ESTIMATION OF PARAMETER OF RECTANGULAR (UNIFORM) PROBABILITY DISTRIBUTION ON (0,θ) BASED ON PRELIMINARY TEST AND TWO SAMPLES

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Abstract

Two independent fixed sized samples are drawn from continuous uniform distributions on (0, θ_1) and (0, θ_2) respectively. Problem is to improve the estimator of θ_1. If the hypothesis that $\theta_1 = \theta_2$ is accepted, we use both the samples to estimate θ_1; otherwise, use its UMVUE based on the first sample alone. The likelihood ratio test is developed to test this hypothesis. Its properties are studied. An improved estimator is suggested and its properties are studied. The mean square error (MSE) of this estimator is compared with the variance of UMVUE based on first sample. The regions in which new estimator is preferable are identified. Graphs for a few specific values are also given.

Keywords: Preliminary test estimation, Estimating parameter of rectangular distribution, improved estimator of parameter of continuous uniform distribution, Likelihood ratio test, Estimating parameter of continuous uniform distribution.

1.INTRODUCTION: Let X_1 ,X_2, … ,X_n and Y_1 ,Y_2 , … ,Y_m be two independent random samples from continuous uniform probability distributions on (0, θ_1) and (0, θ_2) respectively. Let X and Y be the maximum values in these two samples respectively. The problem is to estimate the value of θ_1. It is suspected but not known for sure that $\theta_1 = \theta_2$ Therefore, we use the two samples to test the hypothesis that $\theta_1 = \theta_2$. If the hypothesis is accepted, we use both the samples to estimate θ_1; otherwise, use just the first sample. We provide here the necessary test and the estimator in this situation. We study the properties of the test and the estimator. This estimator is compared with the usual unbiased estimator based on maximum likelihood estimator based on single sample.

2. LIKELIHOOD RATIO TEST TO TEST Ho: $\theta_1 = \theta_2$: The pdf's of X and Y are given by

f_{θ_1} (x)= (n/ θ_1^n)x^{n-1}, 0<x< θ_1 , θ_1>0 (2.1)

And g_{θ_2} (y)= (m/ θ_2^m)y^{m-1}, 0<x<θ_2 , θ_2>0. (2.2)

The joint pdf of (x,y) is

f_{θ_1,θ_2}(x,y)=[mn/((θ_1^n)(θ_2^m))]$x^{n-1}y^{m-1}I_{(0,\theta_1)}(x)I_{(0,\theta_2)}$ (y) (2.3)

The parameter space is,

$$\Theta = \{(\theta_1,\theta_2),\theta_1 = \theta_2,\theta_1 > 0,\theta_2 > 0\}$$

To test Ho: $\theta_1 = \theta_2$against H_1: $\theta_1 \neq \theta_2$,we have,

$$\Theta_0 = \{(\theta_1,\theta_2),\theta_1 = \theta_2,\theta_1 > 0,\theta_2 > 0\}$$

and$\Theta_1 = \{(\theta_1,\theta_2);\theta_1 \neq \theta_2,\theta_1 > 0,\theta_2 > 0\}$

If $\theta_1 \neq \theta_2$ and both are unknown, x and y are maximum likelihood estimators (mles) of θ_1 and θ_1 respectively. Therefore, Sup ($\theta \in \Theta$)) = f_{θ_1,θ_2} (x,y) = (nm/(x^n y^m)) x^{n-1} y^{m-1} = nm/(xy).

If $\theta_1 \neq \theta_2 = \theta$, say, the mle of θ is max(x, y). Therefore,

Sup ($\theta \in \Theta_0$) { f_{θ_1,θ_2} (x,y) } = [nm x^{n-1} y^{m-1}/{ $(\max(x,y))^{m+n}$}]

Therefore, the likelihood ratio

$$\text{L.R.}(x,y) = \frac{Sup\ (\theta\in\Theta)\ \{f_{\theta_1\theta_2}(x,y)\}}{Sup\ (\theta\in\Theta_0)\ \{f_{\theta_1\theta_2}(x,y)\}}$$

=[[nm x^{n-1} y^{m-1}/{ $(\max(x,y))^{m+n}$}]/[nm/xy]]

= $(y/x)^m$, if x $\geq$ y (2.4)

Putting T= X/Y, we have,

L.R.(x ,y)=t^n, if $t \leq 1$

= t^{-m}, if t> 1.

Thus, the likelihood ratio test to test Ho against H_1 is: Reject Ho iff L.R.(x ,y) < c , where, c is some constant so chosen that size of the test becomes α. The test is equivalent to: Reject Ho iff L.R.(x,y)= LR(t) < c, i.e.; iff $t^n < c$ for $0< t \leq 1$ and $(1/t^m) < c$ for t> 1.That is, reject Ho if $t < c^{1/n}$ for $0< t \leq 1$ and if $t > c^{-1/m}$ for t>1.

Thus, the test is

$\Phi(t) = 1$, if $t < c^{1/n}$ for $0< t \leq 1$ and if $t > c^{-1/m}$ for t>1.

=0,otherwise. (2.5)

Applying equal tail criterion,

$$P_{Ho}\{ t < c^{1/n}\} = P_{Ho}\{ t > c^{-1/m}\} = \alpha/2 \quad (2.6)$$

To calculate probabilities in (2.6) we have to know the probability distribution of T. For this, let T = X/Y and U = Y. If $0< t \leq (\theta_1/\theta_2)$, $0< u < \theta_2$.If $(\theta_1/\theta_2) < t < \infty$, $0 < u < (\theta_1/t)$. X = TU. The Jacobian of transformation

$$J = \begin{vmatrix} \frac{dx}{dt} & \frac{dy}{dt} \\ \frac{dx}{du} & \frac{dy}{du} \end{vmatrix} = \begin{vmatrix} u & 0 \\ t & 1 \end{vmatrix} = u\,. \quad (2.7)$$

The joint pdf of T and U, using (2.3) and (2.7) is

h_{θ_1,θ_2}(u,t) = (n/ $\theta_1^{\,n}$)(m/ $\theta_2^{\,m}$)$t^{n-1}u^{n+m-1}$, $0<t<\infty$, $0<u<\theta_2$, $0 < tu < \theta_1$ (2.8)

= 0 , otherwise.

Integrating (2.8) w.r.t. u, we get, the marginal pdf of T as

$$h_{\theta_1,\theta_2}(x,y) = \frac{nm}{n+m}\left(\frac{\theta_2}{\theta_1}\right)^n t^{n-1}$$

h_{θ_1,θ_2} (t)= (nm/(n +m))($\theta_2/\theta_1)^n$ t^{n-1}, $0< t \le (\theta_1/\theta_2)$

= (nm/(n +m))($\theta_1/\theta_2)^m(1/t^{m+1})$, $(\theta_1/\theta_2) < t < \infty$. (2.9)

If $\theta_1 = \theta_2 = \theta$, i.e.; under Ho, we have,

h_θ (t) = (nm/(n +m))t^{n-1}, $0< t \le 1$

= (nm/(n +m))(1/t^{m+1}) , $1< t < \infty$. (2.10)

Note that this is independent of θ.

If the samples are of equal size, i.e.; if n = m, (2.9) becomes,

h_{θ_1,θ_2} (t)= (n/2)($\theta_2/\theta_1)^n$ t^{n-1}, $0< t \le (\theta_1/\theta_2)$

= (n/2)($\theta_1/\theta_2)^m(1/t^{m+1})$, $(\theta_1/\theta_2) < t < \infty$. (2.11)

If $\theta_1 = \theta_2 = \theta$, as well as n = m, this reduces to

h(t) = (n/2) t^{n-1}, $0 < t \le 1$

= (n/2) (1/ t^{n+1}), $1< t < \infty$. (2.12)

From (2.5), we have,

(nm/(n +m))$\int_0^c t^{n-1}dt$ = (nm/(n +m))$\int_{c2}^{\infty}(1/t^{m+1})dt$ = α/2.This gives

the critical region to be

$R_\alpha = \{(0,(\alpha(m+n)/(2m))^{(1/n)})U((2n/\alpha(n+m))^{(1/m)},\infty)\}$

$= \{(0,\sqrt[n]{(\alpha(m+n)/(2m))}\,U(\sqrt[m]{(2n/\alpha(n+m))},\infty)\}$ (2.13)

If m=n, the critical region reduces to, $R_\alpha = \{(0,\sqrt[n]{\alpha})\ U\ (\sqrt[n]{1/\alpha}\ ,\infty)\}$. (2.14)

The power of the test (2.5) is given by ,

$\beta_\phi(\theta_1,\theta_2) = E_{\theta 1,\theta 2}(\Phi(T)) = P[\ (t < c^{1/n})U\ (t > c^{-1/m})]$

$$= \int_0^{\sqrt[n]{\left(\frac{\alpha(m+n)}{2m}\right)}} h(t;\theta 1,\theta 2)dt + \int_{\sqrt[m]{\left(\frac{2n}{\alpha(n+m)}\right)}}^{\infty} h(t;\theta 1,\theta 2)dt$$

$$= \frac{nm}{n+m}\left(\frac{\theta 2}{\theta 1}\right) n\int_0^{\sqrt[n]{\left(\frac{\alpha(m+n)}{2m}\right)}} t^{n-1}\,dt + \frac{nm}{n+m}\left(\frac{\theta 1}{\theta 2}\right) m \int_{\sqrt[m]{\left(\frac{2n}{\alpha(n+m)}\right)}}^{\infty}(1/t^{m+1})dt$$

$$= \frac{nm}{n+m}\left(\frac{\theta 2}{\theta 1}\right) n\alpha(m+n)/(2mn) + \frac{nm}{n+m}\left(\frac{\theta 1}{\theta 2}\right) m \quad \alpha(m+n)/(2mn)$$

$$= \alpha/2[(\theta_2/\theta_1)^n + (\theta_1/\theta_2)^m] = \frac{\alpha}{2}\left[\frac{\theta_1^{m+n}+\theta_2^{m+n}}{\theta_1^n\theta_2^m}\right] \qquad (2.15)$$

$$= \frac{\alpha}{2}\left[\frac{\theta_1^{2n}+\theta_2^{2n}}{(\theta_1\theta_2)^n}\right], \text{ if } n=m. \qquad (2.16)$$

