1393920$$

Where

$$U_1 = -1605y_{s+1} - 5y_{s+3} + 31680)$$

$$V_1 = \frac{1}{594}(-1615y_{3s+3} + 5y_{3s+5} + 31878 + 594 f_n{}^3)$$

2) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the parabola.

$$(594)^2 N^2 = (264\sqrt{5})^2 M - 1393920$$

where

$$M = (1615y_{2s+2} - 5y_{2s+4} - 30690)$$

$$N = (-1605y_{s+1} + 5y_{s+3} + 31680)$$

3.1 NOTE:

Treat (1) as a quadratic in y and solve for y .Employing the procedure similar to the above, the other sets of non-zero distinct integer solutions to (1) are obtained as

$$x_{s+1} = \frac{1}{5}(Y_{s+1} + 66)$$

$$y_{s+1} = \frac{1}{2}(3x_{s+1} \pm \alpha_{s+1})$$

4. Conclusion:

In this paper,we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude, one may search for other choices of solutions to the considered binary quadratic equation and further quadratic equations with multi-variables.

Acknowledgement:

The financial support from the UGC, New Delhi (F.MRP-5122/14 (SERO/UGC) dated March 2015) for a part of this work is gratefully acknowledged.

References

[1]. Banumathy.T.S., A Modern Introduction to Ancient Indian Mathematics,Wiley Eastern Limited, London. (1995)

[2]. Carmichael, R.D.,The Theory of Numbers and Diophantine Analysis, Dover Publications, New York. (1950)

[3]. Dickson. L. E., History of The Theory of Numbers, Vol.II, Chelsia Publicating Co, New York,(1952).

[4].Mordell,L,J., Diophantine Equations, Acadamic Press,London,(1969).

[5].Nigel,P.Smart., TheAlgorithm Resolutions of Diaphantine eqations,Cambridge University, Press, London,(1999).

[6]. Telang ,S.G., Number theory, Tata Mc Graw-Hill Publishing Company , NewDelhi,(1996).

[7]. Gopalan,M.A., and Parvathy,G., "Integral Points On The Hyperbola $x^2 + 4xy + y^2 - 2x - 10y + 24 = 0$,"Antarctica J.Math,Vol 1(2),2010,149-155.

[8]. Gopalan,M.A, Vidhyalakahmi,S, Sumathi.G and Lakshmi.K, Sep "Integral Pionts On The Hyperbola $x^2 + 6xy + y^2 + 40x + 8y + 40 = 0$" , Bessel J.Math. Vol 2(3),2010,159-164.

[9].Gopalan,M.A.,Gokila,K.,andVidhyalakahmi,S ., "On the Diophantine Equation

$x^2+4xy+y^2-2x+2y-6=0$”, ActaCienciaIndica, Vol.XXXIIIM No2,2007, p. 567-570.

[10].Gpalan,M.A.,Vidhyalakahmi,S.,andDevibala, S., On The Diophantine Equation $3x^2+xy=14$,Acta CienciaIndica,Vol.XXXIII M.No2, 2007,P.645-646.

[11].Gopalan.M.A., and Janaki.G., “Observations on $x^2-y^2+x+y+xy=2$”, ImpactJ.Sci.,Tech, Vol2(3),2008, p.143-148.

[12].Gopalan.M.A., Shanmuganadham.P and Vijayashankar,A., “On Binary Quadratic Equation $x^2-5xy+y^2+8x-20y+15=0$”, Acta Ciencia Indica,Vol . XXXIVM. No.4, 2008,p.1803-1805.

[13]. Gopalan,M.A, Vidhyalakahmi,S, Lakshmi.K and Sumathi.G, “ Observation on $3x^2+10xy+4y^2-4x+2y-7=0$” , Diophantus J.Maths.Vol.1(2), 2012, 123-125.

[14] Mollion,R.A, “All Solutions of the Diophatine Equations $X^2-DY^2=n$” Far East J,Math.Sci., Speical Volume,Part III,1998,p.257-293.

[15].Vidhyalakahmi,S, Gopalan,M.A and Lakshmi.K, “ Observation On The Binary Quadratic Equation $3x^2-8xy+3y^2+2x+2y+6=0$”, Scholar Journal of Physics,Mathematics and Statistics, Vol.1(2),2014, 41-45.

[16].Vidhyalakahmi. S, Gopalan,M.A and Lakshmi.K, “Integer Solution of the Binary QuadraticEquation $x^2-5xy+y^2+33x=0$”, International Journal of Innovative Science Engineering &Technology, Vol.1(6), 2014,450-453.