3. ESTIMATOR TO ESTIMATE θ_1:

Consider the following estimator

$$T^*=\begin{cases} X, & if\ T < \sqrt[n]{\frac{\alpha(n+m)}{2m}}\ or\ T > \sqrt[m]{\frac{2n}{\alpha(m+n)}} \\ Z, & otherwise \end{cases} \qquad (3.1)$$

where Z= max(X,Y)

The pdf of Z is given by,

$$f(z I \theta_1,\theta_2)=\begin{cases} \frac{(n+m)z^{n+m+1}}{\theta_1^n\theta_2^m}, 0 < z < \theta_1 \\ \frac{mz^{m-1}}{\theta_2^m}, \theta_1 \le z < \theta_2 \\ 0, otherwise \end{cases} \quad ,\theta_1<\theta_2 \qquad (3.2)$$

And

$$f(z I \theta_1,\theta_2)=\begin{cases} \frac{(n+m)z^{n+m+1}}{\theta_1^n\theta_2^m}, 0 < z < \theta_2 \\ \frac{nz^{n-1}}{\theta_1^n}, \theta_2 \le z < \theta_1 \\ 0, otherwise \end{cases} \quad ,\theta_2\le \theta_1 \qquad (3.3)$$

The expected value of Z is,

$$E(Z)=\begin{cases} \frac{n\theta_1^{m+1}}{(m+1)(n+m+1)\theta_2^m} + \frac{m\theta_2}{m+1}, if\ \theta_1 < \theta_2 \\ \frac{m\theta_2^{n+1}}{(n+1)(n+m+1)\theta_1^n} + \frac{n\theta_1}{n+1},\ if\ \theta_2 \le \theta_1 \end{cases} \qquad (3.4)$$

and

$$E(Z^2)=\begin{cases}\frac{2n\theta_1^{m+2}}{(m+2)(n+m+2)\theta_2^m}+\frac{m\theta_2^2}{m+2}, if\ \theta_1<\theta_2\\ \frac{2m\theta_2^{n+2}}{(n+2)(n+m+2)\theta_1^n}+\frac{n\theta_1^2}{n+2},\ \ if\ \theta_2\le\theta_1\end{cases}$$

If $\theta_1=\theta_2$,

1. $E(Z)=\frac{n+m}{n+m+1}\theta_1$ (3.5)

Which is expectation of maximum of (n+ m) observations.

2. $E(Z^2)=\frac{m+n}{m+n+2}\theta_1^2$

Which is expectation of square of maximum of (m+ n) observations.

We have the p.d.f.s,

$f_X(x \mid \theta_1)=\frac{nx^{n-1}}{\theta_1^n}I_{(0,\theta_1)}(x)$

$f_Y(y \mid \theta_2)=\frac{my^{m-1}}{\theta_2^m}I_{(0,\theta_2)}(y)$

$$h_T(t \mid \theta_1,\theta_2)=\begin{cases}\frac{nm}{n+m}\left(\frac{\theta_2}{\theta_1}\right)^n t^{n-1}\ \ ,\ 0<t<\frac{\theta_1}{\theta_2}\\ \frac{nm}{n+m}\left(\frac{\theta_1}{\theta_2}\right)^m\frac{1}{t^{m+1}}\ ,\ \frac{\theta_1}{\theta_2}\le t<\infty\end{cases}$$

$h_{Y,T}(y,t \mid \theta_1,\ \theta_2)=$

$$\begin{cases}\frac{mn}{\theta_1^n\theta_2^m}y^{m+n-1}t^{n-1}\ ,0<t<\infty,\ \ 0<y<\theta_2,\ \ 0<ty<\theta_1\\ \qquad 0\quad , otherwise\end{cases} \tag{3.6}$$

If $\frac{\theta_1}{\theta_2}\le t<\infty, 0<y<\frac{\theta_1}{t}$ and if $0<t<\frac{\theta_1}{\theta_2}, 0<y<\theta_2$

$$h(y \mid t,\theta_1,\theta_2)=\begin{cases}\frac{n+m}{\theta_1^{n+m}}y^{m+n-1}t^{m+n},\ \ 0<y<\frac{\theta_1}{t},\ \ \frac{\theta_1}{\theta_2}\le t<\infty\\ \frac{n+m}{\theta_2^{m+n}}y^{n+m-1},\ \ 0<t<\frac{\theta_1}{\theta_2},\ \ 0<y<\theta_2\\ \qquad 0\ \ ,\ otherwise\end{cases} \tag{3.7}$$

Let us put $Y=\frac{X}{Y}$, $T=V,\ \ 0<\frac{X}{T}<\theta_2\ ,\ \ 0<V<\infty$

Jacobean of transformation is,

$$|J| = \begin{vmatrix} \frac{dy}{dx} & \frac{dt}{dx} \\ \frac{dy}{dv} & \frac{dt}{dv} \end{vmatrix} = \frac{1}{v}$$

Therefore,

$$h(x,v \mid \theta_1, \theta_2) = \begin{cases} \frac{nm}{\theta_1^n \theta_2^m} \frac{x^{n+m-1}}{v^{m+1}}, & 0 < x < \theta_1,\ 0 < v < \infty,\ 0 < \frac{x}{v} < \theta_2 \\ 0, & otherwise \end{cases}$$

In our notation, V=T= $\frac{X}{Y}$, therefore,

h(x,t| θ_1, θ_2) =

$$\begin{cases} \frac{nm}{\theta_1^n \theta_2^m} x^{n+m-1} t^{-(m+1)}, & 0 < x < \theta_1,\ 0 < t < \infty,\ 0 < \frac{x}{t} < \theta_2 \\ 0, & \text{otherwise} \end{cases} \tag{3.8}$$

$0 < \frac{x}{t} < \theta_2$ $0 < x < \theta_2 t$, also $0 < x < \theta_1$, therefore $0 < x < \min\{\theta_1, \theta_2\}$

If $0 < t < \frac{\theta_1}{\theta_2}$, *this implies that* $0 < x < \theta_2 t$

If $\frac{\theta_1}{\theta_2} \leq t < \infty$, $0 < x < \theta_1$

Therefore, the conditioned density of X given T is given by,

$$h(x \mid t, \theta_1, \theta_2) = \begin{cases} \frac{n+m}{\theta_2^{n+m}} x^{m+n-1} t^{-(m+n)}, & 0 < t < \frac{\theta_1}{\theta_2},\ 0 < x < \theta_2 t \\ \frac{n+m}{\theta_1^{n+m}} x^{m+n-1}, & \frac{\theta_1}{\theta_2} \leq t < \infty,\ 0 < x < \theta_1 \end{cases} \tag{3.9}$$

let $A_1 = \left(0, \sqrt[n]{\frac{\alpha(n+m)}{2m}}\right)$, $A_2 = \left(\sqrt[m]{\frac{2n}{\alpha(m+n)}}, \infty\right)$ and $A = A_1 \cup A_2$.

Consider the estimator T^* defined earlier,

$$T^* = \begin{cases} X, & if\ T \epsilon A \\ Z, & if\ T \epsilon A^c \end{cases}$$

Thus, $T^* = X\, I_A(t) + Z\, I_{A^c}(t)$

Therefore,

$E(T^*) = E(X\, I_A(t)) + E(Z\, I_{A^c}(t))$

On A_1, $0 < T < \sqrt[n]{\frac{\alpha(n+m)}{2m}} < \frac{\theta_1}{\theta_2}$, therefore,

$E(X \mid t\ \epsilon\ A_1) = \frac{n+m}{n+m+1} \theta_2 t$

$$E(X\, I_{A_1}(t)) = \frac{n+m}{n+m+1}\theta_2 \frac{nm}{n+m}\left(\frac{\theta_2}{\theta_1}\right)^n \int_0^{\sqrt[n]{\frac{\alpha(n+m)}{2m}}} t^n\, dt$$

$$= \frac{\alpha}{2}\left(\frac{m+n}{n+m+1}\right)\left(\frac{n}{n+1}\right)\left\{\frac{\alpha(n+m)}{2m}\right\}^{\frac{1}{n}} \frac{\theta_2^{n+1}}{\theta_1^n} \qquad (3.10)$$

$$E(X^2\, I\, t \in A_1) = \frac{n+m}{n+m+2}\theta_2^2\, t^2$$

$$E(X^2\, I_{A_1}(t)) = \frac{n+m}{n+m+2}\theta_2^2 \frac{nm}{n+m}\left(\frac{\theta_2}{\theta_1}\right)^n \int_0^{\sqrt[n]{\frac{\alpha(n+m)}{2m}}} t^{n+1}\, dt$$

$$= \frac{\alpha}{2}\left(\frac{m+n}{n+m+2}\right)\left(\frac{n}{n+2}\right)\left\{\frac{\alpha(n+m)}{2m}\right\}^{\frac{2}{n}} \frac{\theta_2^{n+2}}{\theta_1^n} \qquad (3.11)$$

In general,

$$E(X^r\, I_{A_1}(t)) = \frac{\alpha}{2}\left(\frac{m+n}{n+m+r}\right)\left(\frac{n}{n+r}\right)\left\{\frac{\alpha(n+m)}{2m}\right\}^{\frac{r}{n}} \frac{\theta_2^{n+r}}{\theta_1^n}$$

On A_2 , $t > \frac{\theta_1}{\theta_2}$ and

$$E(X \mid t \in A_2) = \frac{n+m}{n+m+1}\theta_1$$

$$E(X\, I_{A_2}(t)) = \frac{n+m}{n+m+1}\theta_1 \frac{nm}{n+m}\left(\frac{\theta_1}{\theta_2}\right)^m \int_{\sqrt[m]{\frac{2m}{\alpha(n+m)}}}^{\infty} \frac{1}{t^{m+1}}\, dt$$

$$= \frac{\alpha}{2}\left(\frac{m+n}{n+m+1}\right)\left(\frac{\theta_1^{m+1}}{\theta_2^m}\right) \qquad (3.12)$$

$$E(X^2 \mid t \in A_2) = \frac{n+m}{n+m+2}\theta_1^2$$

$$E(X^2\, I_{A_2}^2(t)) = \frac{\alpha}{2}\left(\frac{m+n}{n+m+2}\right)\left(\frac{\theta_1^{m+2}}{\theta_2^m}\right) \qquad (3.13)$$

In general,

$$E(X^r\, I_{A_2}^r(t)) = \frac{\alpha}{2}\left(\frac{m+n}{n+m+r}\right)\left(\frac{\theta_1^{m+r}}{\theta_2^m}\right), r > 0$$

But $T^* = X$ if $T \epsilon A_1$ or $T \epsilon A_2$, i.e. if $T \epsilon A$

Therefore actually we want,

$$E(X\, I_A(t)) = E[X\, I_{A_1}(t) + X\, I_{A_2}(t)]$$

$$= \frac{\alpha}{2}\left(\frac{m+n}{n+m+1}\right)\left\{\frac{n}{n+1}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{1}{n}}\frac{\theta_2^{n+1}}{\theta_1^{n}}+\frac{\theta_1^{m+1}}{\theta_2^{m}}\right\} \quad (3.14)$$

$E(X^r I_A^r(t)) = E(X^r I_{A_1}(t)) + E(X^r I_{A_2}(t))$

$$= \frac{\alpha}{2}\left(\frac{m+n}{n+m+r}\right)\left\{\frac{n}{n+r}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{r}{n}}\frac{\theta_2^{n+r}}{\theta_1^{n}}+\frac{\theta_1^{m+r}}{\theta_2^{m}}\right\}$$

Consider,

$E[(X I_A(t) - \theta_1)^2] = E[X^2 I_A(t) - 2\theta_1 X I_A(t) + \theta_1^2]$

$= E[X^2 I_A(t)] - 2\theta_1 E[X I_A(t)] + \theta_1^2$

$$= -\frac{\alpha}{2}\left(\frac{m+n}{n+m+1}\right)\left(\frac{m+n+3}{m+n+2}\right)\frac{\theta_1^{m+2}}{\theta_2^{m}} + \frac{\alpha}{2}n(n+m)\left\{\frac{\alpha(m+n)}{2m}\right\}^{\frac{1}{n}}\frac{\theta_2^{n+1}}{\theta_1^{n}}\left[\left\{\frac{\alpha(m+n)}{2m}\right\}^{\frac{1}{n}}\frac{\theta_2}{(n+2)(m+n+2)} - \frac{2\theta_1}{(n+1)(m+n+1)}\right] + \theta_1^2 \quad (3.15)$$

$$T^* = \begin{cases} X, & \text{if } 0 < t < c_1, \quad c_2 < t < \infty \\ Z, & \text{if } c_1 \le t \le c_2 \end{cases}$$

$Z = \max(T,1)\, I_{A^c}(t)$, $\quad A^c = (c_1, c_2)$

Note that $c_1 < \frac{\theta_1}{\theta_2} < c_2$, $c_1 < 1 < c_2$

4. CASE – I $\quad \theta_1 < \theta_2$

$$T^* = \begin{cases} X, & 0 < t < c_1 \\ Y, & c_1 \le t \le \frac{\theta_1}{\theta_2} \\ Y, & \frac{\theta_1}{\theta_2} \le t < 1 \\ X, & 1 \le t < c_2 \\ X, & c_2 \le t < \infty \end{cases}$$

Thus,

$$T^* = X\{ I_{(0,c_1)}(t) + I_{(c_2,\infty)}(t)\} + X\, I_{(1,c_2)}(t) + Y\, I_{(c_1,\frac{\theta_1}{\theta_2})}(t) + Y\, I_{(\frac{\theta_1}{\theta_2},1)}(t) \quad (4.1)$$

Therefore, $E(T^*) = a+b+c+d$;

Where, $a = E[X\{ I_{(0,c_1)}(t) + I_{(c_2,\infty)}(t)\}]$, $\quad b = E[X\, I_{(1,c_2)}(t)]$,

$c = E[Y\, I_{(c_1,\frac{\theta_1}{\theta_2})}(t)]$, $\quad d = E[Y\, I_{(\frac{\theta_1}{\theta_2},1)}(t)]$

a= E[X { $I_{(0,c1)}$(t) + $I_{(c2,\infty)}$(t) }]

$$= \frac{\alpha}{2}\left(\frac{m+n}{n+m+1}\right)\left\{\frac{n}{n+1}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{1}{n}}\frac{\theta_2^{n+1}}{\theta_1^n}+\frac{\theta_1^{m+1}}{\theta_2^m}\right\}$$

E[X^2 {$I_{(0,c_1)}(t)$ + $I_{(c_2,\infty)}(t)$ }] = $\frac{\alpha}{2}\left(\frac{m+n}{n+m+2}\right)\left\{\frac{n}{n+2}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{1}{n}}\frac{\theta_2^{n+2}}{\theta_1^n}+\frac{\theta_1^{m+2}}{\theta_2^m}\right\}$

b= E[X $I_{(1,c_2)}(t)$] = E[$I_{(1,c_2)}(t)$ E[X I t ϵ (1,c$_2$)]]

$$= \left(\frac{m+n}{n+m+1}\right)\theta_1\int_1^{c_1}\frac{nm}{n+m}\left(\frac{\theta_1}{\theta_2}\right)^m\frac{1}{t^{m+1}}dt$$

$$= \frac{n\theta_1^{m+1}}{(n+m+1)\theta_2^m}-\frac{\alpha}{2}\frac{m+n}{m+n+1}\frac{\theta_1^{m+1}}{\theta_2^m}$$

E[X^2 $I_{(1,c_2)}$ (t)] = $\frac{n\theta_1^{m+2}}{(n+m+2)\theta_2^m}-\frac{\alpha}{2}\frac{m+n}{m+n+2}\frac{\theta_1^{m+2}}{\theta_2^m}$.

Now, we consider

c= E[Y $I_{(c_1,\frac{\theta_1}{\theta_2})}$(t)] = E [$I_{(c_1,\frac{\theta_1}{\theta_2})}$(t) E[Y I t ϵ $\left(c_1,\frac{\theta_1}{\theta_2}\right)$]

$$= \left(\frac{m+n}{n+m+1}\right)\theta_2\frac{mn}{m+}\left(\frac{\theta_2}{\theta_1}\right)^n\int_{c_1}^{\frac{\theta_1}{\theta_2}}t^{n-1}dt = \frac{m\theta_2}{n+m+1}-\frac{\alpha}{2}\frac{m+n}{m+n+1}\frac{\theta_2^{n+1}}{\theta_1^n}$$

E[Y^2 $I_{(c_1,\frac{\theta_1}{\theta_2})}$(t)]= $\frac{m\theta_2^2}{n+m+2}-\frac{\alpha}{2}\frac{m+n}{m+n+2}\frac{\theta_2^{n+2}}{\theta_1^n}$

d= E[Y $I_{(\frac{\theta_1}{\theta_2},1)}$(t)] = E[$I_{(\frac{\theta_1}{\theta_2},1)}$(t) E[Y I t ϵ $\left(\frac{\theta_1}{\theta_2},1\right)$]

$$= \left(\frac{m+n}{n+m+1}\right)\theta_1\left(\frac{\theta_1}{\theta_2}\right)^m\frac{mn}{m+n}\int_{\frac{\theta_1}{\theta_2}}^{1}\frac{1}{t^{m+2}}dt$$

$$= \frac{mn\theta_2}{(m+1)(m+n+1)}-\frac{mn\theta_1^{m+1}}{(m+1)(m+n+1)\theta_2^m}$$

E(Y^2$I_{(\frac{\theta_1}{\theta_2},1)}$(t))= $\frac{mn\theta_2^2}{(m+2)(m+n+2)}-\frac{mn\theta_1^{m+2}}{(m+2)(m+n+2)\theta_2^m}$

From a, b, c and d

$$E(T^*)= \frac{\alpha}{2}\left(\frac{m+n}{n+m+1}\right)\left\{\frac{n}{n+1}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{1}{n}}\frac{\theta_2^{n+1}}{\theta_1^n}+\frac{\theta_1^{m+1}}{\theta_2^m}\right\} + \frac{n\theta_1^{m+1}}{(n+m+1)\theta_2^m} - \frac{\alpha}{2}\frac{m+n}{m+n+1}\frac{\theta_1^{m+1}}{\theta_2^m} + \frac{m\theta_2}{n+m+1} - \frac{\alpha}{2}\frac{m+n}{m+n+1}\frac{\theta_2^{n+1}}{\theta_1^n} + \frac{mn\theta_2}{(m+1)(m+n+1)}$$

$$-\frac{mn\theta_1^{m+1}}{(m+1)(m+n+1)\theta_2^m}$$

Thus,

$$E(T^*)= \frac{\alpha}{2}\frac{m+n}{m+n+1}\frac{\theta_2^{n+1}}{\theta_1^n}\left\{\frac{n}{n+1}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{1}{n}}-1\right\}+\frac{n\theta_1^{m+1}}{(m+1)(n+m+1)\theta_2^m}+\frac{m\theta_2}{m+1} \qquad (4.2)$$

In estimating θ_1 by T^* the bias is,

$$b_{T^*}\left(\frac{\theta_1}{\theta_2}\right)= E(T^*) - \theta_1$$

$$= \frac{\alpha}{2}\frac{m+n}{m+n+1}\frac{\theta_2^{n+1}}{\theta_1^n}\left\{\frac{n}{n+1}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{1}{n}}-1\right\}+\frac{n\theta_1^{m+1}}{(m+1)(n+m+1)\theta_2^m}+\frac{m\theta_2}{m+1} - \theta_1 \qquad (4.3)$$

We also have,

$$E(T^{*2}) = E[X^2\{I_{(0,c_1)}(t) + I_{(c_2,\infty)}(t)\}] + E[X^2 I_{(1,c_2)}(t)] + E[Y^2 I_{(c_1,\frac{\theta_1}{\theta_2})}(t)]$$

$$+E[Y^2 I_{(\frac{\theta_1}{\theta_2},1)}(t)]$$

$$= \frac{\alpha}{2}\frac{m+n}{m+n+2}\frac{\theta_2^{n+2}}{\theta_1^n}\left\{\frac{n}{n+2}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{2}{n}}-1\right\}$$

$$+\frac{n\theta_1^{m+2}}{(n+m+2)\theta_2^m}\left[1-\frac{m}{m+2}\right]+\frac{m\theta_2^2}{m+n+2}\left[1+\frac{n}{m+2}\right]$$

$$E(T^{*2})= \frac{\alpha}{2}\frac{m+n}{m+n+2}\frac{\theta_2^{n+2}}{\theta_1^n}\left\{\frac{n}{n+2}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{2}{n}}-1\right\}+\frac{2n\theta_1^{m+2}}{(m+2)(n+m+2)\theta_2^m}+\frac{m\theta_2^2}{m+2} \qquad (4.4)$$

The MSE (mean squared error) of T^* in estimating θ_1 is given by,

$$E(T^{*}-\theta_1)^2 = E(T^{*})^2 - 2\,\theta_1\, E(T^{*}) + \theta_1^{\,2} = \frac{\alpha}{2}\frac{m+n}{m+n+2}\frac{\theta_2^{n+2}}{\theta_1^{n}}\left\{\frac{n}{n+2}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{2}{n}} - 1\right\}$$

$$-\alpha\frac{n+m}{n+m+1}\frac{\theta_2^{n+1}}{\theta_1^{n-1}}\left[\frac{n}{n+1}\left\{\alpha\frac{(n+m)}{2m}\right\}^{\frac{1}{n}} - 1\right]$$

$$-\frac{2n(2m+n+3)\theta_1^{m+2}}{(m+1)(m+2)(m+n+1)(n+m+2)\theta_2^{m}} + \frac{m\theta_2^2}{m+2} - \frac{2m\theta_1\theta_2}{m+1} + \theta_1^2 \qquad (4.5)$$

If m=n,

$$b_{T^*}\left(\frac{\theta_1}{\theta_2}\right) = \frac{n}{n+1}\left[\frac{\alpha n}{2n+1}\frac{\theta_2}{\theta^{n}}\left\{\alpha^{\frac{1}{n}} - \frac{n+1}{n}\right\} + \theta_1\left\{\frac{\theta^{n}}{2n+1} - \frac{n+1}{n}\right\} + \theta_2\right], \qquad (4.6)$$

where, $\theta = \frac{\theta_1}{\theta_2}$.

5.CASE- II: $\theta_1 > \theta_2$

In this situation, $\frac{\theta_1}{\theta_2} > 1.\ c_1 < 1 < c_2, c_1 < \frac{\theta_1}{\theta_2} < c_2, c_1 < 1 < \frac{\theta_1}{\theta_2} < c_2$

$$T^* = \begin{cases} X \ , if\ 0 < t < c_1 \\ Y \ , if\ c_1 \le t < 1 \\ X \ , if\ 1 \le t < \frac{\theta_1}{\theta_2} \\ X \ , if\ \frac{\theta_1}{\theta_2} \le t < \infty \end{cases}$$

Thus,

$$T^* = X\, I_{(0,c_1)} + Y\, I_{(c_1,1)}(t) + X\, I_{\left(1,\frac{\theta_1}{\theta_2}\right)}(t) + X\, I_{\left(\frac{\theta_1}{\theta_2},\infty\right)}(t) \qquad (5.1)$$

$$= (I) + (II) + (III) + (IV)$$

$E(I) = E[X\, I_{(0,\, c_1)}(t)]$

$$= E[\, I_{(0,\, c_1)}(t)\ E\{X\ I\ t \in (0, c_1)\}\ ]$$

$$= \frac{n+m}{n+m+1}\,\theta_2 \int_0^{c_1} t\, h_T(t \,|\, \theta_1, \theta_2)\,dt\,, \quad c_1 = \left[\frac{\alpha(m+n)}{2m}\right]^{\frac{1}{n}}$$

$$= \frac{mn}{m+n+1}\frac{\theta_2^{n+1}}{\theta_1^{n}}\frac{1}{n+1}\left\{\frac{\alpha(m+n)}{2m}\right\}^{\frac{n+1}{n}}$$

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$E(I^2) = E[X^2 \, I_{(0,\, c_1)}(t)]$

$$= \frac{mn}{m+n+2} \frac{\theta_2^{n+2}}{\theta_1^n} \frac{1}{n+2} \left\{\frac{\alpha(m+n)}{2m}\right\}^{\frac{n+2}{n}}$$

$E(II) = E[\, I_{(c_1,1)}(t)\; E\{Y \mid t \in (c_1,1)\}\,]$

$$= \frac{n+m}{n+m+1} \theta_1 \int_{c_1}^{1} \frac{1}{t} \, h_T(t|\theta_1, \theta_2)\, dt$$

$$= \frac{n+m}{n+m+1} \theta_2 \frac{nm}{m+n} \left(\frac{\theta_2}{\theta_1}\right)^n \frac{1}{n} \left(1 - c_1^n\right)$$

$$= \frac{m}{n+m+1} \frac{\theta_2^{n+1}}{\theta_1^n} \left[1 - \frac{\alpha(m+n)}{2m}\right]$$

$$= \frac{m\theta_2^{n+1}}{(n+m+1)\theta_1^n} - \frac{\alpha}{2} \frac{m+n}{m+n+1} \frac{\theta_2^{n+1}}{\theta_1^n}$$

$E(II^2) = E\,[Y^2 \, I_{(c_1,1)}(t)\,]$

$$= \frac{m\theta_2^{n+2}}{(n+m+2)\theta_1^n} - \frac{\alpha}{2} \frac{m+n}{m+n+2} \frac{\theta_2^{n+2}}{\theta_1^n}$$

$E(III) = E\,[I_{\left(1,\frac{\theta_1}{\theta_2}\right)}(t)\; E\{X \mid t \in (1,\frac{\theta_1}{\theta_2})\}]$

$$= \frac{m+n}{m+n+1} \frac{\theta_2^{n+1}}{\theta_1^n} \frac{mn}{m+n} \frac{1}{n+1} \left(\frac{\theta_1^{n+1}}{\theta_2^{n+1}} - 1\right)$$

$$= \frac{mn\theta_1}{(n+1)(m+n+1)} - \frac{mn\theta_2^{n+1}}{(n+1)(m+n+1)\theta_1^n}$$

$E(III^2) = E[X^2 \, I_{\left(1,\frac{\theta_1}{\theta_2}\right)}(t)]$

$$= \frac{mn\theta_1^2}{(n+2)(m+n+2)} - \frac{mn\theta_2^{n+2}}{(n+2)(m+n+2)\theta_1^n}$$

$E(IV) = E\,[X \, I_{\left(\frac{\theta_1}{\theta_2},\infty\right)}(t)]$

$$= E\,[I_{\left(\frac{\theta_1}{\theta_2},\infty\right)}(t)\; E\{X \mid t \in \left(\frac{\theta_1}{\theta_2}, \infty\right)\}\,]$$

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$$= \frac{m+n}{m+n+1} \frac{\theta_1^{m+1}}{\theta_2^m} \frac{nm}{m+n} \int_{\theta_1/\theta_2}^{\infty} \frac{1}{t^{m+1}}\, dt$$

$$= \frac{n\theta_1^{m+1}}{(m+n+1)\theta_2^m} \left(\frac{\theta_2^m}{\theta_1^m}\right)$$

$$E(IV) = \frac{n\theta_1}{m+n+1}$$

$$E(IV^2) = E\left[X^2 I_{\left(\frac{\theta_1}{\theta_2},\infty\right)}(t)\right]$$

$$= \frac{m+n}{m+n+2} \frac{\theta_1^{m+2}}{\theta_2^m} \frac{nm}{n+m} \frac{1}{m} \left(\frac{\theta_2^m}{\theta_1^m}\right)$$

$$= \frac{n\theta_1^2}{m+n+2}$$

Thus,

$$E(T^*) = \frac{mn}{m+n+1} \frac{\theta_2^{n+1}}{\theta_1^n} \frac{1}{n+1} \left\{\frac{\alpha(m+n)}{2m}\right\}^{\frac{n+1}{n}} + \frac{m\theta_2^{n+1}}{(n+m+1)\theta_1^n} - \frac{\alpha}{2} \frac{m+n}{m+n+1} \frac{\theta_2^{n+1}}{\theta_1^n}$$
$$+ \frac{mn\theta_1}{(n+1)(m+n+1)} - \frac{mn\theta_2^{n+1}}{(n+1)(m+n+1)\theta_1^n} + \frac{n\theta_1}{m+n+1}.$$

$$= \frac{\alpha}{2} \frac{\theta_2^{n+1}(m+n)}{(m+n+1)\theta_1^n} \left[\frac{n}{n+1} \left\{\frac{\alpha(n+m)}{2m}\right\}^{\frac{1}{n}} - 1\right] + \frac{m\theta_2^{n+1}}{(m+n+1)\theta_1^n} \left[1 - \frac{n}{n+1}\right] + \frac{n\theta_1}{m+n+1} \left\{\frac{m}{n+1} + 1\right\}$$

$$E(T^*) = \frac{\alpha}{2} \frac{m+n}{m+n+1} \frac{\theta_2^{n+1}}{\theta_1^n} \left[\frac{n}{n+1} \left\{\frac{\alpha(n+m)}{2m}\right\}^{\frac{1}{n}} - 1\right] + \frac{m\theta_2^{n+1}}{(n+1)(m+n+1)\theta_1^n} + \frac{n\theta_1}{n+1} \quad (5.2)$$

$$E(T^{*2}) = \frac{mn}{m+n+2} \frac{\theta_2^{n+2}}{\theta_1^n} \frac{1}{n+2} \left\{\frac{\alpha(m+n)}{2m}\right\}^{\frac{n+2}{n}} + \frac{m\theta_2^{n+2}}{(n+m+2)\theta_1^n} - \frac{\alpha}{2} \frac{m+n}{m+n+2} \frac{\theta_2^{n+2}}{\theta_1^n}$$
$$+ \frac{mn\theta_1^2}{(n+2)(m+n+2)} - \frac{mn\theta_2^{n+2}}{(n+2)(m+n+2)\theta_1^n} + \frac{n\theta_1^2}{m+n+2}$$

$$= \frac{\alpha}{2}\frac{\theta_2^{n+2}(m+n)}{(m+n+2)\theta_1^{n}}\left[\frac{n}{n+2}\left\{\frac{\alpha(n+m)}{2m}\right\}^{\frac{2}{n}}-1\right]+\frac{m\theta_2^{n+2}}{(m+n+2)\theta_1^{n}}\left[1-\frac{n}{n+2}\right]+\frac{n\theta_1^2}{m+n+2}\left\{\frac{m}{n+2}+1\right\}$$

$$E({T^*}^2)=\frac{\alpha}{2}\frac{m+n}{m+n+2}\frac{\theta_2^{n+2}}{\theta_1^{n}}\left[\frac{n}{n+2}\left\{\frac{\alpha(n+m)}{2m}\right\}^{\frac{2}{n}}-1\right]+\frac{2m\theta_2^{n+2}}{(n+2)(m+n+2)\theta_1^{n}}+\frac{n\theta_1^2}{n+2} \qquad (5.3)$$

In this situation, i.e., when $\theta_1 > \theta_2$ the bias of T^* in estimating θ_1 is given by,

$$b_{T^*}\left(\frac{\theta_1}{\theta_2}\right)= E(T^*) - \theta_1$$

$$= \frac{\alpha}{2}\frac{m+n}{m+n+1}\frac{\theta_2^{n+1}}{\theta_1^{n}}\left\{\frac{n}{n+1}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{1}{n}}-1\right\}+\frac{m\theta_2^{n+1}}{(n+1)(n+m+1)\theta_1^{n}}-\frac{\theta_1}{n+1} \qquad (5.4)$$

Also

$$E(T^*-\theta_1)^2= E(T^*)^2 - 2\,\theta_1\, E(T^*) + {\theta_1}^2$$

$$= \frac{\alpha}{2}\frac{m+n}{m+n+2}\frac{\theta_2^{n+2}}{\theta_1^{n}}\left\{\frac{n}{n+2}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{2}{n}}-1\right\} - \alpha\frac{n+m}{n+m+1}\frac{\theta_2^{n+1}}{\theta_1^{n-1}}\left[\frac{n}{n+1}\left\{\alpha\frac{(n+m)}{2m}\right\}^{\frac{1}{n}}-1\right]+\frac{2m\theta_2^{n+2}}{(n+2)(n+m+2)\theta_1^{n}}+\frac{n\theta_1^2}{n+2}-\frac{m\theta_2^{n+1}}{(n+1)(m+n+1)\theta_1^{n-1}}-\frac{2n\theta_1^2}{n+1}+\theta_1^2$$

$$= \frac{\alpha}{2}\frac{m+n}{m+n+2}\frac{\theta_2^{n+2}}{\theta_1^{n}}\left\{\frac{n}{n+2}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{2}{n}}-1\right\} - \alpha\frac{n+m}{n+m+1}\frac{\theta_2^{n+1}}{\theta_1^{n-1}}\left[\frac{n}{n+1}\left\{\alpha\frac{(n+m)}{2m}\right\}^{\frac{1}{n}}-1\right]+\frac{m\theta_2^{n+1}}{\theta_1^{n}}\left[\frac{2\theta_2}{(n+2)(n+m+2)}-\frac{\theta_1}{(n+1)(m+n+1)}\right]+\theta_1^2\left[\frac{n}{n+2}-\frac{2n}{n+1}+1\right]$$

$$= \frac{\alpha}{2}\frac{m+n}{m+n+2}\frac{\theta_2^{n+2}}{\theta_1^{n}}\left\{\frac{n}{n+2}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{2}{n}}-1\right\}\alpha\frac{n+m}{n+m+1}\frac{\theta_2^{n+1}}{\theta_1^{n-1}}\left[\frac{n}{n+1}\left\{\alpha\frac{(n+m)}{2m}\right\}^{\frac{1}{n}}-1\right]+\frac{m\theta_2^{n+1}}{\theta_1^{n}}\left[\frac{2\theta_2}{(n+2)(n+m+2)}-\frac{\theta_1}{(n+1)(m+n+1)}\right]+\frac{2\theta_1^2}{(n+1)(n+2)} \qquad (5.5)$$

6. CASES I AND II TOGETHER.

Thus, from the cases I and II above we conclude that,

$$T^* = \begin{cases} X\{I_{(0,c_1)}(t) + I_{(1,\infty)}(t)\} + Y\left\{I_{\left(c_1,\frac{\theta_1}{\theta_2}\right)}(t) + I_{\left(\frac{\theta_1}{\theta_2},1\right)}(t)\right\}, & if\, \theta_1 < \theta_2 \\ X\{I_{(0,c_1)}(t) + I_{(1,\infty)}(t)\} + YI_{(c_1,1)}(t), & if\, \theta_1 \geq \theta_2 \end{cases} \quad (6.1)$$

Its bias is given by,

$$b_{T^*}(\theta_1,\theta_2) = \begin{cases} \frac{\alpha}{2}\frac{m+n}{m+n+1}\frac{\theta_2^{n+1}}{\theta_1^n}\left\{\frac{n}{n+1}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{1}{n}} - 1\right\} + \frac{n\theta_1^{m+1}}{(m+1)(n+m+1)\theta_2^m} + \frac{m\theta_2}{m+1} - \theta_1, if\, \theta_1 < \theta_2 \\ \frac{\alpha}{2}\frac{m+n}{m+n+1}\frac{\theta_2^{n+1}}{\theta_1^n}\left\{\frac{n}{n+1}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{1}{n}} - 1\right\} + \frac{m\theta_2^{n+1}}{(n+1)(n+m+1)\theta_1^n} - \frac{\theta_1}{n+1}, & if\, \theta_1 \geq \theta_2 \end{cases} \quad (6.2)$$

The MSE of T^* is given by,

$$E(T^*-\theta_1)^2 = \begin{cases} b_2 - \frac{2n(2m+n+3)\theta_1^{m+2}}{(m+1)(m+2)(n+m+1)(n+m+2)\theta_2^m} + \frac{m\theta_2^2}{m+2} - \frac{2m\theta_1\theta_2}{m+1} + \theta_1^2, if\, \theta_1 < \theta_2 \\ b_2 + \frac{m\theta_2^{n+1}}{\theta_1^n}\left[\frac{2\theta_2}{(n+2)(n+m+2)} - \frac{\theta_1}{(n+1)(n+m+1)}\right] + \frac{2\theta_1^2}{(n+1)(n+2)}, if\, \theta_1 \geq \theta_2 \end{cases} \quad (6.3)$$

Where

$$b_2 = \frac{\alpha}{2}\frac{m+n}{m+n+2}\frac{\theta_2^{n+2}}{\theta_1^n}\left\{\frac{n}{n+2}\left(\frac{\alpha(m+n)}{2m}\right)^{\frac{2}{n}} - 1\right\} - \alpha\frac{n+m}{n+m+1}\frac{\theta_2^{n+1}}{\theta_1^{n-1}}\left[\frac{n}{n+1}\left\{\frac{\alpha(n+m)}{2m}\right\}^{\frac{1}{n}} - 1\right] \quad (6.4)$$

If n=m, $c_1=\alpha^{\frac{1}{n}}$, $c_2=\left(\frac{1}{\alpha}\right)^{1/m}$ and

$$b_{T^*}(\theta_1,\theta_2) = \begin{cases} \frac{\alpha}{2}\frac{2n}{2n+1}\frac{\theta_2^{n+1}}{\theta_1^n}\left\{\frac{n}{n+1}(\alpha)^{\frac{1}{n}} - 1\right\} + \frac{n\theta_1^{n+1}}{(n+1)(2n+1)\theta_2^n} + \frac{n\theta_2}{n+1} - \theta_1, if\, \theta_1 < \theta_2 \\ \frac{\alpha}{2}\frac{2n}{2n+1}\frac{\theta_2^{n+1}}{\theta_1^n}\left\{\frac{n}{n+1}(\alpha)^{\frac{1}{n}} - 1\right\} + \frac{n\theta_2^{n+1}}{(n+1)(2n+1)\theta_1^n} - \frac{\theta_1}{n+1}, if\, \theta_1 \geq \theta_2 \end{cases}$$

$$b_{T^*}(\theta_1,\theta_2) = \begin{cases} \frac{\alpha n}{2n+1}\frac{\theta_2^{n+1}}{\theta_1^n}\left\{\frac{n}{n+1}(\alpha)^{\frac{1}{n}} - 1\right\} + \frac{n\theta_1^{n+1}}{(n+1)(2n+1)\theta_2^n} + \frac{n\theta_2}{n+1} - \theta_1, if\, \theta_1 < \theta_2 \\ \frac{\alpha n}{2n+1}\frac{\theta_2^{n+1}}{\theta_1^n}\left\{\frac{n}{n+1}(\alpha)^{\frac{1}{n}} - 1\right\} + \frac{n\theta_2^{n+1}}{(n+1)(2n+1)\theta_1^n} - \frac{\theta_1}{n+1}, if\, \theta_1 \geq \theta_2 \end{cases} \quad (6.5)$$

If α=0, T^* gives always pooled estimator and $c_1=0$, $c_2=\infty$. This gives

$$b_{T^*}(\theta_1,\theta_2) = \begin{cases} \frac{n\theta_1^{m+1}}{(m+1)(n+m+1)\theta_2^m} + \frac{m\theta_2}{m+1} - \theta_1, & if\ \theta_1 < \theta_2 \\ \frac{m\theta_2^{n+1}}{(n+1)(n+m+1)\theta_1^n} - \frac{\theta_1}{n+1}, & if\ \theta_1 \geq \theta_2 \end{cases} \tag{6.6}$$

If n=m, this becomes,

$$b_{T^*}(\theta_1,\theta_2) = \begin{cases} \frac{n\theta_1^{n+1}}{(n+1)(2n+1)\theta_2^m} + \frac{n\theta_2}{n+1} - \theta_1, if\ \theta_1 < \theta_2 \\ \frac{n\theta_2^{n+1}}{(n+1)(2n+1)\theta_1^n} - \frac{\theta_1}{n+1}, if\ \theta_1 \geq \theta_2 \end{cases} \tag{6.7}$$

In this putting $\theta_1 = \theta_2$, we have

$$b_{T^*}(\theta_1) = \frac{n\theta_1^{n+1}}{(n+1)(2n+1)\theta_1^n} + \frac{n\theta_1}{n+1} - \theta_1$$

$$= \frac{n\theta_1 - 2n\theta_1 - \theta_1}{(n+1)(2n+1)}$$

$$= \frac{-\theta_1(n+1)}{(n+1)(2n+1)}$$

$$= -\frac{\theta_1}{2n+1}$$: the bias of the maximum of a sample of size 2n.

If m=n the MSE of T^* is given by,

$$E(T^*-\theta_1)^2 = \begin{cases} b_2' - \frac{n\theta_1^{n+2}}{(n+2)(2n+1)(2n+2)\theta_2^n} + \frac{n\theta_2^2}{n+2} - \frac{2n\theta_1\theta_2}{n+1} + \theta_1^2, if\ \theta_1 < \theta_2 \\ b_2' + \frac{n\theta_2^{n+1}}{\theta_1^n}\left[\frac{2\theta_2}{(n+2)(2n+2)} - \frac{\theta_1}{(n+1)(2n+1)}\right] + \frac{2\theta_1^2}{(n+1)(n+2)}, if\ \theta_1 \geq \theta_2 \end{cases} \tag{6.8}$$

Where

$$b_2' = \frac{\alpha}{2}\frac{n}{n+1}\frac{\theta_2^{n+2}}{\theta_1^n}\left[\frac{n}{n+2}\alpha^{\frac{2}{n}} - 1\right] - \alpha\frac{2n}{2n+1}\frac{\theta_2^{n+1}}{\theta_1^{n-1}}\left[\frac{n}{n+1}\alpha^{\frac{1}{n}} - 1\right]$$

If α=0, in this, then, b_2'=0 and

$$E(T^*-\theta_1)^2 = \begin{cases} -\frac{6n\theta_1^{n+2}}{(n+2)(2n+1)(2n+2)\theta_2^n} + \frac{n\theta_2^2}{n+2} - \frac{2n\theta_1\theta_2}{n+1} + \theta_1^2, if\ \theta_1 < \theta_2 \\ \frac{n\theta_2^{n+1}}{(n+1)\theta_1^n}\left[\frac{\theta_2}{(n+2)} - \frac{\theta_1}{(2n+1)}\right] + \frac{2\theta_1^2}{(n+1)(n+2)}, if\ \theta_1 \geq \theta_2 \end{cases} \tag{6.9}$$

$$= \frac{\theta_1^2}{(n+1)(2n+1)}. \tag{6.10}$$

(6.10)is the MSE of the maximum in the 2n observations, if $\theta_1 = \theta_2$.

If α=1, m=n we have c_1=c_2=1 and T^* becomes never pool estimator. In this case

$$b_{T^*}(\theta_1,\theta_2)=\begin{cases}\frac{n}{2n+1}\frac{\theta_2^{n+1}}{\theta_1^n}\left\{\frac{n}{n+1}-1\right\}+\frac{n\theta_1^{n+1}}{(n+1)(2n+1)\theta_2^n}+\frac{n\theta_2}{n+1}-\theta_1, if\ \theta_1<\theta_2\\ \frac{n}{2n+1}\frac{\theta_2^{n+1}}{\theta_1^n}\left\{\frac{n}{n+1}-1\right\}+\frac{n\theta_2^{n+1}}{(n+1)(2n+1)\theta_1^n}-\frac{\theta_1}{n+1}, if\ \theta_1\geq\theta_2\end{cases} \tag{6.11}$$

If $\theta_1 = \theta_2$, this becomes,

$$b_{T^*}(\theta_1) = -\frac{n\theta_1}{(2n+1)(n+1)} + \frac{n\theta_1}{(n+1)(2n+1)} - \frac{\theta_1}{n+1}$$

$$= -\frac{\theta_1}{n+1}. \tag{6.12}$$

(6.12) gives bias of the maximum in n observations in single sample.

If m=n, α=1,

$$b_2=\frac{n\theta_2^{n+2}}{(n+1)\theta_1^n}\left[\frac{2n(\theta_1-\theta_2)+4\theta_1-\theta_2}{(n+2)(2n+1)\theta_2}\right]$$

If $\theta_1 = \theta_2$, this becomes,

$$b_2=\frac{3n\theta_1^2}{(n+1)(n+2)(2n+1).}$$

If n=m, α=1, $\theta_1 = \theta_2$

$E(T^*-\theta_1)^2 = \frac{2\theta_1^2}{(n+1)(n+2).}$: MSE of maximum of sample of size n (i.e., of X).

Note that,

$$\frac{\theta_1^2}{(n+1)(2n+1)} < \frac{2\theta_1^2}{(n+1)(n+2)}, \quad \forall\ n\geq 1 \text{ and } \theta_1>0.$$

That is, MSE of proposed estimator T* is smaller than that of the sample maximum of size n ∀ n≥1 and θ_1>0.

7. COMPARISON OF ESTIMATORS:

To have an idea of comparative values of mean square error (MSE) of the proposed estimator T* and the variance $V = \frac{\theta_1^2}{n(n+2)}$ of uniformly minimum variance unbiased (UMVU) Estimator based on the single sample, $T = \frac{(n+1)}{n}X$, we calculate them for some particular values. T* would provide better estimator when θ_1=θ_2. For the sake of convenience, let us take θ_2=1 and θ_1=0.4, 0.5,…,1.3 etc. Using these values of the parameters we evaluate MSE of T* and the variance V of UMVUE based on single sample. For small samples T* gives smaller MSE than V. This is illustrated by choosing some values of the sample sizes n and m. Following tables and the graphs make it clear that whenever θ_1=θ_2, T* can be used in a short span of the values. But, we can not say that T* is uniformly better than T for all θ_1,θ_2, and for all n, m.

Thus, the proposed estimator T* can be profitably used in the specific region, with care. If the sample sizes are more than 15, the UMVUE is consistently better than T* as its variance is less than the corresponding MSE of T*. In this discussion I did not consider the magnitude of the bias of T*. But, I have derived expressions for bias of T* in various situations.

1. **Here we consider n=3, m=8 and θ_2=1. The MSE(T*) and Variance V are as below**

Table 7.1

MSE(T*) , VARIANCE V WHEN n=3,m=8 AND θ_2=1

θ_1	n	m	mse	var
0.4	3	8	0.155928	0.010667
0.5	3	8	0.141231	0.016667
0.6	3	8	0.097822	0.024
0.7	3	8	0.058256	0.032667
0.8	3	8	**0.032224**	**0.042667**
0.9	3	8	**0.022331**	**0.054**
1	3	8	**0.027665**	**0.066667**
1.1	3	8	0.181955	0.080667
1.2	3	8	0.183311	0.096
1.3	3	8	0.193911	0.112667

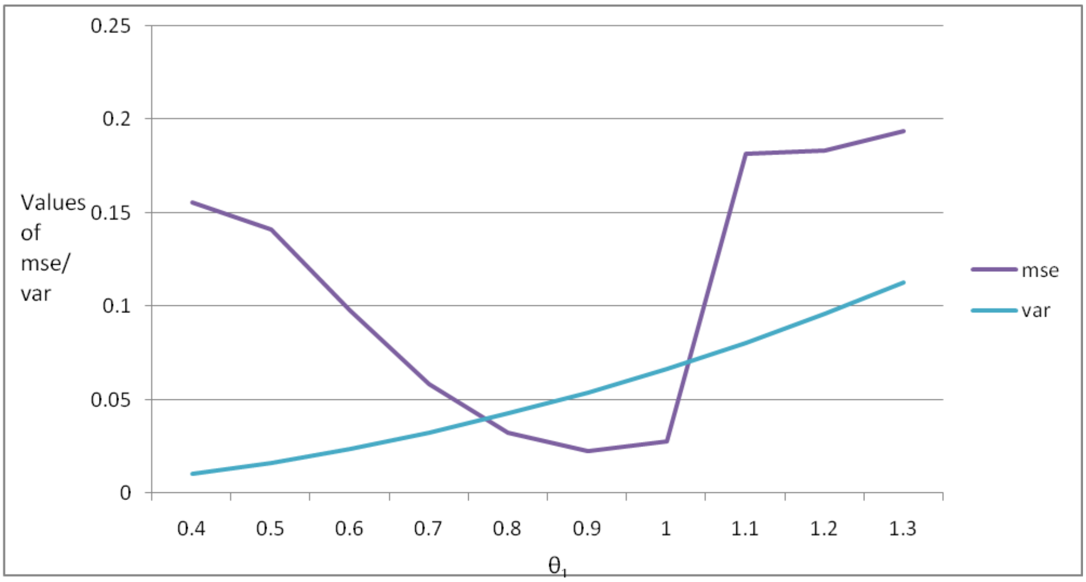

Figure 7.1

Figure 7.1 shows the span where T* provides a better estimate than UMVUE based on single sample.

2. Here we consider n=5, m=7 and θ_2=1. The MSE(T*) and Variance V are as below.

Table 7.2

MSE(T*) , VARIANCE V WHEN n=5,m=7 AND θ_2=1

θ_1	n	m	mse	var
0.6	5	7	0.065936	0.010286
0.7	5	7	0.047346	0.014
0.8	5	7	0.026049	0.018286
0.9	5	7	**0.016461**	**0.023143**
1	5	7	**0.019571**	**0.028571**
1.1	5	7	0.091949	0.034571
1.2	5	7	0.088216	0.041143
1.3	5	7	0.091916	0.048286
1.4	5	7	0.100039	0.056

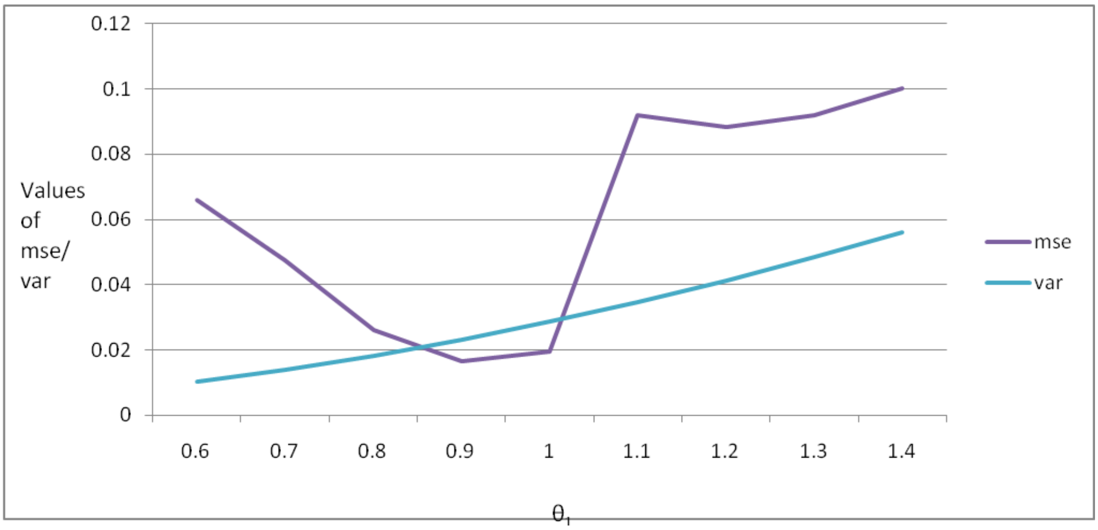

Figure 7.2

3. Here we consider n=8, m=3 and θ_2=1. The MSE(T*) and Variance V are as below.

Table 7.3

MSE(T*) , VARIANCE V WHEN n=8,m=3 AND θ_2=1

θ_1	n	m	mse	var
0.8	8	3	0.015295	0.008
0.9	8	3	0.013654	0.010125
1	8	3	0.016591	0.0125
1.1	8	3	0.036653	0.015125
1.2	8	3	0.036585	0.018
1.3	8	3	0.039823	0.021125
1.4	8	3	0.044726	0.0245
1.5	8	3	0.050627	0.028125
1.6	8	3	0.057235	0.032
1.7	8	3	0.064418	0.036125
1.8	8	3	0.072113	0.0405
1.9	8	3	0.080288	0.045125
2	8	3	0.088928	0.05
2.1	8	3	0.098023	0.055125
2.2	8	3	0.107569	0.0605
2.3	8	3	0.117564	0.066125
2.4	8	3	0.128005	0.072
2.5	8	3	0.138891	0.078125
2.6	8	3	0.150224	0.0845
2.7	8	3	0.162001	0.091125
2.8	8	3	0.174222	0.098
2.9	8	3	0.186889	0.105125

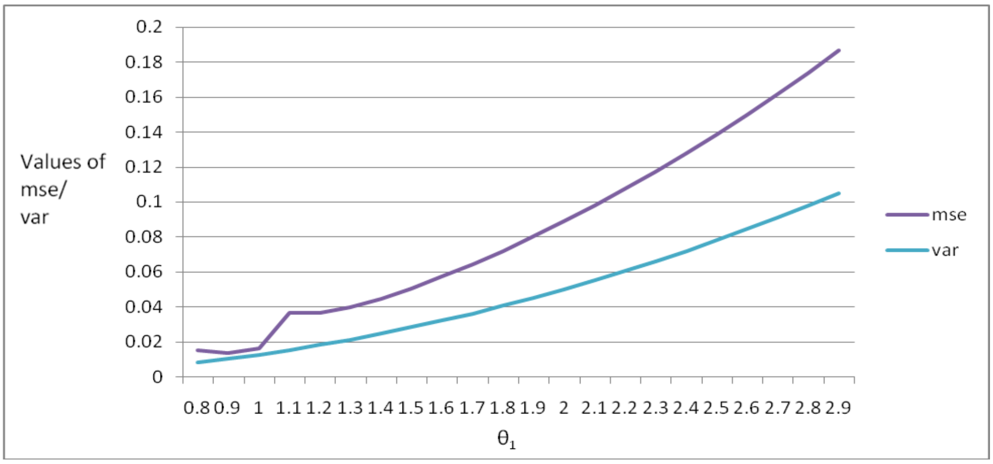

Figure 7.3

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4. **Here we consider n=m=3 and θ_2=1. The MSE(T*) and Variance V are as below.**

Table 7.4

MSE(T*) , VARIANCE V WHEN n=m=3 AND θ_2=1

θ_1	n=m	mse	var
0.4	3	0.084078	0.010667
0.5	3	0.084268	0.016667
0.6	3	0.061419	0.024
0.7	3	0.042279	0.032667
0.8	3	**0.033758**	**0.042667**
0.9	3	**0.036705**	**0.054**
1	3	**0.049507**	**0.066667**
1.1	3	0.157842	0.080667
1.2	3	0.167972	0.096
1.3	3	0.18439	0.112667
1.4	3	0.205548	0.130667
1.5	3	0.230507	0.15
1.6	3	0.258679	0.170667

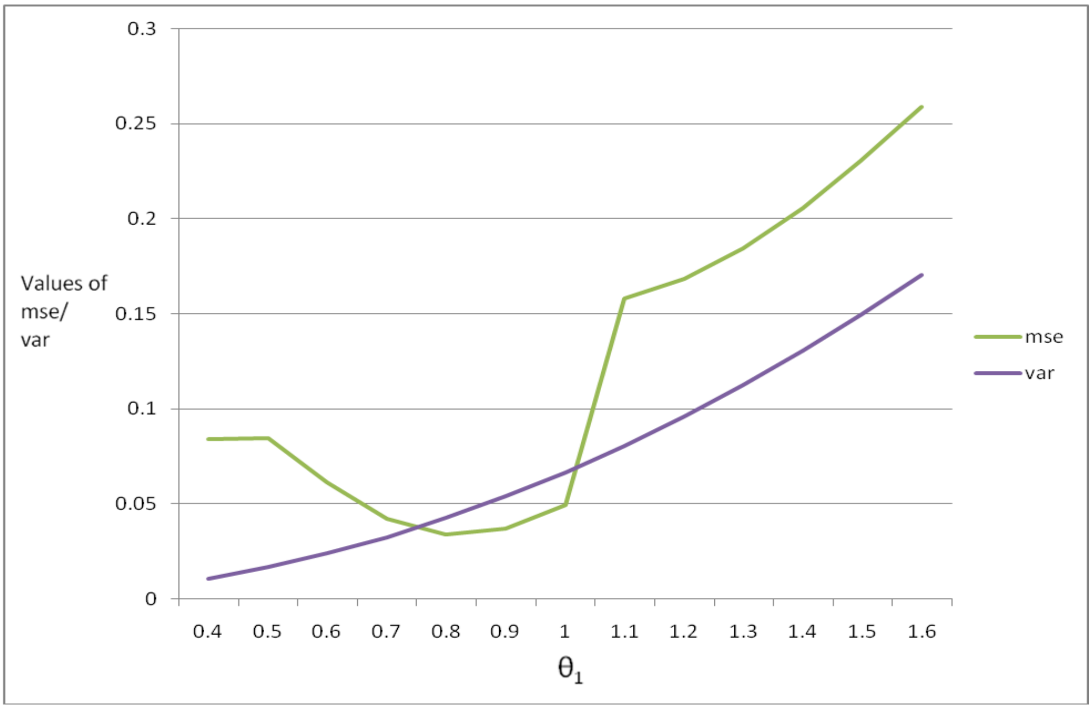

Figure 7.4

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5. Here we consider n=m=5 and θ_2=1. The MSE(T*) and Variance V are as below.

Table 7.5

MSE(T*) , VARIANCE V WHEN n=m=5 AND θ_2=1

θ_1	n=m	mse	var
0.6	5	0.053385	0.010286
0.7	5	0.040361	0.014
0.8	5	0.024444	0.018286
0.9	5	**0.018647**	**0.023143**
1	5	**0.023456**	**0.028571**
1.1	5	0.086482	0.034571
1.2	5	0.085198	0.041143
1.3	5	0.090243	0.048286
1.4	5	0.099125	0.056
1.5	5	0.110572	0.064286
1.6	5	0.123901	0.073143
1.7	5	0.138733	0.082571

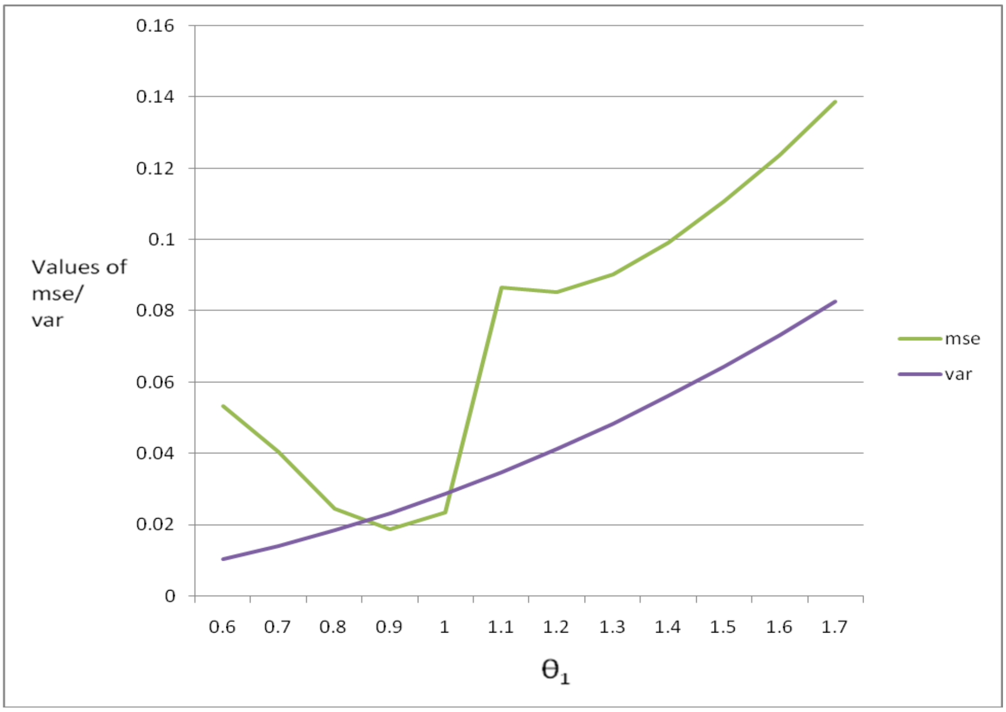

Figure 7.5

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6. Here we consider n=8, m=8 and θ_2=1. The MSE(T*) and Variance V are as below.

Table 7.6

MSE(T*) , VARIANCE V WHEN n=8,m=8 AND θ_2=1

θ_1	n=m	mse	var
0.7	8	0.027597	0.006125
0.8	8	0.020979	0.008
0.9	8	0.010685	0.010125
1	8	**0.011014**	**0.0125**
1.1	8	0.044467	0.015125
1.2	8	0.039973	0.018
1.3	8	0.041341	0.021125

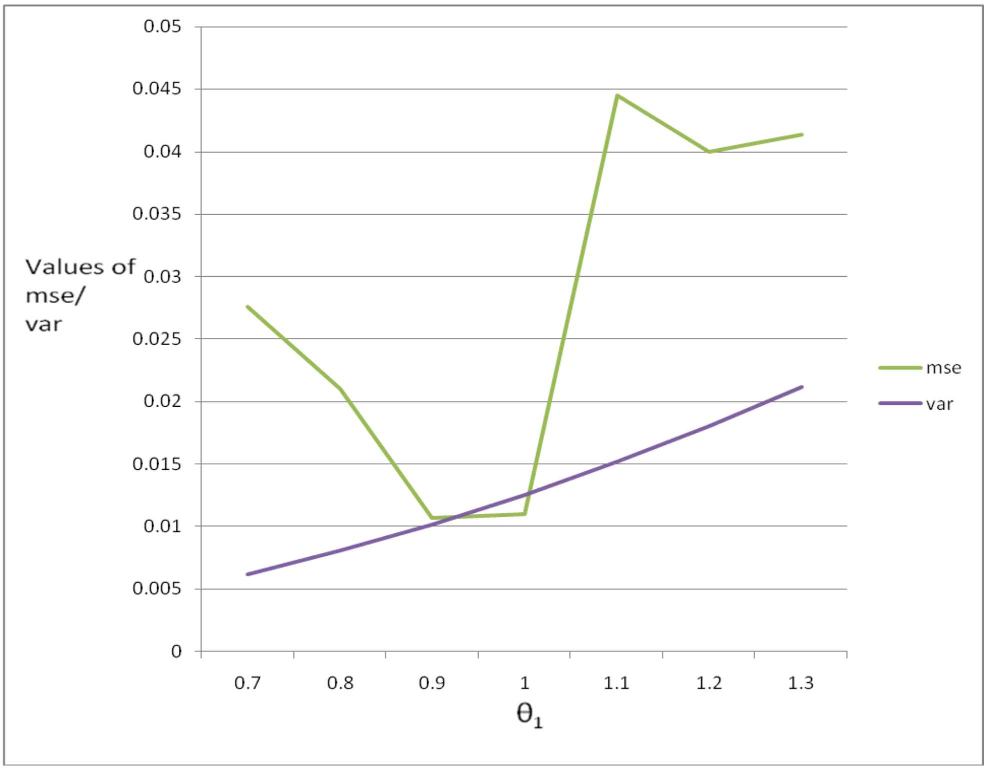

Figure 7.6

7. Here we consider n=15, m=15 and θ_2=1. The MSE(T*) and Variance V are as below.

Table 7.7

MSE(T*) , VARIANCE V WHEN n=15,m=15 AND θ_2=1

θ_1	n=m	mse	var
0.9	15	0.006613	0.003176
1	15	**0.003685**	**0.003922**
1.1	15	0.014804	0.004745
1.2	15	0.012066	0.005647

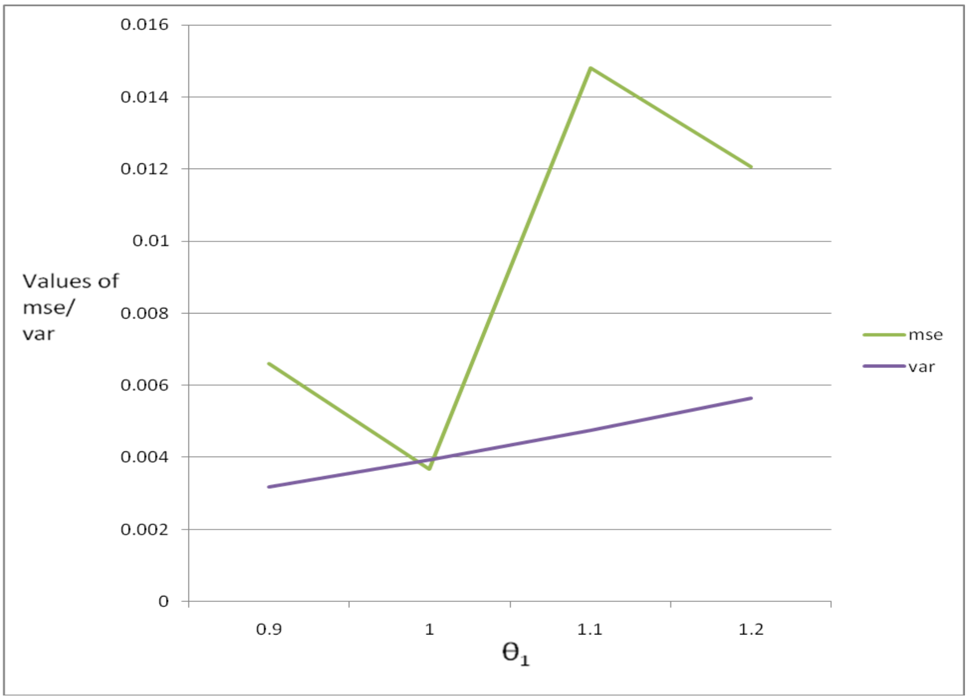

Figure 7.7

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